Continuous dependence of solutions on a parameter of abstract generalized linear differential equations (Opial type results)

Giselle A. Monteiro and Milan Tvrdý

Institute of Mathematics
Academy Sciences of the Czech Republic

Workshop on Differential Equations 2014
In honour of Milan Tvrdý

Universidade de São Paulo, 2008

Praha, 2010

- Generalized linear differential equations in a Banach space: Continuous dependence on a parameter. *Discrete Contin. Dyn. Syst.* **33** (1) (2013), 283–303.
- Continuous dependence of solutions of abstract generalized linear differential equations with potential converging uniformly with a weight. **accepted***(Boundary Value Problems)*

The results

Generalized linear differential equations in a Banach space: Continuous dependence on a parameter. *Discrete Contin. Dyn. Syst.* **33** (1) (2013), 283–303.

Continuous dependence of solutions of abstract generalized linear differential equations with potential converging uniformly with a weight. **accepted***(Boundary Value Problems)*

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0 \; + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$$\left.\begin{array}{c} A_k \stackrel{?}{\to} A_0 \\ f_k \stackrel{?}{\to} f_0 \end{array}\right\} \quad \Longrightarrow \quad x_k \rightrightarrows x_0$$

Integral equations in Banach space

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + f(t) - f(a), \quad t \in [a,b]\,.$$

- $\widetilde{x} \in X$, where X is a Banach space;
- $f : [a,b] \to X$ is regulated on $[a,b]$;
- $A : [a,b] \to \mathcal{L}(X)$ has bounded variation on $[a,b]$.

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + f(t) - f(a), \quad t \in [a, b]\,.$$

- $\widetilde{x} \in X$, where X is a Banach space;
- $f : [a, b] \to X$ is regulated on $[a, b]$;
- $A : [a, b] \to \mathcal{L}(X)$ has bounded variation on $[a, b]$.

Notation:

- $G([a, b], X)$ is the Banach space of regulated functions $f : [a, b] \to X$ with the norm $\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|$
- $BV([a, b], X) = \{f : [a, b] \to X;\ \mathrm{var}_a^b f < \infty\}$
- $f_n \rightrightarrows f$ on $[\alpha, \beta]$: f_n tends to f uniformly on $[\alpha, \beta]$

(starting point) Prague, 2010

Definition

Let $F : [a, b] \to \mathcal{L}(X)$ *and* $g : [a, b] \to X$.

The K-S integral $\int_a^b \mathrm{d}[F]g$ *exists if there is* $I \in X$ *such that for every* $\varepsilon > 0$ *there exists a gauge* δ *on* $[a, b]$ *such that*

$$\Big\| I - \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})]\, g(\xi_j) \Big\|_X < \varepsilon$$

provided the partition

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b, \qquad \xi_j \in [\alpha_{j-1}, \alpha_j]$$

is δ*-fine, i.e.* $\alpha_j - \alpha_{j-1} < \delta(\xi_j)$

Definition

Let $F: [a,b] \to \mathcal{L}(X)$ *and* $g: [a,b] \to X$.

The K-S integral $\int_a^b \mathrm{d}[F]g$ *exists if there is* $I \in X$ *such that for every* $\varepsilon > 0$ *there exists a gauge* δ *on* $[a,b]$ *such that*

$$\Big\| I - \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})]\, g(\xi_j) \Big\|_X < \varepsilon$$

provided the partition

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b, \qquad \xi_j \in [\alpha_{j-1}, \alpha_j]$$

is δ*-fine, i.e.* $\alpha_j - \alpha_{j-1} < \delta(\xi_j)$

Existence: $F \in BV([a,b], \mathcal{L}(X))$ and $g \in G([a,b], X)$

Definition

Let $F : [a, b] \to \mathcal{L}(X)$ *and* $g : [a, b] \to X$.

The K-S integral $\int_a^b \mathrm{d}[F]g$ *exists if there is* $I \in X$ *such that for every* $\varepsilon > 0$ *there exists a gauge* δ *on* $[a, b]$ *such that*

$$\Big\| I - \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})]\, g(\xi_j) \Big\|_X < \varepsilon$$

provided the partition

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b, \qquad \xi_j \in [\alpha_{j-1}, \alpha_j]$$

is δ*-fine, i.e.* $\alpha_j - \alpha_{j-1} < \delta(\xi_j)$

Existence: $F \in G([a, b], \mathcal{L}(X))$ and $g \in BV([a, b], X)$

Definition

Let $F : [a, b] \to \mathcal{L}(X)$ *and* $g : [a, b] \to X$.

