ON POSITIVE SOLUTIONS TO A SINGULAR INITIAL VALUE PROBLEM

Svatoslav Staněk, Czech Republic
e-mail: svatoslav.stanek@upol.cz

Workshop z diferenciálních rovnic

konaný u příležitosti životního jubilea

Doc. RNDr. Milana Tvrdého, CSc.

Malá Morávka, 2014

1. Motivation

A model for the time dependent flow of water transported in variably-saturated porous medium with exponential diffusivity such as rock, soil or building materials

W. Brutsaert, *Universal constants for scaling the exponential soil water diffusivity*, Water Resour. Res., 15(2) 1979, 481–483.

is given by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(D(u)\frac{\partial u}{\partial y}\right),$$

where $u = u(y,t)$ is called saturation and represents the volume function of the pore space occupied by liquid, and

$$D(u) = D_0 e^{\beta u}, \quad D_0 > 0, \ \beta > 0.$$

The self-similar solutions are stable attractors and take the form

$$u(y,t) = \xi(x), \quad x = y/t^{1/2}, \ 0 < x < \infty.$$

If we set $w(x) = e^{\beta\xi(x)}$ and make a trivial rescaling, then it follows that w is a solution of the initial value problem

$$\left.\begin{aligned} w'' &= -\frac{xw'}{w}, \\ w(0) = 1, \quad w'(0) &= -\gamma < 0 \;\; (\gamma = \gamma(\beta)). \end{aligned}\right\} \tag{1}$$

Asymptotic behaviour and asymptotical computations for problem (1) are given in

- P. Amodio, C.J. Budd, O. Koch, G. Settanni, E. Weinmüller, *Asymptotic computations for a model of flow in concrete*, submitted to Comput. Math. Appl.
- C.J. Budd, J.M. Stockie, *Asymptotic behaviour of wetting fronts in porous media with exponential moisture diffusivity*, submitted

Motivated by (1), we investigate the initial value problem

$$y''(t) = a(t)\frac{p(y'(t))}{g(y(t))}, \tag{2}$$

$$y(0) = 1, \quad y'(0) = -\gamma, \quad \gamma \geq 0. \tag{3}$$

$(a(t) = t,\ p(x) = -x,\ g(y) = y)$

$$y''(t) = a(t)\frac{p(y'(t))}{g(y(t))} \qquad y(0) = 1,\ y'(0) = -\gamma, \quad \gamma \geq 0$$

Assumptions

(H_1) $a \in C^1[0,\infty)$, $a(0) \geq 0$ and $a' > 0$ on $(0,\infty)$,

(H_2) $p \in C^1(-\infty, 0]$, $p(0) = 0$, $p' < 0$ on $(-\infty, 0]$ and there exist positive constants m, M such that

$$m \leq |p'(x)| \leq M \quad \text{for } x \in (-\infty, 0],$$

(H_3) $g \in C^1[0,1]$, $g(0) = 0$ and $g' > 0$ on $[0,1]$.

A function y is called a solution of problem (2), (3) on an interval J, $J \subset [0,\infty)$, $0 \in J$, if $y \in C^2(J)$, $y > 0$ on J, y satisfies (3) and (2) holds for $t \in J$.

EXAMPLE 1. Let $\alpha, \beta \in [1, \infty)$, $k, c \in [0, \infty)$, $h \in C(-\infty, 0]$, $m \leq h(x) \leq M$ for $x \in (-\infty, 0]$, where m, M are positive constants. The equation

$$y'' = \frac{t^\alpha + ce^t}{y^\beta + k \tan y} \int_{y'}^{0} h(s)\,\mathrm{d}s \tag{4}$$

satisfies conditions $(H_1) - (H_3)$ for $a(t) = t^\alpha$, $p(x) = \int_x^0 h(s)\,\mathrm{d}s$ and $g(v) = v^\beta + k \tan v$.

REMARK 1.

