Boundary value problems for systems coming from models of house burglary

JEAN MAWHIN

(joint work with MARTA GARCIA-HUIDOBRO and RAUL MANÁSEVICH)

The motivation

- mathematical model (M) for burglary of houses (PITCHER, 2010)
- $\Omega \subset \mathbb{R}^n$ open, bounded, smooth boundary
- $\eta > 0, \quad A^0 > 0, \; \psi > 0, \; \omega > 0$ *given*
- A attractiveness for a house to be burgled, N density of burglars
- $\eta\Delta(A - A^0) - A + A^0 + \psi NA(1 - A) = 0$ *in* Ω

  $\Delta N - \nabla \cdot \left(2N\frac{\nabla A}{A}\right) - \omega^2(N - 1) = 0$ *in* Ω

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  $\partial_\nu A \ = \ \partial_\nu N = 0$ *on* $\partial\Omega$
- $A > 0, \ N > 0$ natural assumptions
- equilibria of the associated parabolic evolution problem
- *positive constant solution*

  $$A \ = (2\psi)^{-1} \left[\psi - 1 + \sqrt{(\psi - 1)^2 + 4A^0\psi}\right], \quad N = 1$$

Generalizations

- $A^0,\ \psi,\ \eta \in C(\overline{\Omega}, \mathbb{R}^+)$
- $\eta(x)\Delta[A - A^0(x)] - A + A^0(x) + \psi(x)NA(1-A) = 0$ *in* Ω

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- **question** : existence of positive (non constant) solution ?
- more general problem :
  $$\eta(x)\Delta[A - A^0(x)] - A + A^0(x) + Nf(x,A) = 0 \ \textit{ in } \ \Omega$$
  $$\Delta N + \nabla \cdot [N\nabla h(A)] - \omega^2(N-1) = 0 \ \textit{ in } \ \Omega$$
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- $f \in C(\overline{\Omega} \times \mathbb{R}_0^+, \mathbb{R}),\ h \in C^2(\mathbb{R}^+, \mathbb{R})$

Radial solutions on an annulus

- $0 < l < L, \ \Omega = \{x \in \mathbb{R}^n : l < \|x\| < L\}$
- $\eta = \eta(\|x\|), \ A_0 = A_0(\|x\|), \ f = f(\|x\|, u)$
- radial solutions of the Neumann problem on Ω for
  $\eta(\|x\|)\Delta[A - A^0(\|x\|)] - A + A^0(\|x\|) + Nf(\|x\|, A) = 0$ *in* Ω
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- equivalent to ODE problem
  $\eta(r)[r^{n-1}(A - A^0(r))']' + r^{n-1}[-A + A^0(r) + Nf(r, A)]$
  $= 0$ *in* (l, L)
  $(r^{n-1}N')' + \{r^{n-1}[h(A)]'N\}' - r^{n-1}\omega^2(N - 1) = 0$ *in* (l, L)
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- still more general second equation
  $(r^{n-1}N')' + [r^{n-1}g(r, A, A')N]' - r^{n-1}\omega^2(N-1) = 0$ *in* (l, L)
- $g \in C^1([l, L] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$

The case where $n = 1$ (for simplicity)

- r *rebaptized* x
- $0 \leq l < L,\ \omega > 0,\ \eta \in C([l, L], \mathbb{R}^+),\ A^0 \in C^2([l, L], \mathbb{R}^+)$
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- **Neumann problem** *:*
  $$\eta(x)[A - A^0(x)]'' - A + A^0(x) + Nf(x, A) = 0$$
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  $$N'' + [g(x,A,A')N]' - \omega^2(N-1) = 0$$
  $$A'(l) = A'(L) = N'(l) = N'(L) = 0$$
- *solution :*
  $$(A,N) \in C^2([0,T]) \times C^2([0,T])\ :\ A(x) > 0\ ,\ \forall\, x \in [l,L]$$
- *positive solution :*
  $$(A,N)\ :\ A(x) > 0,\ N(x) > 0\ ,\ \forall\, x \in [l,L]$$

