Generalized elementary functions

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The exponential function

Classical exponential function:

$$z'(t) = z(t), \quad z(0) = 1 \quad \Rightarrow \quad z(t) = \exp t$$

$$z'(t) = p(t)z(t), \quad z(t_0) = 1 \quad \Rightarrow \quad z(t) = \exp\left(\int_{t_0}^{t} p(s)\,ds\right)$$

Equivalent integral form:

$$z(t) = 1 + \int_{t_0}^{t} p(s)z(s)\,ds$$

Generalized exponential function:

$$z(t) = 1 + \int_{t_0}^{t} z(s)\,dP(s)$$

Kurzweil-Stieltjes integral

A function $f : [a, b] \to \mathbb{C}$ is KS-integrable with respect to $g : [a, b] \to \mathbb{C}$ if there exists a number $I \in \mathbb{C}$ such that given an $\varepsilon > 0$, there is a function $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\left| \sum_{j=1}^{k} f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1})) - I \right| < \varepsilon$$

for every partition with division points

$$a = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k = b$$

and tags $\tau_j \in [\alpha_{j-1}, \alpha_j]$, $j \in \{1, \ldots, k\}$, such that

$$[\alpha_{j-1}, \alpha_j] \subset (\tau_j - \delta(t_j), \tau_j + \delta(\tau_j)), \quad j \in \{1, \ldots, k\}.$$

Notation: $I = \int_a^b f(s)\, dg(s)$.

Notation (jumps)

We work with regulated functions defined on an interval $[a, b]$ and use the following notation:

$$\Delta^+ g(t) = \begin{cases} g(t+) - g(t) & \text{if } t \in [a, b), \\ 0 & \text{if } t = b, \end{cases}$$

$$\Delta^- g(t) = \begin{cases} g(t) - g(t-) & \text{if } t \in (a, b], \\ 0 & \text{if } t = a. \end{cases}$$

Indefinite Kurzweil-Stieltjes integral

Consider a pair of functions $f, g : [a, b] \to \mathbb{C}$ such that g is regulated and $\int_a^b f(s)\, dg(s)$ exists. Then, for every $t_0 \in [a, b]$, the function

$$h(t) = \int_{t_0}^{t} f(s)\, dg(s), \quad t \in [a, b]$$

is regulated and satisfies

$$\begin{aligned} \Delta^+ h(t) &= f(t)\Delta^+ g(t), \quad t \in [a, b), \\ \Delta^- h(t) &= f(t)\Delta^- g(t), \quad t \in (a, b]. \end{aligned}$$

Notation (countable sums)

If $I \subset \mathbb{R}$ is an interval, $h : I \to \mathbb{C}$ is a function which is zero except a countable set $\{t_1, t_2, \ldots\} \subset I$, and the sum $S = \sum_i h(t_i)$ is absolutely convergent, we use the notation

$$S = \sum_{x \in I} h(x).$$

Integration by parts

Theorem (integration by parts for Riemann-Stieltjes)

If at least one the integrals $\int_a^b f\,dg$, $\int_a^b g\,df$ exists, then

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a).$$

Theorem (integration by parts for Kurzweil-Stieltjes)

If $f, g : [a, b] \to \mathbb{C}$ are regulated and at least one of them has bounded variation, then

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a)$$

$$+ \sum_{x \in [a,b]} (\Delta^- f(x)\Delta^- g(x) - \Delta^+ f(x)\Delta^+ g(x)).$$

Substitution theorem

Theorem

Assume that $h : [a, b] \to \mathbb{C}$ *is bounded and* $f, g : [a, b] \to \mathbb{C}$ *are such that* $\int_a^b f\, dg$ *exists. Then*

$$\int_a^b h(x)\, d\left[\int_a^x f(z)\, dg(z)\right] = \int_a^b h(x) f(x)\, dg(x),$$

whenever either side of the equation exists.

Linear equations (existence and uniqueness)

Theorem

Let $t_0 \in [a,b]$*. Consider a function* $P : [a,b] \to \mathbb{C}$*, which has bounded variation on* $[a,b]$ *and satisfies*

$$1 + \Delta^+ P(t) \neq 0, \qquad t \in [a, t_0),$$
$$1 - \Delta^- P(t) \neq 0, \qquad t \in (t_0, b].$$

Then, for every $z_0 \in \mathbb{C}$*, there exists a unique function* $z : [a,b] \to \mathbb{C}$ *such that*

$$z(t) = z_0 + \int_{t_0}^{t} z(s)\,dP(s), \quad t \in [a,b].$$

The function z *has bounded variation on* $[a,b]$*. If* P *and* z_0 *are real, then* z *is real as well.*

Generalized exponential function (definition)