Under condition (H_2), $p > 0$ on $(-\infty, 0)$ and since

$$-mx \le p(x) \le -Mx \quad \text{for } x \in (-\infty, 0], \tag{5}$$

we have

$$\int_{-\infty}^{-1} \frac{1}{p(x)} \, \mathrm{d}x = \infty, \quad \int_{-1}^{0} \frac{1}{p(x)} \, \mathrm{d}x = \infty.$$

REMARK 2. We can also discuss equation (2) under more generally initial conditions

$$y(0) = \rho, \quad y'(0) = -\gamma_1, \quad \rho > 0, \ \gamma_1 \ge 0.$$

2. Preliminaries

LEMMA 1. If $\gamma = 0$, then $y = 1$ is the unique solution of (2), (3) on $[0, \infty)$. If $\gamma > 0$, then problem (2), (3) has a unique solution y on an interval $[0, \phi(\gamma))$, where

$$\phi(\gamma) = \sup\{t \geq 0 : y(s) > 0 \text{ for } s \in [0, t]\},$$

and $y' < 0$ on $[0, \phi(\gamma))$, $y'' > 0$ on $(0, \phi(\gamma))$.

Proof. The function $f(t, x, y) = a(t)\frac{p(y)}{g(x)}$ and its partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous on on the set $[0, \infty) \times (0, 1] \times (-\infty, 0]$, and $f > 0$ on $(0, \infty) \times (0, 1] \times (-\infty, 0)$. Hence problem (2), (3) has a unique solution y on an interval $J \subset [0, \infty)$.

If $\gamma = 0$, then $y = 1$ is the unique solution of (2), (3) on $[0, \infty)$.

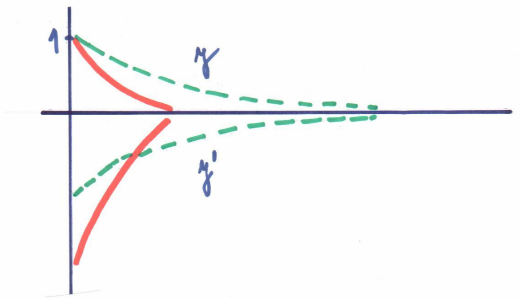
Let $\gamma > 0$. We claim that $y' < 0$ on J. In the opposite case there exists $\xi \in J$ such that $y'(\xi) = 0$ and $y' < 0$ on $[0, \xi)$. Consider the initial value problem

$$\left.\begin{aligned} u'' &= \frac{a(t)}{g(y(t))} p(u'), \\ u(\xi) &= y(\xi), \quad u'(\xi) = 0. \end{aligned}\right\} \tag{6}$$

Problem (6) has a unique solution $u(t) = y(\xi)$ for $t \in J$. Since y is also a solution of (6), we have $y(t) = y(\xi)$ for $t \in J$, which is impossible. Hence $y' < 0$ on J, and therefore $y > 0$ and y is defined on $[0, \phi(\gamma))$. It follows from $(H_1) - (H_3)$ that $y'' > 0$ on the interval $(0, \phi(\gamma))$.

Keeping in mind Lemma 1, we denote by y_γ the unique solution of (2), (3). Then y_γ is defined on $[0, \phi(\gamma))$, $y_0 = 1$, $\phi(0) = \infty$, and for each $\gamma > 0$ we have $y_\gamma > 0$, $y'_\gamma < 0$ on $[0, \phi(\gamma))$ and $y''_\gamma > 0$ on $(0, \phi(\gamma))$.

LEMMA 2. If $\phi(\gamma) < \infty$ for some $\gamma > 0$, then $\lim_{t \to \phi(\gamma)} y_\gamma(t) = 0$, $\lim_{t \to \phi(\gamma)} y'_\gamma(t) = 0$.
If $\phi(\gamma) = \infty$ for some $\gamma > 0$, then $\lim_{t \to \infty} y_\gamma(t) \geq 0$, $\lim_{t \to \infty} y'_\gamma(t) = 0$.



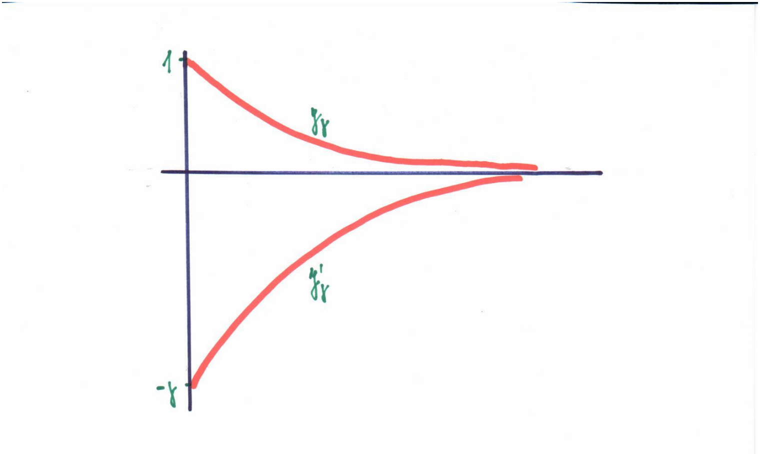
LEMMA 3. Let $0 \le \gamma_1 < \gamma_2$. Then $\phi(\gamma_1) \ge \phi(\gamma_2)$ and