The K-S integral $\int_a^b \mathrm{d}[F]g$ *exists if there is* $I \in X$ *such that for every* $\varepsilon > 0$ *there exists a gauge* δ *on* $[a, b]$ *such that*

$$\Big\| I - \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})]\, g(\xi_j) \Big\|_X < \varepsilon$$

provided the partition

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b, \qquad \xi_j \in [\alpha_{j-1}, \alpha_j]$$

is δ*-fine, i.e.* $\alpha_j - \alpha_{j-1} < \delta(\xi_j)$

On Kurzweil-Stieltjes integral in Banach space. *Math. Bohem.*, **137** (2012), 365–381.

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + f(t) - f(a), \quad t \in [a,b], \qquad (1)$$

where $\widetilde{x} \in X$, $A \in BV([a,b], \mathcal{L}(X))$ and $f \in G([a,b], X)$.

Š. Schwabik [Math. Bohemica, 1999 and 2000]

Existence and uniqueness:

$$\big[I - \Delta^- A(t)\big]^{-1} \in \mathcal{L}(X) \quad \text{for all } t \in (a,b]. \qquad \text{(E)}$$

where $\Delta^- A(t) = A(t) - A(t-)$

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + f(t) - f(a), \quad t \in [a,b], \qquad (1)$$

where $\widetilde{x} \in X$, $A \in BV([a,b], \mathcal{L}(X))$ and $f \in G([a,b], X)$.

Existence and uniqueness:

$$\left[I - \Delta^- A(t)\right]^{-1} \in \mathcal{L}(X) \quad \text{for all } t \in (a,b]. \qquad \text{(E)}$$

Proposition

Let $x \in G([a,b], X)$ be the solution of (1). Then

1. $(x - f) \in BV([a,b], X)$,
2. $\|x(t)\|_X \le c_A \left(\|\widetilde{x}\|_X + 2\,\|f\|_\infty \right) \exp(c_A \operatorname{var}_a^t A)$, $t \in [a,b]$, *with* $0 < c_A < \infty$

Continuous dependence on a parameter

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$$A_k \in BV([a,b],\mathcal{L}(X)), \quad f_k \in G([a,b],X), \quad \tilde{x}_k \in X \quad \text{for } k \in \mathbb{N}_0\,.$$

[Monteiro & Tvrdý, DCDS 33, 2013]

Assume

(i) $[I - \Delta^- A_0(t)]^{-1} \in \mathcal{L}(X)$ *for all* $t \in (a,b]$

(ii) $f_k \rightrightarrows f_0$ *on* $[a,b]$ *and* $\tilde{x}_k \to \tilde{x}_0$

(iii) *There exists* $\gamma > 0$ *such that* $\mathrm{var}_a^b A_k \le \gamma$, *for* $k \in \mathbb{N}$

(iv) $A_k \rightrightarrows A_0$

Then $x_k \rightrightarrows x_0$ *on* $[a,b]$.

Continuous dependence on a parameter: Opial type results

Consider

$$x_k' = P_k(t)\,x_k, \qquad x_k(a) = \widetilde{x}$$

$$x_0' = P_0(t)\,x_0, \qquad x_0(a) = \widetilde{x}$$

where $P_k : [a,b] \to \mathbb{R}^m$ are Lebesgue integrable.

[Opial, J. Diff. Eq., 1967]

Assume:

$$\lim_{k\to\infty} \left[\left\| \int_a^t P_k \,\mathrm{d}s - \int_a^t P_0 \,\mathrm{d}s \right\|_\infty \left(1 + \|P_k\|_1\right) \right] = 0$$

Then: $x_k \rightrightarrows x_0$ *on* $[a,b]$.

Consider

$$x_k' = P_k(t)\, x_k, \qquad x_k(a) = \widetilde{x}$$

$$x_0' = P_0(t)\, x_0, \qquad x_0(a) = \widetilde{x}$$

where $P_k : [a, b] \to \mathbb{R}^m$ are Lebesgue integrable.