Definition

Let $t_0 \in [a,b]$. Consider a function $P : [a,b] \to \mathbb{C}$, which has bounded variation on $[a,b]$. Assume that $1 + \Delta^+ P(t) \neq 0$ for every $t \in [a,t_0)$, and $1 - \Delta^- P(t) \neq 0$ for every $t \in (t_0,b]$.
We define the generalized exponential function

$$t \mapsto e_{dP}(t,t_0), \quad t \in [a,b],$$

as the unique solution $z : [a,b] \to \mathbb{C}$ of the linear equation

$$z(t) = 1 + \int_{t_0}^{t} z(s)\, dP(s), \quad t \in [a,b].$$

- P is real $\Rightarrow$ e_{dP} is real
- $P(s) = s$ $\Rightarrow$ $e_{dP}(t,t_0) = e^{t-t_0}$

Generalized exponential function (basic properties)

Let $P : [a, b] \to \mathbb{C}$ be a function with bounded variation. The generalized exponential function has the following properties:

1. If P is constant, then $e_{dP}(t, t_0) = 1$ for every $t \in [a, b]$.
2. $e_{dP}(t, t) = 1$ for every $t \in [a, b]$.
3. $t \mapsto e_{dP}(t, t_0)$ is regulated on $[a, b]$ and satisfies

$$\begin{aligned} \Delta^+ e_{dP}(t, t_0) &= \Delta^+ P(t) e_{dP}(t, t_0), \quad t \in [a, b), \\ \Delta^- e_{dP}(t, t_0) &= \Delta^- P(t) e_{dP}(t, t_0), \quad t \in (a, b], \end{aligned}$$

4. $t \mapsto e_{dP}(t, t_0)$ has bounded variation on $[a, b]$.
5. $e_{dP}(t, s) e_{dP}(s, r) = e_{dP}(t, r)$ for every $t,\ s,\ r \in [a, b]$.
6. $e_{dP}(t, s) = e_{dP}(s, t)^{-1}$ for every $t,\ s \in [a, b]$.
7. $\overline{e_{dP}(t, t_0)} = e_{d\overline{P}}(t, t_0)$ for every $t \in [a, b]$.
8. If P is continuous, then $e_{dP}(t, t_0) = e^{P(t) - P(t_0)}$.

Product of exponentials

Assume that $P, Q : [a, b] \to \mathbb{C}$ have bounded variation,

$$\begin{aligned}(1 + \Delta^+ P(t))(1 + \Delta^+ Q(t)) &\neq 0 \text{ for every } t \in [a, t_0),\\ (1 - \Delta^- P(t))(1 - \Delta^- Q(t)) &\neq 0 \text{ for every } t \in (t_0, b].\end{aligned}$$

Then

$$e_{dP}(t, t_0) e_{dQ}(t, t_0) = e_{d(P \oplus Q)}(t, t_0), \quad t \in [a, b],$$

where

$$(P \oplus Q)(t) = P(t) + Q(t) + \sum_{s \in [t_0, t)} \Delta^+ Q(s) \Delta^+ P(s)$$

$$- \sum_{s \in (t_0, t]} \Delta^- Q(s) \Delta^- P(s).$$

Proof outline

We need

$$e_{dP}(t,t_0)e_{dQ}(t,t_0) = 1 + \int_{t_0}^{t} e_{dP}(s,t_0)e_{dQ}(s,t_0)\, d(P \oplus Q)(s).$$

Integration by parts + substitution:

$$e_{dP}(t,t_0)e_{dQ}(t,t_0) = e_{dP}(t_0,t_0)e_{dQ}(t_0,t_0) +$$

$$+ \int_{t_0}^{t} e_{dP}(s,t_0)\, d[e_{dQ}(s,t_0)] + \int_{t_0}^{t} e_{dQ}(s,t_0)\, d[e_{dP}(s,t_0)]$$

$$+ \sum_{s\in[t_0,t]} (\Delta^+ e_{dP}(s,t_0)\Delta^+ e_{dQ}(s,t_0) - \Delta^- e_{dP}(s,t_0)\Delta^- e_{dQ}(s,t_0)).$$

$$\int_{t_0}^{t} e_{dP}(s,t_0)\, d[e_{dQ}(s,t_0)] = \int_{t_0}^{t} e_{dP}(s,t_0)\, d\left[1 + \int_{t_0}^{s} e_{dQ}(u,t_0)\, dQ(u)\right]$$

$$\int_{t_0}^{t} e_{dQ}(s,t_0)\, d[e_{dP}(s,t_0)] = \int_{t_0}^{t} e_{dQ}(s,t_0)\, d\left[1 + \int_{t_0}^{s} e_{dP}(u,t_0)\, dP(u)\right]$$