$$y_{\gamma_1}(t) > y_{\gamma_2}(t) \quad \text{for } t \in (0, \phi(\gamma_2)). \tag{7}$$

3. Qualitative properties of solutions

Theorem 1. For each $\gamma \geq 0$,

$$\phi(\gamma) = \infty, \quad \lim_{t\to\infty} y_\gamma(t) > 0, \quad \lim_{t\to\infty} y'_\gamma(t) = 0.$$



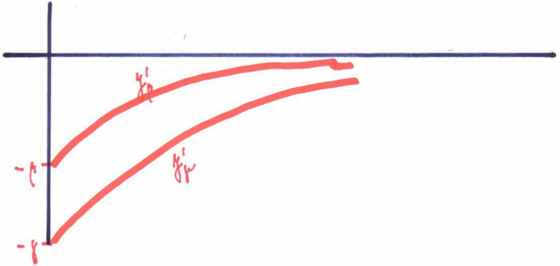
The following result states the properties of y'_γ on $[0, \infty)$ with different values of γ.

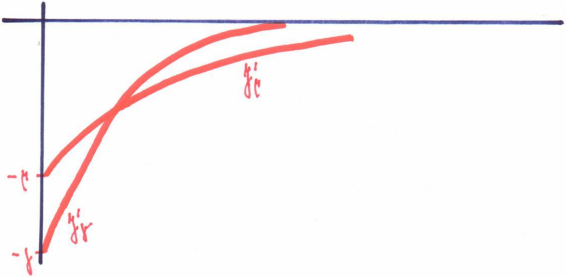
Lemma 4. Let $0 < \alpha < \beta$. Then either

$$y'_\alpha(t) > y'_\beta(t) \quad \text{for } t \in [0, \infty)$$

or there exists $\xi > 0$ such that

$$y'_\alpha(\xi) = y'_\beta(\xi),\ y'_\alpha > y'_\beta \text{ on } [0, \xi) \text{ and } y'_\alpha < y'_\beta \text{ on } (\xi, \infty).$$





It follows from Theorem 1 that $\phi(\gamma) = \infty$ for all $\gamma \geq 0$. In order to investigate the values of $\lim_{t\to\infty} y_\gamma(t)$, we introduce a function $\Delta : [0, \infty) \to (0, 1]$ by the formula

$$\Delta(\gamma) = \lim_{t\to\infty} y_\gamma(t).$$

The properties of Δ are collected in the following result.

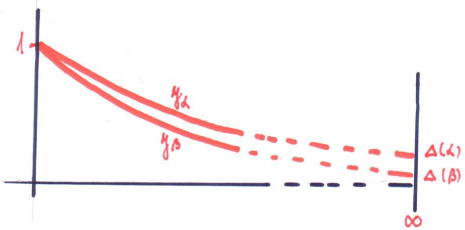
Theorem 2. $\Delta \in C[0, \infty)$, Δ is nonincreasing, $\Delta(0) = 1$, $\Delta(\gamma) > 0$ for $\gamma \geq 0$ and $\lim_{\gamma\to\infty} \Delta(\gamma) = 0$.

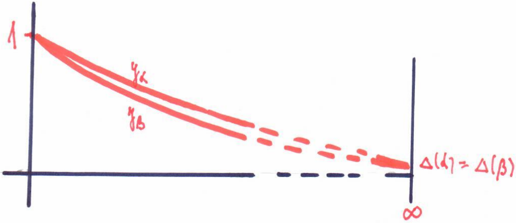
By Theorem 2, for each $\rho \in (0,1]$ there exists at least one $\gamma = \gamma(\rho) \geq 0$ such that $\lim_{t\to\infty} y_\gamma(t) = \Delta(\rho)$. We are interested in the set of all such γ. To this end, we introduce a multi-valued function $\Lambda : (0,1] \to 2^{\mathbb{R}}$ as

$$\Lambda(\rho) = \{\gamma \in [0,\infty) : \Delta(\gamma) = \rho\}.$$

The following result gives properties of Λ.

