On the pumping effect in a pipe/tank flow configuration with friction

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Definition

Let $T > 0$, $g : \mathbb{R}^3 \to \mathbb{R}$ and let $e : \mathbb{R} \to \mathbb{R}$ be nonconstant and T-periodic. Then the equation

$$x'' = g(x, x', e(t))$$

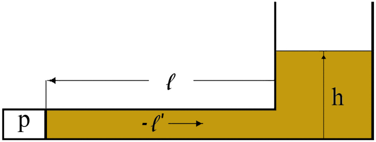
generates a *T-periodically forced pump* if it has a T-periodic solution x such that

$$g(\bar{x}, 0, \bar{e}) \neq 0$$

(i.e. the mean value $\bar{x}$ of x is not an equilibrium of $x'' = g(x, x', \bar{e})$.

G. Propst: Pumping effects in models of periodically forced flow configurations. *Physica D* **217** (2006), 193–201.

ρ	... density of the liquid (constant)
$p(t)$	... periodic pressure
g	... acceleration of gravity
r_0	... friction coefficient
ζ	... junction coefficient
A_P/A_T	... cross sections of pipe/tank
V_0	... constant total volume of liquid
$w = -\ell'$	... velocity in the pipe



$$A_P\,\ell(t) + A_T\,h(t) \equiv V_0 \quad \Longrightarrow \quad h(t) \equiv \frac{1}{A_T}\left(V_0 - A_P\,\ell(t)\right).$$

Momentum balance with Poiseuille's law and Bernoulli's equation

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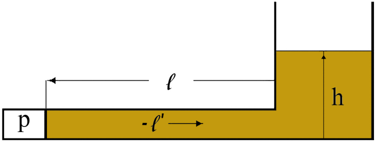

$$A_P\,\ell(t) + A_T\,h(t) \equiv V_0 \quad \Longrightarrow \quad h(t) \equiv \frac{1}{A_T}\big(V_0 - A_P\,\ell(t)\big)\,.$$

Momentum balance with Poiseuille's law and Bernoulli's equation $\Longrightarrow$

$$\ell\,\ell'' + a\,\ell\,\ell' + b\,(\ell')^2 + c\,\ell = e(t),$$

where

$$T > 0, \quad a = \frac{r_0}{\rho} > 0, \quad b = \Big(1 + \frac{\zeta}{2}\Big) \geq 3/2,$$

$$e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho} \text{ is } T - \text{periodic}, \quad 0 < c = \frac{g\,A_p}{A_T} < 1.$$

This leads to singular periodic problem:

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)\,,$$

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Multiplying the equation by u and integrating over $[0, T]$ gives

THEOREM 1

(1) has a positive solution only if $\overline{e} \geq 0$ (i.e. $\overline{p} \leq \rho\, g\, \frac{V_0}{A_T}$).

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THEOREM 2

If (1) has a positive solution, then its generates a T-periodically forced pump.

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

Theorem 3

ASSUME:

- $a \geq 0, \quad b > 1, \quad c > 0,$
- e is continuous and T-periodic on $\mathbb{R}$, $e_* > 0$,
- $\dfrac{(b+1)\,c^2}{4\,e_*} < \left(\dfrac{\pi}{T}\right)^2 + \dfrac{a^2}{4}.$

THEN: (1) has a positive solution.

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Definition

A T-periodic function $\sigma_1 \in C^2[0,T]$ is a *lower function* for

$$u'' + a\,u' + f(t,u) = 0 \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

if

$$\sigma_1''(t) + a\,\sigma_1'(t) + f(t,\sigma_1(t)) \geq 0 \quad \text{for } t \in [0,T],$$

while an *upper function* is defined analogously, but with reversed inequality.