Inverse of exponential

Assume that $P : [a, b] \to \mathbb{C}$ has bounded variation,

$$\begin{aligned} 1 + \Delta^+ P(t) &\neq 0 \text{ for every } t \in [a, b), \\ 1 - \Delta^- P(t) &\neq 0 \text{ for every } t \in (a, b]. \end{aligned}$$

Then

$$e_{dP}(t, t_0)^{-1} = e_{d(\ominus P)}(t, t_0), \quad t \in [a, b],$$

where

$$(\ominus P)(t) = -P(t) + \sum_{s \in [t_0, t)} \frac{(\Delta^+ P(s))^2}{1 + \Delta^+ P(s)} - \sum_{s \in (t_0, t]} \frac{(\Delta^- P(s))^2}{1 - \Delta^- P(s)}.$$

Note: $P \oplus (\ominus P) = (\ominus P) \oplus P = 0$.
But: Besides $\ominus P$, there are other functions Q such that $e_{dP} e_{dQ} = 1$.

Group properties

Let $\mathcal{P}$ be the class of all functions $P : [a, b] \to \mathbb{C}$, which have bounded variation and satisfy

$$\begin{aligned} 1 + \Delta^+ P(t) &\neq 0 \text{ for every } t \in [a, b), \\ 1 - \Delta^- P(t) &\neq 0 \text{ for every } t \in (a, b]. \end{aligned}$$

- $\mathcal{P}$ is closed under $\oplus$ and $\ominus$.
- $\oplus$ is commutative and associative.

Write $P \sim Q$ iff P, Q differ by a constant.

Then $(\mathcal{P}/\sim, \oplus, \ominus, 0)$ is a commutative group.

Sign of the exponential

Assume that $P : [a, b] \to \mathbb{C}$ has bounded variation,

$$
\begin{aligned}
1 + \Delta^+ P(t) &\neq 0 \text{ for every } t \in [a, b), \\
1 - \Delta^- P(t) &\neq 0 \text{ for every } t \in (a, b].
\end{aligned}
$$

Then, for every $t_0 \in [a, b]$, the following statements hold:

1. $e_{dP}(t, t_0) \neq 0$ for all $t \in [a, b]$.
2. $t \mapsto e_{dP}(t, t_0)$ changes sign at all points t where $1 + \Delta^+ P(t) < 0$ or $1 - \Delta^- P(t) < 0$.
3. If $1 + \Delta^+ P(t) > 0$ and $1 - \Delta^- P(t) > 0$ for all t, then $t \mapsto e_{dP}(t, t_0)$ remains positive. Moreover, the set of all functions satisfying the two conditions is a subgroup of $(\mathcal{P}/\sim, \oplus, \ominus, 0)$.

Generalized hyperbolic functions (definition)

Definition

Consider a function $P : [a, b] \to \mathbb{C}$, which has bounded variation on $[a, b]$. Let $t_0 \in [a, b]$ and assume that

$$\begin{aligned} 1 - (\Delta^+ P(t))^2 &\neq 0, \quad t \in [a, t_0), \\ 1 - (\Delta^- P(t))^2 &\neq 0, \quad t \in (t_0, b]. \end{aligned}$$

We define the generalized hyperbolic functions $t \mapsto \cosh_{dP}(t, t_0)$ and $t \mapsto \sinh_{dP}(t, t_0)$, $t \in [a, b]$, by the formulas

$$\cosh_{dP}(t, t_0) = \frac{e_{dP}(t, t_0) + e_{d(-P)}(t, t_0)}{2},$$

$$\sinh_{dP}(t, t_0) = \frac{e_{dP}(t, t_0) - e_{d(-P)}(t, t_0)}{2}.$$

The generalized hyperbolic functions have the following properties:

1. If P is real, then $\sinh_{dP}$ and $\cosh_{dP}$ are real.
2. If P is continuous, then $\cosh_{dP}(t,t_0)=\cosh(P(t)-P(t_0))$ and $\sinh_{dP}(t,t_0)=\sinh(P(t)-P(t_0))$.
3. $\cosh_{dP}(t_0,t_0)=1$, $\sinh_{dP}(t_0,t_0)=0$.
4. $\cosh_{dP}(t,t_0)=1+\int_{t_0}^{t}\sinh_{dP}(s,t_0)\,dP(s)$, $t\in[a,b]$.
5. $\sinh_{dP}(t,t_0)=\int_{t_0}^{t}\cosh_{dP}(s,t_0)\,dP(s)$, $t\in[a,b]$.
6. $\cosh^2_{dP}(t,t_0)-\sinh^2_{dP}(t,t_0)=e_{dQ}(t,t_0)$, $t\in[a,b]$, where

$$Q(t)=(P\oplus(-P))(t)=\sum_{s\in(t_0,t]}(\Delta^-P(s))^2-\sum_{s\in[t_0,t)}(\Delta^+P(s))^2.$$

