On the periodic problem for singular second-order ordinary differential equations

Alexander Lomtatidze

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega), \; u'(0) = u'(\omega)$$

Definition. We say that the function $p \in L_\omega$ belongs to the set $\mathcal{V}^-(\omega)$ (resp., $\mathcal{V}^+(\omega)$) if, for any function $u \in AC'([0, \omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{pro} \quad t \in [0, \omega]; \quad u(0) = u(\omega), \; u'(0) = u'(\omega),$$

the inequality

$$u(t) \leq 0 \quad \text{pro} \quad t \in [0, \omega] \quad \Big(\text{resp., } u(t) \geq 0 \quad \text{pro} \quad t \in [0, \omega]\Big)$$

is fulfilled.

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has a (nontrivial) sign-constant solution.

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has a (nontrivial) sign-constant solution.

Remark. The inclusion $p \in \mathcal{V}_0(\omega)$ holds if and only if the function p admits the representation

$$p(t) = g(t) + \big(\ell(g)(t)\big)^2 \quad \text{pro} \quad t \in \mathbb{R},$$

where $g \in L_\omega$, $\overline{g} = 0$, and

$$\ell(g)(t) \overset{\text{def}}{=} -\frac{1}{\omega}\int_t^{t+\omega}\int_t^s \big(g(\xi) - \overline{g}\big)\mathrm{d}\xi\mathrm{d}s \quad \text{pro} \quad t \in \mathbb{R}.$$

Theorem. *Let $p \in L_\omega$. Then the inclusion $p \in \mathcal{V}^-(\omega)$ holds if and only if there exists a function $\gamma \in AC'([0,\omega])$ satisfying*

$$\gamma''(t) \leq p(t)\gamma(t) \quad \textit{pro} \quad t \in [0,\omega],$$
$$\gamma(t) > 0 \quad \textit{pro} \quad t \in [0,\omega],$$
$$\gamma(0) \geq \gamma(\omega), \qquad \frac{\gamma'(\omega)}{\gamma(\omega)} \geq \frac{\gamma'(0)}{\gamma(0)},$$

and

$$\gamma(0) - \gamma(\omega) + \frac{\gamma'(\omega)}{\gamma(\omega)} - \frac{\gamma'(0)}{\gamma(0)} + \operatorname{mes}\Big\{t \in [0,\omega] : \gamma''(t) < p(t)\gamma(t)\Big\} > 0.$$

Corollary. *Let* $p \in L_\omega$, $p \not\equiv 0$,

$$\|[p]_-\|_L < \frac{4}{\omega},$$

and

$$\|[p]_+\|_L \geq \|[p]_-\|_L \left(1 - \frac{\omega}{4}\|[p]_-\|_L\right)^{-1}.$$

Then $p \in \mathcal{V}^-(\omega)$.

On the set $\mathcal{V}^-(\omega)$

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Corollary. *Let* $p \in L_\omega$, $[p]_-^2 \in L_\omega$, $p \not\equiv 0$,

$$k^*(\omega)\big\|[p]_-^2\big\|_L < 1,$$

and

$$\big\|[p]_+\big\|_L \geq \big\|[p]_-\big\|_L + \frac{\omega}{4}\big\|[p]_-\big\|_L^2 \left(1 - \sqrt{k^*(\omega)\big\|[p]_-^2\big\|_L}\right)^{-1}.$$

Then $p \in \mathcal{V}^-(\omega)$.

Corollary. *Let* $p \in L_\omega$, $\overline{p} > 0$, *and*

$$|\ell(p)(t)| \leq \sqrt{\overline{p}} \quad \textit{pro} \quad t \in [0, \omega],$$

where

$$\ell(p)(t) \stackrel{\text{def}}{=} -\frac{1}{\omega} \int_t^{t+\omega} \int_t^s \big(p(\xi) - \overline{p}\big) \mathrm{d}\xi \mathrm{d}s \quad \textit{pro} \quad t \in \mathbb{R}.$$

Then $p \in \mathcal{V}^-(\omega)$.