STEP 1:

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)\,,$$

$x = u^{1/\mu},\ \mu = \frac{1}{b+1} \implies \quad (1) \rightsquigarrow$

$$(2) \qquad x'' + a\,x'(t) + s(t)\,x^{\beta} - r(t)\,x^{\alpha} = 0\,, \quad x(0) = x(T),\ x'(0) = x'(T)\,,$$

where

$$r(t) = \frac{e(t)}{\mu}, \quad s(t) = \frac{c}{\mu}, \ \alpha = 1 - 2\,\mu,\ \beta = 1 - \mu.$$

Proposition

$u : [0, T] \to \mathbb{R}$ is a positive solution of (1) iff $x = u^{1/\mu}$ is a positive solution of (2).

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$u : [0, T] \to \mathbb{R}$ is a positive solution of (1) iff $x = u^{1/\mu}$ is a positive solution of (2).

We have:

$$r_* > 0,\ s_* > 0, \quad 0 < \alpha < \beta < 1\,.$$

(2) $x'' + a\,x'(t) + f(t,x) = 0\,,\quad x(0) = x(T),\ x'(0) = x'(T),$ where $f(t,x) = s(t)\,x^{\beta} - r(t)\,x^{\beta}.$

<u>STEP 2</u>: We have

$$f(t,x) < 0 \ \text{ for } \ t \in [0,T] \ \text{ and } \ 0 < x < x_0 = (r_*/s^*)^{1/(\beta-\alpha)}\ ,$$

$$f(t,x) > 0 \ \text{ for } \ t \in [0,T] \ \text{ and } \ x > x_1 = (r^*/s_*)^{1/(\beta-\alpha)}\ .$$

Thus, there are constant lower and upper functions σ_1 and σ_2 of (2) such that

$$0 < \sigma_2 < x_0 < x_1 < \sigma_1.$$

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STEP 3: We put $\lambda_0 = \dfrac{(b+1)\,c^2}{4\,e_*}$ and show that there is $\delta_0 \in (0,\sigma_2)$ such that

$$\lambda\,(x-\delta) - f(t,x) \geq 0 \text{ for } t \in [0,T],\ \delta \in (0,\delta_0),\ \lambda \geq \lambda_0 \text{ and } x \in [\delta,\infty).$$

(2) $x'' + a\,x'(t) + f(t,x) = 0\,, \quad x(0) = x(T),\ x'(0) = x'(T),$ where $f(t,x) = s(t)\,x^{\beta} - r(t)\,x^{\beta}.$

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$$\lambda\,x - f(t,x) \ge x^{1-2\mu}\,(\lambda\,x^{2\mu} - s^*\,x^{\mu} + r_*)$$

(2) $x'' + a\,x'(t) + f(t,x) = 0\,,\quad x(0) = x(T),\ x'(0) = x'(T),$ where $f(t,x) = s(t)\,x^{\beta} - r(t)\,x^{\beta}.$

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STEP 4: Choose $\delta \in (0,\delta_0)$, define $\lambda^* = \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}$ and

$$\widetilde{f}(t,x) = \begin{cases} f(t,\delta) + \lambda^*\,(x-\delta) & \text{for } x < \delta\,, \\ f(t,x) & \text{for } x \ge \delta \end{cases}$$

and consider auxiliary problem

$$(3) \qquad x'' + a\,x'(t) + \widetilde{f}(t,x) = 0\,, \qquad x(0) = x(T),\ x'(0) = x'(T)\,,$$

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Lemma (Bonheure & De Coster, 2003)

ASSUME:

- $\widetilde{f} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous,
- σ_1 and σ_2 are lower and upper functions of (3),
- $\sigma_2 < \sigma_1$ on $[0,T]$,
- there is p continuous on $[0,T]$ such that

$$\limsup_{x\to-\infty} \widetilde{f}(t,x) \le p(t) \quad \text{and} \quad \limsup_{x\to\infty} \frac{\widetilde{f}(t,x)}{x} \le \frac{\pi^2}{T^2} \quad \text{uniformly in } t \in [0,T].$$

THEN: problem (3) has a solution x such that

$$\sigma_2(t_1) \le x(t_1) \le \sigma_1(t_1) \quad \text{for some } t_1 \in [0,T].$$

(3) $$x'' + a\,x'(t) + \widetilde{f}(t,x) = 0\,, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

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STEP 5: We show that $x \ge \delta$ for any solution x of (3).