Assumptions upon $f, \; g$ and statement

- $(f_1) \;\; \exists\, R > osc\, A^0 \;:\; \forall\, x \in [l, L] \;:$
  - $f(x, A) \geq 0$ *when* $0 \leq A \leq R$
  - $f(x, A) \leq 0$ *when* $A \geq R$
- $(f_2) \;\; A^0 - \eta(A^0)''$ *is positive and* $\exists\, R > 0 \;:\; \forall\, x \in [l, L] \;:$
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- $(g) \quad g(l, A, 0) = 0$ *and* $g(L, A, 0) = 0 \,,\; \forall\, A \in \mathbb{R}^+$

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- (g) $g(l, A, 0) = 0$ *and* $g(L, A, 0) = 0$, $\forall\, A \in \mathbb{R}^+$
- **Theorem.** $(f_1) + (g)$ *or* $(f_2) + (g)$ $\Rightarrow$ *problem*
  $$\eta(x)[A - A^0(x)]'' - A + A^0(x) + N f(x, A) = 0$$
  $$N'' + [g(x, A, A')N]' - \omega^2(N - 1) = 0$$
  $$A'(l) = A'(L) = N'(l) = N'(L) = 0$$
  has at least one positive solution

Homotopy and *a priori* estimates

- **proof.** fixed point reduction and Leray-Schauder degree
- homotopy : (H)

$$-\eta(x)[A - A^0(x)]'' + A - A^0(x) = \lambda N f(x, A)$$

$$-N'' + \omega^2 N = \omega^2 + \lambda[g(x, A, A')N]'$$

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- *a priori* estimates through a combination of pointwise estimates based upon extremum properties and L^1-estimates
- $B \in C([l, L]),\ \min B := \min_{[l,L]} B,\ \max B := \max_{[l,L]} B$
- estimates for possible positive solution of (H) for any $\lambda \in [0, 1]$
  - **Lemma 1.** $\int_l^L N(x)\,dx = L - l$
  - **Lemma 2.** $\min N > 0$

More *a priori* estimates

estimates for possible positive solution of (H) *for some* $\lambda \in [0,1]$ *:*

- **Lemma 3.** $(f_1) \Rightarrow \forall\, x \in [l, L]$:
  $0 < A_{0,1} := \min\{\min A^0, R - osc\, A^0\} \leq A(x)$
  $\leq \max\{\max A^0, R + osc\, A^0\} := A_{1,1}$

- **Lemma 4.** $(f_2) \Rightarrow \forall\, x \in [l, L]$:
  $0 < A_{0,2} := \min\{\min[A^0 - \eta(A^0)''], R\} \leq A(x)$
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- **Lemma 5.** $\forall\, x \in [l, L]\,,\; |A'(x)|$
  $\le L \max |(A^0)''| + \frac{L}{\min \eta} \left[A_1 + \max A^0 + \max |f|\right] := A_2$

- **Lemma 6.** $\forall\, x \in [l, L]\;:\; N(x)$
  $\le 1 + 2\omega^2 L^2 + L \max_{[l,L]\times[A_0,A_1]\times[-A_2,A_2]} |g| := A_3$

Linear results

- **Lemma 7.** $\forall\, f \in C([l,L])$
  $-\eta(x)A'' + A = f(x),\ A'(l) = 0 = A'(L)$
  has a unique solution $A \in C^2([l,L])$ *, and*
  $\exists\, C > 0\ :\ \|A\|_\infty + \|A'\|_\infty \leq C\|f\|_\infty$ *. Furthermore,*
  $f(x) > 0\ ,\ \forall\, x \in [l,L]\ \Rightarrow\ A(x) > 0\ ,\ \forall\, x \in [l,L]$

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- **Lemma 8.** $-N'' + \omega^2 N = \omega^2,\ \ N'(l) = 0 = N'(L)$
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- **Lemma 9.** $\forall\, f \in C([l,L])$
  $-N'' + \omega^2 N = f(x),\ \ N'(l) = 0 = N'(L)$
  has the unique solution $N(x) = \int_l^L G(x,y) f(y)\, dy$ *, where*

$$G(x,y) = \begin{cases} \dfrac{\cosh\omega(L-y)\ \cosh\omega(x-l)}{\omega\sinh\omega(L-l)} & \textit{if}\quad l \leq x \leq y \leq L \\ \dfrac{\cosh\omega(L-x)\ \cosh\omega(y-l)}{\omega\sinh\omega(L-l)} & \textit{if}\quad l \leq y < x \leq L \end{cases}$$