On the set $\mathcal{V}^+(\omega)$

Definition. We say that the function $p \in L_\omega$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$u'' = p(t)u; \quad u(\alpha) = 0, \ u(\beta) = 0$$

has no nontrivial solution for any $\alpha < \beta$ satisfying $\beta - \alpha < \omega$.

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Theorem. *Let $p \in L_\omega$. Then the inclusion $p \in \mathcal{V}^+(\omega)$ holds if and only if $p \in \mathcal{D}(\omega)$ and there exists a function $\gamma \in AC'([0,\omega])$ satisfying*

$$\gamma''(t) \geq p(t)\gamma(t) \quad \textit{pro} \quad t \in [0,\omega],$$
$$\gamma(t) > 0 \quad \textit{pro} \quad t \in [0,\omega],$$
$$\gamma(0) = \gamma(\omega), \quad \gamma'(0) \geq \gamma'(\omega),$$

and

$$\gamma'(0) - \gamma'(\omega) + \operatorname{mes}\Big\{t \in [0,\omega] : \gamma''(t) > p(t)\gamma(t)\Big\} > 0.$$

Theorem. *Let $p \in L_\omega$. Then the inclusion $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ holds if and only if $p \in \operatorname{Int}\mathcal{D}(\omega)$ and there exists $\gamma \in AC'([0,\omega])$ satisfying*

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Theorem. *Let $p \in L_\omega$ be such that*

$$p \not\equiv 0, \qquad \int_0^\omega p(s)\mathrm{d}s \leq 0.$$

Then $p \in \mathcal{V}^+(\omega)$ ($p \in \operatorname{Int} \mathcal{V}^+(\omega)$) if and only if $p \in \mathcal{D}(\omega)$ ($p \in \operatorname{Int} \mathcal{D}(\omega)$).

Corollary. *Let* $p \in L_\omega$, $p \not\equiv 0$, $\overline{p} \le 0$, *and*

$$k^*(\omega) \int_0^\omega [p(s)]_-^2 \, \mathrm{d}s < 1.$$

Then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$.

On the set $\mathcal{V}^+(\omega)$

Corollary. *Let* $p \in L_\omega$, $p \not\equiv 0$, $\overline{p} \leq 0$, *and*

$$k^*(\omega) \int_0^\omega [p(s)]_-^2 \,\mathrm{d}s < 1.$$

Then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$.

Corollary. *Let* $p \in L_\omega$, $p \not\equiv$ *Const.*, $\overline{p} \leq 0$, *and*

$$\frac{1}{\ell} \left(\mathrm{e}^{\omega \ell} - 1 \right) \sqrt{|\overline{p}| + \ell^2} \leq \pi.$$

Then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$.

Corollary. *Let* $p \in L_\omega$, $p \not\equiv Const.$,

$$\ell \leq \frac{1}{\omega}\,\ln(1+\pi)$$

and

$$0 \leq \overline{p} \leq \frac{\ln(1+\pi)}{\omega^2\pi}\left(\int_0^\omega |\ell(p)(\xi)|\mathrm{d}\xi\right)^2,$$

where

$$\ell(p)(t) \stackrel{\text{def}}{=} -\frac{1}{\omega}\int_t^{t+\omega}\int_t^s \big(p(\xi)-\overline{p}\big)\mathrm{d}\xi\mathrm{d}s \quad \textit{pro} \quad t \in \mathbb{R}.$$

Then $p \in \operatorname{Int}\mathcal{V}^+(\omega)$*.*

Example. Let $\omega = 2\pi$, $p(t) = c + \lambda \cos t$, $\lambda \neq 0$. Then $\overline{p} = c$, $\ell(p)(t) = \lambda \sin t$, $\ell = |\lambda|$, $\int_0^\omega |\ell(p)(s)| \mathrm{d}s = 4|\lambda|$. If

$$\lambda^2 - \lambda^2 \frac{\pi^2}{(\mathrm{e}^{2\pi|\lambda|} - 1)^2} \leq c \leq 0$$

then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$. On the other hand, if

$$|\lambda| \leq \frac{1}{2\pi} \ln(1 + \pi)$$

and

$$0 \leq c \leq \frac{4\lambda^2}{\pi^3} \ln(1 + \pi)$$

then $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, as well.