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<u>STEP 5</u>: We show that $x\ge\delta$ for any solution x of (3).

Put $u=x-\delta$. Then

$$u''(t)+a\,u'(t)+\lambda^*\,u(t)=h(t)\ \text{ for } t\in[0,T],\ \ u(0)=u(T),\ u'(0)=u'(T),$$

where

$$h(t):=\lambda^*\,(x(t)-\delta)-\widetilde{f}(t,x(t))\ge 0\ \text{ on } [0,T]$$

due to STEP 3 and due to our assumption $\lambda^*=\left(\frac{\pi}{T}\right)^2+\frac{a^2}{4}>\lambda_0=\frac{(b+1)\,c^2}{4\,e_*}$.

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Lemma (Omari & Trombetta, 1992)

<u>ASSUME</u>: $a,\lambda\in\mathbb{R},\ 0<\lambda\le\left(\frac{\pi}{T}\right)^2+\frac{a^2}{4},\ h:[0,T]\to\mathbb{R}$ is continuous.

<u>THEN</u>: $u''+a\,u'+\lambda\,u=h(t),\ \ u(0)=u(T),\ u'(T)=u'(T)$

$\Longrightarrow\quad u\ge 0$ for all $h\ge 0$ on $[0,T]$.

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Put $u = x - \delta$. Then

$$u''(t) + a\,u'(t) + \lambda^*\,u(t) = h(t) \ \text{ for } t \in [0,T],\ u(0) = u(T),\ u'(0) = u'(T),$$

where

$$h(t) := \lambda^*\,(x(t) - \delta) - \widetilde{f}(t, x(t)) \ge 0 \ \text{ on } [0,T]$$

due to STEP 3 and due to our assumption $\lambda^* = \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} > \lambda_0 = \frac{(b+1)\,c^2}{4\,e_*}$.

Lemma (Omari & Trombetta, 1992)

ASSUME: $a, \lambda \in \mathbb{R}$, $0 < \lambda \le \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}$, $h : [0,T] \to \mathbb{R}$ is continuous.

THEN: $\quad u'' + a\,u' + \lambda\,u = h(t),\ u(0) = u(T),\ u'(T) = u'(T)$

$$\Longrightarrow \quad u \ge 0 \ \text{ for all } h \ge 0 \ \text{ on } [0,T].$$

Hence, $u \ge 0$ on $[0,T]$, i.e. $x \ge \delta$ on $[0,T]$. $\square$

(2) $$x'' + a\,x'(t) + f(t,x) = 0\,, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

Theorem 3

ASSUME:

- $f(t,x) = s(t)\,x^{\beta} - r(t)\,x^{\alpha}$,
- $r\,,s$ are continuous and positive on $[0,T]$, $0 < \alpha < \beta < 1$,

THEN: (2) has a positive solution whenever

a^2 is large enough or T is small enough.

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Recall that for

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the sufficient condition reads as follows $$\frac{(b+1)\,c^2}{4\,e_*} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}.$$

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To get a similar explicit bound also for (2), we need a good upper estimate for the roots of the algebraic equation

$$\lambda\,x^{1-\alpha}-s^*\,\beta\,x^{\beta-\alpha}+r_*\,\alpha=0\,.$$

Application of Krasnoselskii compresion/expansion theorem

(4) $\qquad x''+a\,x'+m^2\,x=0, \quad x(0)-x(T), \; x'(0)=x'(T)$

where $\qquad a\geq 0 \quad \text{and} \quad 0<m^2<\left(\frac{\pi}{T}\right)^2+\left(\frac{a}{2}\right)^2.$

Then (4) is non-resonant and possesses Green's function $G_m(t,s)$ such that

- $G_m(t,s)>0$ for all $t,s\in[0,T]$,
- $\displaystyle\int_0^T G_m(t,s)\,d\,s=\frac{1}{m^2},$
- there exists $c_m\in(0,1)$ such that $G_m(s,s)\geq c_m\,G(t,s)$ for all $t,s\in[0,T]$.