[Opial, J. Diff. Eq., 1967]

Assume:

$$\lim_{k\to\infty} \left[\left\| \int_a^t P_k \,\mathrm{d}s - \int_a^t P_0 \,\mathrm{d}s \right\|_\infty \Big(1 + \|P_k\|_1 \Big) \right] = 0$$

Then: $x_k \rightrightarrows x_0$ *on* $[a, b]$.

$$A_k(t) = \int_a^t P_k \,\mathrm{d}s, \quad t \in [a, b].$$

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k, \quad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0, \quad t \in [a,b]$$

$$A_k \in BV([a,b],\mathcal{L}(X)), \quad \tilde{x}_k \in X \quad \text{for } k \in \mathbb{N}_0\,.$$

[Monteiro & Tvrdý, DCDS 33, 2013]

Assume

(i) $[I - \Delta^- A_0(t)]^{-1} \in \mathcal{L}(X)$ *for all* $t \in (a,b]$

(ii) $\tilde{x}_k \to \tilde{x}_0$

(iii) $\lim\limits_{k\to\infty} \|A_k - A_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$

Then $x_k \rightrightarrows x_0$ *on* $[a,b]$.

Continuous dependence on a parameter: finite dimensional case

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$A_k \in BV([a,b], \mathcal{L}(\mathbb{R}^m))$, $f_k \in BV([a,b], \mathbb{R}^m)$, $\tilde{x}_k \in \mathbb{R}^m$ for $k \in \mathbb{N}_0$.

Continuous dependence on a parameter: finite dimensional case

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\, x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\, x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$A_k \in BV([a,b], \mathcal{L}(\mathbb{R}^m))$, $f_k \in BV([a,b], \mathbb{R}^m)$, $\tilde{x}_k \in \mathbb{R}^m$ for $k \in \mathbb{N}_0$.

$B_k(t) = \begin{pmatrix} A_k(t) & f_k(t) \\ 0 & 0 \end{pmatrix}$ and $\widetilde{y}_n = \begin{pmatrix} \widetilde{x}_k \\ 1 \end{pmatrix}$ for $t \in [a,b]$, $k \in \mathbb{N}_0$.

$$y_k(t) = \begin{pmatrix} x_k(t) \\ 1 \end{pmatrix} = \widetilde{y}_k + \int_a^t \mathrm{d}\,[B_k]\, y_k, \quad k \in \mathbb{N}_0$$

Continuous dependence on a parameter: finite dimensional case

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$A_k \in BV([a,b],\mathcal{L}(\mathbb{R}^m))$, $f_k \in BV([a,b],\mathbb{R}^m)$, $\tilde{x}_k \in \mathbb{R}^m$ for $k \in \mathbb{N}_0$.

Theorem

Assume

(i) $\det[I - \Delta^- A_0(t)] \neq 0$ *for all* $t \in (a,b]$

(ii) $\tilde{x}_k \to \tilde{x}_0$

(iii) $\lim_{k\to\infty} \|A_k - A_0\|_\infty \left(1 + \mathrm{var}_a^b A_k + \mathrm{var}_a^b f_k\right) = 0$

(iv) $\lim_{k\to\infty} \|f_k - f_0\|_\infty \left(1 + \mathrm{var}_a^b A_k + \mathrm{var}_a^b f_k\right) = 0.$

Then $x_k \rightrightarrows x_0$ *on* $[a,b]$.

Křtiny 2010

Brazil, 2011

Brazil, 2011

Continuous dependence on a parameter: finite dimensional case

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$A_k \in BV([a,b],\mathcal{L}(\mathbb{R}^m))$, $f_k \in BV([a,b],\mathbb{R}^m)$, $\tilde{x}_k \in \mathbb{R}^m$ for $k \in \mathbb{N}_0$.

Theorem

Assume

(i) $\det[I - \Delta^- A_0(t)] \neq 0$ *for all* $t \in (a,b]$

(ii) $\tilde{x}_k \to \tilde{x}_0$

(iii) $\lim_{k\to\infty} \|A_k - A_0\|_\infty \left(1 + \mathrm{var}_a^b A_k + \mathrm{var}_a^b f_k\right) = 0$

(iv) $\lim_{k\to\infty} \|f_k - f_0\|_\infty \left(1 + \mathrm{var}_a^b A_k + \mathrm{var}_a^b f_k\right) = 0.$

Then $x_k \rightrightarrows x_0$ *on* $[a,b]$.