Fixed point reduction

- $K : C([l, L]) \to C^1([l, L]), f \mapsto$ *unique solution* A *of*
  $-\eta(x)A'' + A = f(x),\ A'(l) = 0 = A'(L)$
  K *is continuous and takes positive functions into positive functions*

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- $E := C^1([l,L]) \times C([l,L])$
  $\|(A,N)\|_E = \|A\|_\infty + \|A'\|_\infty + \|N\|_\infty$

- $\mathcal{F} : \{(A,U,\lambda) \in E \times [0,1] \mid A > 0, N > 0\} \to E$
  $\mathcal{F}(A,N,\lambda) = \big(K[-\eta(A^0)'' + A^0 + \lambda N f(\cdot, A)],$
  $-\lambda \int_l^L \partial_y G(\cdot,y) g(y, A(y), A'(y)) N(y)\, dy + \omega^2 \int_l^L G(\cdot,y)\, dy\big)$

- **Lemma 10.** $\forall\, \lambda \in [0,1]$ *fixed,* (A,N) *is a positive solution of (H)*
  $\Leftrightarrow$ (A,N) *is a fixed point of* $\mathcal{F}$

Existence proof

- fix $0 < R_0 < A_0, R_1 > A_1, R_2 > A_2, R_3 > A_3$
  define $D = \{(A, N) \in E : R_0 < A(x) < R_1,\ \|A'\|_\infty < R_2,$
  $0 < N(x) < R_3\ \ (x \in [l, L])\}$

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- apply Leray-Schauder's continuation theorem to $\mathcal{F}$ on $\overline{D} \times [0, 1]$

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- apply Leray-Schauder's continuation theorem to $\mathcal{F}$ on $\overline{D} \times [0, 1]$
- Lemmas 1 to 6 $\Rightarrow \forall\, \lambda \in [0, 1]$ and for any possible fixed point (A, N) of $\mathcal{F}$, $(A, N) \notin \partial D$
- $d_{LS}[I - \mathcal{F}(\cdot, 1), D, 0] = d_{LS}[I - \mathcal{F}(\cdot, 0), D, 0] = 1$
- $\mathcal{F}(\cdot, 1)$ has a fixed point in Ω

Application to the model

- **Corollary 1.** *If either* $osc\, A^0 < 1$ *or* $\min(A^0 - \eta(A^0)'') > 0$ *,*

$$\eta(x)[A - A^0(x)]'' - A + A^0(x) + \psi(x) N A(1 - A) = 0$$

$$N'' - 2\left(\frac{A'}{A} N\right)' - \omega^2 (N - 1) = 0$$

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  has at least one positive solution
- *similar results for arbitrary* n
- *for* $n = 1$ *, similar results under* **periodic** *boundary conditions*
- **open problems**
  - conditions for **uniqueness** of the solution in the model case
  - radial problem for $n \geq 2$ on a **ball** (i.e. case $l = 0$)
  - other symmetries or **non radial** case

Other models

- similar results also obtained for models of the type

$$\eta[A - A^0(x)]'' - A + A^0(x) + NA = 0$$

$$\left(N' - 2N\frac{A'}{A}\right)' + A^1(x) - A^0(x) - NA = 0 \qquad (*)$$

$$A'(l) = A'(L) = N'(l) = N'(L) = 0$$

- **Theorem 2**. *If* $L > 0$, $\eta > 0$, $A^0 \in C^2([0,L],\mathbb{R}^+)$ *,* $(A^0)'(0) = (A^0)'(L) = 0$ *,* $A^1 \in C([0,L],\mathbb{R}^+)$ *,* $A^1(x) > A^0(x)\ \forall\, x \in [0,L]$*, the Neumann problem (*) has at least one positive solution.*
- similar but more involved method of proof because homotopy to a nonlinear problem inspired by the technique of the continuation theorem of coincidence degree theory

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Happy birthday, Milan,
many more Stieltjes integrals and snapshots !

Thank you for your attention !