Put $\quad P=\{x\in C[0,T]: x(t)\geq 0 \text{ on } [0,T] \text{ and } x(t)\geq c_m\,\|x\| \text{ on } [0,T]\}.$

Krasnoselskii Fixed Point Theorem

Let P be a cone in X, Ω_1 and Ω_2 be bounded open sets in X such that $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$.
Let $F: P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator such that one of the following conditions holds:

- $\|F\,x\|\geq\|x\|$ for $x\in P\cap\partial\Omega_1$ and $\|F\,x\|\leq\|x\|$ for $x\in P\cap\partial\Omega_2$,
- $\|F\,x\|\leq\|x\|$ for $x\in P\cap\partial\Omega_1$ and $\|F\,x\|\geq\|x\|$ for $x\in P\cap\partial\Omega_2$.

Then F has a fixed point in the set $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Application of Krasnoselskii compresion/expansion theorem

(4) $\qquad x'' + a\,x' + m^2 x = 0, \quad x(0) - x(T), \; x'(0) = x'(T)$

where $\qquad a \geq 0 \quad \text{and} \quad 0 < m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2.$

Then (4) is non-resonant and possesses Green's function $G_m(t,s)$ such that

- $G_m(t,s) > 0$ for all $t, s \in [0,T]$,
- $\displaystyle\int_0^T G_m(t,s)\,d\,s = \frac{1}{m^2},$
- there exists $c_m \in (0,1)$ such that $G_m(s,s) \geq c_m\, G(t,s)$ for all $t, s \in [0,T]$.

Put $\quad P = \{x \in C[0,T] : x(t) \geq 0 \text{ on } [0,T] \;\; \text{and} \;\; x(t) \geq c_m\, \|x\| \text{ on } [0,T]\}.$

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(2) $\qquad x''+ax'+s(t)x^{\beta}-r(t)x^{\alpha}=0, \qquad x(0)=x(T),\ x'(0)=x'(T)$

Theorem 4

ASSUME: $a\geq 0,\ r,s\in C[0,T],\ 0<\alpha<\beta<1,$

- there exists $m>0$, with $m^2<\left(\frac{\pi}{T}\right)^2+\left(\frac{a}{2}\right)^2$, such that
$$r(t)x^{\alpha}-s(t)x^{\beta}+m^2x\geq 0 \quad \text{for } t\in[0,T],\ x\geq 0,$$
- $r_*>0$ and $s_*>0$.

THEN: (2) has a positive solution.

Application of Krasnoselskii compresion/expansion theorem

(2) $$x'' + a x' + s(t) x^{\beta} - r(t) x^{\alpha} = 0\,, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

Corollary 1

ASSUME: $a \geq 0,\ r, s \in C[0,T],\ 0 < \alpha < \beta < 1.$

- $r_* > 0$ and $s_* > 0$,
- $s^* < \min\{\left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2, r_*\}.$

THEN: (2) has a positive solution.

(1) $$u'' + a u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)\,,$$

Corollary 2=Theorem 3

ASSUME:

- $a \geq 0, \quad b > 1, \quad c > 0,$
- e is continuous and T-periodic on $\mathbb{R}$, $e_* > 0,$
- $\dfrac{(b+1)\,c^2}{4\,e_*} < \left(\dfrac{\pi}{T}\right)^2 + \dfrac{a^2}{4}.$

THEN: (1) has a positive solution.