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\, x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\, x_0 \; + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$$A_k \in BV([a,b],\mathcal{L}(X)), \quad f_k \in G([a,b],X), \quad \tilde{x}_k \in X \quad \text{for } k \in \mathbb{N}_0\,.$$

Continuous dependence on a parameter

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$$A_k \in BV([a,b],\mathcal{L}(X)), \quad f_k \in G([a,b],X), \quad \tilde{x}_k \in X \quad \text{for } k \in \mathbb{N}_0\,.$$

[Monteiro & Tvrdý, Boundary Value Problems]

Assume $f_0 \in BV([a,b],X)$ *and*

- (i) $[I - \Delta^- A_0(t)]^{-1} \in \mathcal{L}(X)$ *for all* $t \in (a,b]$
- (ii) $\tilde{x}_k \to \tilde{x}_0$
- (iii) $\lim_{k\to\infty} \|A_k - A_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$
- (iv) $\lim_{k\to\infty} \|f_k - f_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$

Then $x_k \rightrightarrows x_0$ *on* $[a,b]$.

Lemma

Let $A_k \in BV([a,b],\mathcal{L}(X))$ *for* $k \in \mathbb{N}_0$ *and assume*

(i) $[I - \Delta^- A_0(t)]^{-1} \in \mathcal{L}(X)$ *for all* $t \in (a,b]$

(ii) $\lim\limits_{k\to\infty} \|A_k - A_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$

Then there exist $r^* > 0$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\|y\|_\infty \le r^* \left(\|y(a)\| + \left(1 + \mathrm{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| y(t) - y(a) - \int_a^t \mathrm{d}[A_k]\, y \right\| \right)$$

for all $y \in G([a,b],X)$ *and* $k \ge k_0$

Kiguradze [J. Sov. Math., 1988]

Hakl, Lomtatidze and Stavrolaukis [Abstr. Appl. Anal., 2004]

Karlštejn, 2013

Theorem

Consider

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\, x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\, x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

$$A_k \in BV([a,b], \mathcal{L}(X)), \quad f_k \in G([a,b], X), \quad \tilde{x}_k \in X \quad \text{for } k \in \mathbb{N}_0\,.$$

Assume $f_0 \in BV([a,b], X)$ *and*

(i) $[I - \Delta^- A_0(t)]^{-1} \in \mathcal{L}(X)$ *for all* $t \in (a,b]$

(ii) $\tilde{x}_k \to \tilde{x}_0$

(iii) $\lim_{k\to\infty} \|A_k - A_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$

(iv) $\lim_{k\to\infty} \|f_k - f_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$

Then $x_k \rightrightarrows x_0$ *on* $[a,b]$.

Example

Let $A : [0,1] \to \mathbb{R}$, $A(t) = 0$, $t \in [0,1]$. For $k \in \mathbb{N}$ put

$$A_k(t) = \begin{cases} 0 & \text{if } \ t \in [0, \tau_{0,k}]\,, \\ a_{m,k}\, t + b_{m,k} & \text{if } t \in [\tau_{m,k}\,, \tau_{m+1,k}], \quad 0 \le m \le n_k - 1\,, \end{cases}$$

where

$$n_k = [k^{3/2}] + 1, \qquad \tau_{m,k} = \frac{1}{2^{\,n_k - m}}, \qquad 0 \le m \le n_k,$$

$$a_{0,k} = \frac{2^{\,n_k}}{k}(-1)^{\,n_k}, \qquad b_{0,k} = \frac{1}{k}\,(-1)^{\,n_k - 1},$$

$$a_{m,k} = \frac{2^{\,n_k - m + 1}}{k}\,(-1)^{\,n_k - m}, \quad b_{m,k} = \frac{3}{k}\,(-1)^{\,n_k - m + 1}, \quad 1 \le m \le n_k - 1$$

[graphics]

Example

Let $A : [0,1] \to \mathbb{R}$, $A(t) = 0$, $t \in [0,1]$. For $k \in \mathbb{N}$ put

$$A_k(t) = \begin{cases} 0 & \text{if } \; t \in [0, \tau_{0,k}]\,, \\ a_{m,k}\, t + b_{m,k} & \text{if } t \in [\tau_{m,k}, \tau_{m+1,k}], \quad 0 \le m \le n_k - 1\,, \end{cases}$$

where $n_k = [k^{3/2}] + 1$, $\tau_{m,k} = 1/2^{\,n_k - m}$, for $0 \le m \le n_k$.