Theorem 3. For each $\rho \in (0,1]$, $\Lambda(\rho)$ is either a one-point set or a compact interval $[a_\rho, b_\rho]$. If Γ is the set of all $\rho \in (0,1]$ such that $\Lambda(\rho)$ is a one-point set, then $(0,1] \setminus \Gamma$ is at most countable.


1
y_α
y_β
$\Delta(\alpha)$
$\Delta(\beta)$
∞


1
y_α
y_β
$\Delta(\alpha) = \Delta(\beta)$
∞

Theorem 4. Let the function a satisfying (H_1) be bounded. Then for all $\rho \in (0, 1]$, the set $\Lambda(\rho)$ is one-point, and therefore Δ is decreasing on $(0, 1]$.

The next result states the limit properties of y_γ and its derivative as $\gamma \to \infty$ at points of the interval $(0, \infty)$, and the properties of a function $\varphi_t : [0, \infty) \to [0, 1)$, $t > 0$, defined as

$$\varphi_t(\gamma) = y_\gamma(t).$$

Theorem 5. Let $t_* > 0$. Then

(*a*) $\lim_{\gamma \to \infty} y_\gamma(t_*) = 0$ and $\lim_{\gamma \to \infty} y'_\gamma(t_*) = 0$,

(*b*) $\varphi_{t_*} \in C[0, \infty)$, φ_{t_*} is decreasing, $\varphi_{t_*}(0) = 1$ and $\lim_{\gamma \to \infty} \varphi_{t_*}(\gamma) = 0$.

4. Boundary value problems for equation (2)

In this section, we apply Theorems 2, 4 and 5 to the solvability of the boundary values problems

$$y'' = a(t)\frac{p(y')}{g(y)}, \quad y(0) = 1, \quad \lim_{t\to\infty} y(t) = c, \quad c \in (0, 1],$$

$$y'' = a(t)\frac{p(y')}{g(y)}, \quad y(0) = 1, \quad y'(0) = -\lim_{t\to\infty} y(t),$$

and

$$y'' = a(t)\frac{p(y')}{g(y)}, \quad y(0) = 1, \quad y(t_*) = c, \quad t^* \in (0, \infty), \; c \in (0, 1].$$

Theorem 6. For each $c \in (0,1]$ there exists at least one solution of problem

$$y'' = a(t)\frac{p(y')}{g(y)}, \quad y(0) = 1, \quad \lim_{t\to\infty} y(t) = c.$$

If in addition a is bounded, then for each $c \in (0,1]$ this problem has a unique solution.

Proof. Choose $c \in (0,1]$. Theorem 2 guarantees that the equation $\Delta(\gamma) = c$ has at least one solution $\gamma = \gamma_0$. Then y_{γ_0} is a solution of our problem.

If a is bounded, then equation $\Delta(\gamma) = c$ has a unique solution $\gamma = \gamma_*$ by Theorems 2 and 4. Hence y_{γ_*} is the unique solution of our problem.

Theorem 7. Problem

$$y'' = a(t)\frac{p(y')}{g(y)}, \quad y(0) = 1, \;\; y'(0) = -\lim_{t\to\infty} y(t),$$

has a unique solution.

Proof. It follows from the definition of the function Δ that our problem is solvable and y is its solution if and only if $y = y_\gamma$, where γ is a solution of the equation $\Delta(\gamma) = \gamma$. By Theorem 2, the last equation has a unique solution.

Theorem 8. For each $c \in (0,1]$ and each $t_* \in (0,\infty)$, there exists a unique solution of problem

$$y'' = a(t)\frac{p(y')}{g(y)}, \quad y(0) = 1, \quad y(t_*) = c.$$

Proof. Choose $c \in (0,1]$ and $t_* \in (0,\infty)$. By Theorem 4, the equation $\varphi_{t_*}(\gamma) = c$ has a unique solution $\gamma = \gamma_*$, $\gamma_* \in [0,\infty)$. Then y_{γ_*} is the unique solution of our problem. □