(2) $\qquad x''+a\,x'(t)+f(t,x)=0\,, \qquad x(0)=x(T),\ x'(0)=x'(T)$

Lemma (Omari & Njoku, 2003)

ASSUME: $a>0$,

- σ_1 is a strict lower function, σ_2 is a strict upper function of (2) and $\sigma_2<\sigma_1$ on $[0,T]$.
- $\frac{\partial}{\partial x}f(t,x)\leq\left(\frac{\pi}{T}\right)^2+\frac{a^2}{4}$ for $t\in[0,T],\ x\in[\sigma_2(t),\sigma_1(t)]$,
- there is a continuous $\gamma:[0,T]\to[0,\infty)$ such that $\bar{\gamma}>0$ and

$$\frac{\partial}{\partial x}f(t,x)\geq\gamma(t) \quad \text{for } t\in[0,T],\ x\in[\sigma_2(t),\sigma_1(t)].$$

Then (2) has at least one asymptotically stable T-periodic solution x fulfilling

$$\sigma_2\leq x\leq\sigma_1 \quad \text{on } [0,T]\,.$$

(2) $$x'' + a x'(t) + f(t,x) = 0, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

Theorem 5

ASSUME: $a > 0$, $f(t,x) = s(t)x^{\beta} - r(t)x^{\alpha}$,

- r, s are continuous and positive on $[0, T]$, $0 < \alpha < \beta < 1$,
- $$\beta s^* \left(\frac{s^*}{r_*}\right)^{(1-\beta)/(\beta-\alpha)} - \alpha r_* \left(\frac{s_*}{r^*}\right)^{(1-\alpha)/(\beta-\alpha)} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4},$$
- $$\frac{\alpha}{\beta}\frac{r^*}{s_*} < \frac{r_*}{s^*}.$$

THEN: (2) has at least one asymptotically stable positive solution.

(2) $$x''+ax'(t)+f(t,x)=0\,, \qquad x(0)=x(T),\ x'(0)=x'(T)$$

Theorem 5

ASSUME: $a>0$, $f(t,x)=s(t)\,x^{\beta}-r(t)\,x^{\alpha}$,

- r, s are continuous and positive on $[0,T]$, $0<\alpha<\beta<1$,
- $\beta\, s^* \left(\frac{s^*}{r_*}\right)^{(1-\beta)/(\beta-\alpha)} - \alpha\, r_* \left(\frac{s_*}{r^*}\right)^{(1-\alpha)/(\beta-\alpha)} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}$,
- $\frac{\alpha}{\beta}\,\frac{r^*}{s_*} < \frac{r_*}{s^*}$.

THEN: (2) has at least one asymptotically stable positive solution.

(1) $$u''+a\,u' = \frac{1}{u}\left(e(t)-b\,(u')^2\right)-c\,, \quad u(0)=u(T),\ u'(0)=u'(T)$$

Corollary

(1) has at least one asymptotically stable positive solution if

$$\frac{c^2\left(b\,(e^*)^2-(b-1)\,(e_*)^2\right)}{e_*\,(e^*)^2} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \quad \text{and} \quad (b-1)\,e^* < b\,e_*.$$

- $\bar{e} > 0$ (i.e. $\bar{p} < g\,\rho\,\dfrac{V_0}{A_T}$) is the necessary condition for the existence of a positive T-periodic solution.
  $e_* > 0$ (i.e. $p^* < g\,\rho\,\dfrac{V_0}{A_T}$) is needed in the sufficient condition for the existence of a positive T-periodic solution.

- $\bar{e} > 0$ (i.e. $\bar{p} < g\,\rho\,\dfrac{V_0}{A_T}$) is the necessary condition for the existence of a positive T-periodic solution.

  $e_* > 0$ (i.e. $p^* < g\,\rho\,\dfrac{V_0}{A_T}$) is needed in the sufficient condition for the existence of a positive T-periodic solution.