Let $f : [0,1] \to \mathbb{R}$

$$f(t) = \begin{cases} \dfrac{(-1)^n}{\sqrt[4]{n}} & \text{if } \; t \in (2^{-n}, 2^{-(n-1)}] \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \; t = 0\,, \end{cases}$$

Example

Let $A:[0,1]\to\mathbb{R}$, $A(t)=0$, $t\in[0,1]$. For $k\in\mathbb{N}$ put

$$A_k(t)=\begin{cases}0 & \text{if } \ t\in[0,\tau_{0,k}]\,,\\ a_{m,k}\,t+b_{m,k} & \text{if } t\in[\tau_{m,k},\tau_{m+1,k}], \quad 0\le m\le n_k{-}1\,,\end{cases}$$

where $n_k=[k^{3/2}]+1$, $\tau_{m,k}=1/2^{\,n_k-m}$, for $0\le m\le n_k$. Consider

$$f(t)=\begin{cases}\dfrac{(-1)^n}{\sqrt[4]{n}} & \text{if } \ t\in(2^{-n},2^{-(n-1)}] \text{ for some } n\in\mathbb{N},\\ 0 & \text{if } \ t=0\,,\end{cases}$$

We have $f\in G[0,1]$, $A_k\in BV[0,1]$, $k\in\mathbb{N}$,

$$\lim_{k\to\infty}\left(1+\mathrm{var}_a^b A_k\right)\|A_k-A\|_\infty=0 \quad \text{BUT} \quad \int_a^t \mathrm{d}[A_k]\,f \not\rightarrow \int_a^t \mathrm{d}[A]\,f$$

Example

Let $A : [0,1] \to \mathbb{R}$, $A(t) = 0$, $t \in [0,1]$. For $k \in \mathbb{N}$ put

$$A_k(t) = \begin{cases} 0 & \text{if } \ t \in [0, \tau_{0,k}]\,, \\ a_{m,k}\, t + b_{m,k} & \text{if } t \in [\tau_{m,k}, \tau_{m+1,k}], \quad 0 \le m \le n_k - 1\,, \end{cases}$$

where $n_k = [k^{3/2}] + 1$, $\tau_{m,k} = 1/2^{\,n_k - m}$, for $0 \le m \le n_k$. Consider

$$f(t) = \begin{cases} \dfrac{(-1)^n}{\sqrt[4]{n}} & \text{if } \ t \in (2^{-n}, 2^{-(n-1)}] \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \ t = 0\,, \end{cases}$$

We have $f \in G[0,1]$, $A_k \in BV[0,1]$, $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \left(1 + \mathrm{var}_a^b\, A_k\right) \|A_k - A\|_\infty = 0$$

$$\lim_{k\to\infty} \left(1 + \mathrm{var}_a^b\, A_k\right) \|f_k - f\|_\infty = 0, \quad \text{for } f_k \equiv f$$

Example

$$x(t) = \int_0^t \mathrm{d}[A]\, x + f(t),\ \ t \in [0,1], \qquad \text{has solution } x \equiv f$$

while for

$$x_k(t) = \int_0^t \mathrm{d}[A_k]\, x_k + f_k(t), \quad t \in [0,1], \quad k \in \mathbb{N},$$

the solution is

$$x_k(t) = \begin{cases} f(t) & \text{if} \ \ t \in [0, \tau_{0,k}]\,, \\ c_{m,k}\ \exp(a_{m,k}\, t) & \text{if} \ \ t \in (\tau_{m,k}, \tau_{m+1,k}], \quad 0 \le m \le n_k - 1, \end{cases}$$

x_k does not converge to x

Example

For each $k \in \mathbb{N}$,

$$x_k(t) = \begin{cases} f(t) & \text{if } \; t \in [0, \tau_{0,k}]\,, \\ c_{m,k}\, \exp(a_{m,k}\, t) & \text{if } \; t \in (\tau_{m,k}, \tau_{m+1,k}], \quad 0 \le m \le n_k - 1, \end{cases}$$