Generalized trigonometric functions (definition)

Definition

Consider a function $P:[a,b]\to\mathbb{C}$, which has bounded variation on $[a,b]$. Let $t_0\in[a,b]$ and assume that

$$\begin{aligned} 1+(\Delta^+P(t))^2 &\neq 0, \quad t\in[a,t_0),\\ 1+(\Delta^-P(t))^2 &\neq 0, \quad t\in(t_0,b]. \end{aligned}$$

We define the generalized trigonometric functions $t\mapsto\cosh_{dP}(t,t_0)$ and $t\mapsto\sinh_{dP}(t,t_0)$, $t\in[a,b]$, by the formulas

$$\begin{aligned} \cos_{dP}(t,t_0) &= \frac{e_{d(iP)}(t,t_0)+e_{d(-iP)}(t,t_0)}{2}=\cosh_{d(iP)}(t,t_0),\\ \sin_{dP}(t,t_0) &= \frac{e_{d(iP)}(t,t_0)-e_{d(-iP)}(t,t_0)}{2i}=-i\sinh_{d(iP)}(t,t_0). \end{aligned}$$

Generalized trigonometric functions (basic properties)

The generalized trigonometric functions have the following properties:

1. If P is real, then $\sin_{dP}$ and $\cos_{dP}$ are real.
2. If P is continuous, then $\cos_{dP}(t,t_0)=\cos(P(t)-P(t_0))$ and $\sin_{dP}(t,t_0)=\sin(P(t)-P(t_0))$.
3. $\cos_{dP}(t_0,t_0)=1$, $\sin_{dP}(t_0,t_0)=0$.
4. $\cos_{dP}(t,t_0)=1-\int_{t_0}^{t}\sin_{dP}(s,t_0)\,dP(s)$, $t\in[a,b]$.
5. $\sin_{dP}(t,t_0)=\int_{t_0}^{t}\cos_{dP}(s,t_0)\,dP(s)$, $t\in[a,b]$.
6. $\cos^2_{dP}(t,t_0)+\sin^2_{dP}(t,t_0)=e_{dQ}(t,t_0)$, $t\in[a,b]$, where

$$Q(t)=(iP\oplus(-iP))(t)=\sum_{s\in[t_0,t)}(\Delta^+P(s))^2-\sum_{s\in(t_0,t]}(\Delta^-P(s))^2.$$

Time scale elementary functions

Let $\mathbb{T}$ be a time scale. Consider a point $t_0 \in [a,b]_{\mathbb{T}}$ and an rd-continuous function $p : [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that $1 + \mu(t)p(t) \neq 0$ for all $t \in [a, t_0)_{\mathbb{T}}$. The time scale exponential function $t \mapsto e_p(t, t_0)$ is the unique solution of the initial-value problem

$$\begin{aligned} x^{\Delta}(t) &= p(t)x(t), \quad t \in [a,b]_{\mathbb{T}}, \\ x(t_0) &= 1. \end{aligned}$$

Time scale hyperbolic functions:

$$\cosh_p = \frac{e_p + e_{-p}}{2}, \quad \sinh_p = \frac{e_p - e_{-p}}{2}$$

Time scale trigonometric functions:

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2}, \quad \sin_p = \frac{e_{ip} - e_{-ip}}{2i}$$

Time scale vs. generalized functions

For every $t \in [a, b]$, let $g(t) = \inf\{s \in [a, b]_{\mathbb{T}}; s \geq t\}$.
Given an rd-continuous function $p : [a, b]_{\mathbb{T}} \to \mathbb{R}$, let

$$P(t) = \int_a^t p(g(s))\, dg(s), \quad t \in [a, b].$$

Then for every $t \in [a, b]_{\mathbb{T}}$, we have

$$e_p(t, t_0) = e_{dP}(t, t_0),$$

$$\cosh_p(t, t_0) = \cosh_{dP}(t, t_0), \quad \sinh_p(t, t_0) = \sinh_{dP}(t, t_0)$$
$$\cos_p(t, t_0) = \cos_{dP}(t, t_0), \quad \sin_p(t, t_0) = \sin_{dP}(t, t_0)$$

- Properties of time scale elementary functions are deducible from the properties of generalized functions.
- Definitions of time scale functions can be extended to the case when p is not rd-continuous but merely Lebesgue Δ-integrable.

References

- G. Antunes Monteiro and A. Slavík, *Generalized elementary functions*, J. Math. Anal. Appl. **411** (2014), 838–852.
- M. Tvrdý, *Stieltjesův integrál. Kurzweilova teorie*, Univerzita Palackého v Olomouci, 2012.