- By Theorem 3,

  $$\frac{(b+1)\,c^2}{4\,e_*} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \implies \text{existence of a positive } T\text{-periodic solution.}$$

  By Theorem 5,

  $$\frac{c^2\,(b\,(e^*)^2 - (b-1)\,(e_*)^2)}{e_*\,(e^*)^2} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \quad \text{and} \quad (b-1)\,e^* < b\,e_*$$

  $\implies$ existence of a positive and asymptotically stable T-periodic solution.

- $\bar{e} > 0$ (i.e. $\bar{p} < g\,\rho\,\dfrac{V_0}{A_T}$) is the necessary condition for the existence of a positive T-periodic solution.

  $e_* > 0$ (i.e. $p^* < g\,\rho\,\dfrac{V_0}{A_T}$) is needed in the sufficient condition for the existence of a positive T-periodic solution.

- If $b = 2$, $c = 1/2$, then by Theorem 3,

  $$\left(\left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\right)\left(p^* - g\,\rho\,\frac{V_0}{A_T}\right) > \frac{3}{16} \implies \text{existence}$$

  and by Theorem 5,

  $$2\,(e^*)^2 - (e_*)^2 < 4\left(\left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\right) \quad \text{and} \quad \frac{e^*}{e_*} < \frac{b}{b-1} = 2$$

  $\implies$ existence and asymptotic stability.

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- **G. Liebau**: Über ein ventilloses Pumpprinzip. *Naturwissenschaften* **41** (1954), 327.
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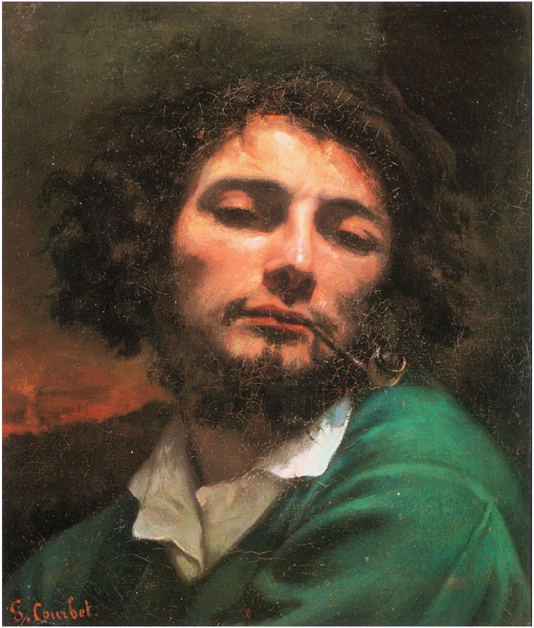
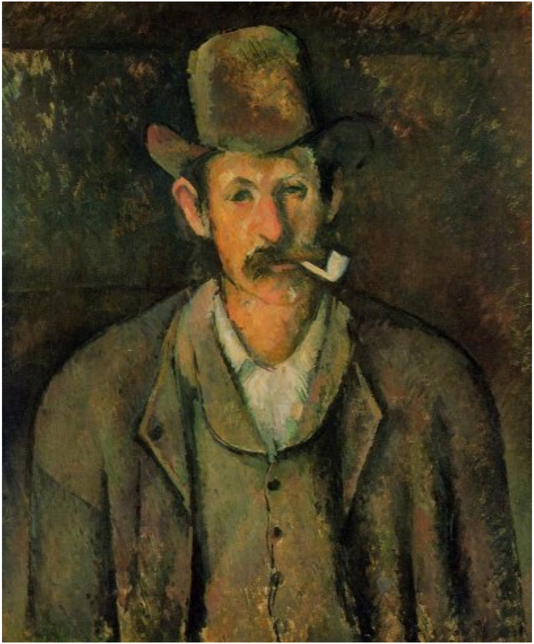
JOAN MIRÓ. The man with a pipe. 1925.



GUSTAVE COURBAT. The man with a pipe. 1849.

JAMES MCNEILL WHISTLER. The man with a pipe. 1859.



PAUL CÉZANNE. The man with a pipe. 1892.

PABLO PICASSO. The man with a pipe. 1915.

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