where

$$c_{0,k} = f(\tau_{1,k})\, \exp(-a_{0,k}\, \tau_{0,k}) = f(\tau_{1,k})\, \exp\left(\frac{1}{k}\, (-1)^{n_k+1}\right)$$

and, for $1 \le m \le n_k - 1$,

$$c_{m,k} = c_{m-1,k}\, \exp\left((a_{m-1,k} - a_{m,k})\, \tau_{m,k}\right) + \left(f(\tau_{m+1,k}) - f(\tau_{m,k})\right)\, \exp(-a_{m,k}\, \tau_{m,k})$$

Example

For each $k \in \mathbb{N}$,

$$x_k(t) = \begin{cases} f(t) & \text{if} \quad t \in [0, \tau_{0,k}], \\ c_{m,k} \, \exp(a_{m,k}\, t) & \text{if} \quad t \in (\tau_{m,k}, \tau_{m+1,k}], \quad 0 \leq m \leq n_k - 1, \end{cases}$$

where $c_{0,k} = f(\tau_{1,k}) \, \exp\left(\frac{1}{k}\,(-1)^{n_k+1}\right)$,

$$c_{1,k} = c_{0,k}\, \exp\left(\frac{4}{k}\,(-1)^{\,n_k}\right) + \left(f(\tau_{2,k}) - f(\tau_{1,k})\right)\, \exp\left(\frac{2}{k}\,(-1)^{\,n_k}\right),$$

for m is even

$$c_{m,k} = \exp\left(\frac{2}{k}\,(-1)^{\,n_k+1}\right)\left(c_{0,k} + \sum_{j=2}^{m+1}(-1)^{j+1}\, f(\tau_{j,k})\right) + \exp\left(\frac{4}{k}\,(-1)^{\,n_k+1}\right)\,\sum_{j=1}^{m}(-1)^{j}\, f(\tau_{j,k}),$$

while for m odd and $m > 1$

$$c_{m,k} = \exp\left(\frac{4}{k}\,(-1)^{n_k}\right)\left(c_{0,k} + \sum_{j=2}^{m}(-1)^{j+1}\, f(\tau_{j,k})\right) + \exp\left(\frac{2}{k}\,(-1)^{n_k}\right)\,\sum_{j=1}^{m+1}(-1)^{j}\, f(\tau_{j,k}).$$

Example

For each $k \in \mathbb{N}$,

$$x_k(t) = \begin{cases} f(t) & \text{if} \quad t \in [0, \tau_{0,k}] \,, \\ c_{m,k} \, \exp(a_{m,k}\, t) & \text{if} \quad t \in (\tau_{m,k}, \tau_{m+1,k}], \quad 0 \le m \le n_k - 1, \end{cases}$$

implies $x_k(1) = c_{n_k-1,k} \, \exp(-4/k)$

Example

$x_k(1) = c_{n_k-1,k} \exp(-4/k)$ where

for n_k odd : $m = n_k - 1$ is even

$$c_{m,k} = \exp\left(\frac{2}{k}(-1)^{n_k+1}\right)\left(c_{0,k} + \sum_{j=2}^{m+1}(-1)^{j+1} f(\tau_{j,k})\right) + \exp\left(\frac{4}{k}(-1)^{n_k+1}\right)\sum_{j=1}^{m}(-1)^{j} f(\tau_{j,k})$$

Example

$$x_k(1) = c_{n_k-1,k} \exp(-4/k) \quad \text{where}$$

for n_k odd

$$c_{n_k-1,k} = \exp\left(\frac{2}{k}\right)\left(c_{0,k} - 1\right) + \exp\left(\frac{2}{k}\right)\left(\exp\left(\frac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1} \frac{1}{\sqrt[4]{m}} + \exp\left(\frac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

Example

$$x_k(1) = c_{n_k-1,k}\ \exp(-4/k) \quad \text{where}$$

for n_k odd

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k}-1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right)-1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} + \exp\left(\tfrac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

and for n_k even : $m = n_k - 1$ is odd

$$c_{m,k} = \exp\left(\frac{4}{k}(-1)^{n_k}\right)\left(c_{0,k} + \sum_{j=2}^{m}(-1)^{j+1} f(\tau_{j,k})\right) + \exp\left(\frac{2}{k}(-1)^{n_k}\right)\sum_{j=1}^{m+1}(-1)^{j} f(\tau_{j,k})\,.$$