$$u'' = p(t)u - \frac{h_0(t)}{u^\lambda}\,; \quad u(0) = u(\omega),\; u'(0) = u'(\omega) \tag{1}$$

Theorem. *Let $\lambda > -1$. Then the problem (1) is solvable if and only if $p \in \mathcal{V}^-(\omega)$. If $p \in \mathcal{V}^-(\omega)$ and $\lambda > 0$ then the problem (1) is uniquely solvable.*

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$$u'' = p(t)u + \frac{h_0(t)}{u^\lambda}; \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2}$$

Theorem. *Let $\lambda \in \,]-1, 1]$. Then the problem (2) is solvable for any h_0 satisfying*

$$h_0(t) \geq 0 \quad \textit{pro} \quad t \in [0, \omega], \quad h_0 \not\equiv 0.$$

if and only if the inclusion $p \in \mathcal{V}^+(\omega)$ holds.

$$u'' = p(t)u - \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{3}$$

$$u'' = p(t)u + \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{4}$$

$$p \in \mathcal{V}^-(\omega)$$

$$u'' = p(t)u - \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega), \; u'(0) = u'(\omega) \tag{3}$$

Theorem. *Let $p \in \mathcal{V}^-(\omega)$ and at least one of the following items be fulfilled:*

- (i) $\lambda > -1$ *and*

$$\rho(p)Q_+ \leq Q_-;$$

- (ii) $\lambda \in \,]-1, 0[$ *and*

$$\|h_0\|_L^{\frac{1}{\lambda+1}} \geq \frac{1}{\lambda+1}\left(\frac{|\lambda|}{\rho(p)\|[p]_+\|_L - \|[p]_-\|_L}\right)^{\frac{|\lambda|}{\lambda+1}} (\rho(p)Q_+ - Q_-);$$

- (iii) $\lambda > 0$ *and*

$$\|h_0\|_L \geq \left(\frac{\omega}{4}\rho(p)Q_+\right)^\lambda (\rho(p)Q_+ - Q_-);$$

- (iv) $\lambda \geq \frac{1}{2}$, $\eta \in [1/\lambda, 2[\,$, $[q]_+^{\frac{1}{2-\eta}} \in L_\omega$, *and*

$$\int_0^\omega \frac{h_0(s)}{|s-a|^{\lambda\eta}}\,\mathrm{d}s = +\infty \quad \textit{pro} \quad a \in [0, \omega[\,.$$

Then the problem (3) *is solvable. Moreover, if $q(t) \leq 0$ for $t \in [0, \omega]$ then the inclusion $p \in \mathcal{V}^-(\omega)$ is necessary for solvability of* (3)*, while if either* (iii) *or* (iv) *holds then the problem* (3) *is uniquely solvable.*

$$u'' = p(t)u + \frac{h_0(t)}{u^\lambda} + q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{4}$$

Theorem. *Let* $p \in \mathcal{V}^-(\omega)$ *and*

$$\lambda > 0, \quad h_0, q \in L_\omega, \quad h_0(t) \geq 0 \quad \textit{pro} \quad t \in [0,\omega], \quad h_0 \not\equiv 0.$$

Let, moreover,

$$Q_- > \rho(p)Q_+$$

and

$$\|h_0\|_L \leq \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \frac{\big(Q_- - \rho(p)Q_+\big)^{\lambda+1}}{\rho(p)\big(\rho(p)\|[p]_+\|_L - \|[p]_-\|_L\big)^\lambda}.$$

Then the problem (4) *is solvable.*