Example

$$x_k(1) = c_{n_k-1,k}\ \exp(-4/k) \quad \text{where}$$

for n_k odd

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k} - 1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1} \frac{1}{\sqrt[4]{m}} + \exp\left(\tfrac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

and for n_k even

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k}\exp\left(\tfrac{2}{k}\right) - 1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1} \frac{1}{\sqrt[4]{m}} - \exp\left(\tfrac{2}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

Example

$$x_k(1) = c_{n_k-1,k} \exp(-4/k) \quad \text{where}$$

for n_k odd

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k} - 1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} + \exp\left(\tfrac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

and for n_k even

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k}\exp\left(\tfrac{2}{k}\right) - 1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} - \exp\left(\tfrac{2}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

$$\exp\left(\tfrac{2}{k}\right)\left(\frac{\exp\left(\tfrac{2}{k}\right) - 1}{\tfrac{2}{k}}\right)\frac{2}{k}\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}}$$

Example

$$x_k(1) = c_{n_k-1,k}\ \exp(-4/k) \quad \text{where}$$

for n_k odd

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k}-1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right)-1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} + \exp\left(\tfrac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

and for n_k even

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k}\exp\left(\tfrac{2}{k}\right)-1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right)-1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} - \exp\left(\tfrac{2}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

$$\exp\left(\tfrac{2}{k}\right)\left(\frac{\exp\left(\frac{2}{k}\right)-1}{\frac{2}{k}}\right)\frac{2}{k}\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} > \exp\left(\tfrac{2}{k}\right)\left(\frac{\exp\left(\frac{2}{k}\right)-1}{\frac{2}{k}}\right)\frac{2}{k}\int_2^{n_k}\frac{1}{\sqrt[4]{t}}\,\mathrm{d}t$$

Example

$$x_k(1) = c_{n_k-1,k}\ \exp(-4/k) \quad \text{where}$$

for n_k odd

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k} - 1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} + \exp\left(\tfrac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

and for n_k even

$$c_{n_k-1,k} = \exp\left(\tfrac{2}{k}\right)\left(c_{0,k}\exp\left(\tfrac{2}{k}\right) - 1\right) + \exp\left(\tfrac{2}{k}\right)\left(\exp\left(\tfrac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} - \exp\left(\tfrac{2}{k}\right)\frac{1}{\sqrt[4]{n_k}}$$

$$\exp\left(\tfrac{2}{k}\right)\left(\frac{\exp\left(\tfrac{2}{k}\right) - 1}{\tfrac{2}{k}}\right)\frac{2}{k}\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} > \exp\left(\tfrac{2}{k}\right)\left(\frac{\exp\left(\tfrac{2}{k}\right) - 1}{\tfrac{2}{k}}\right)\frac{8}{3\,k}\left(\sqrt[4]{(n_k)^3} - \sqrt[4]{2^3}\right)$$

Theorem

Let $A_k \in BV([a,b],\mathcal{L}(X)),\quad f_k \in G([a,b],X),\quad \tilde{x}_k \in X \quad$ *for* $k \in \mathbb{N}_0$

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}\,[A_k]\,x_k + f_k(t) - f_k(a), \qquad t \in [a,b]$$

$$x_0(t) = \tilde{x}_0 + \int_a^t \mathrm{d}\,[A_0]\,x_0 + f_0(t) - f_0(a), \qquad t \in [a,b]$$

Assume $f_0 \in BV([a,b],X)$ *and*

(i) $[I - \Delta^- A_0(t)]^{-1} \in \mathcal{L}(X)$ *for all* $t \in (a,b]$

(ii) $\tilde{x}_k \to \tilde{x}_0$

(iii) $\lim_{k\to\infty} \|A_k - A_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$

(iv) $\lim_{k\to\infty} \|f_k - f_0\|_\infty \left(1 + \mathrm{var}_a^b A_k\right) = 0.$

Then $x_k \rightrightarrows x_0$ *on* $[a,b]$.

Thank you!