An application of the Kurzweil-Stieltjes integral in financial markets

joint work with Harbir Lamba, Sergey Melnik, Dmitrii Rachinskii, in progress.

Pavel Krejčí

Matematický ústav AV ČR

Workshop on Differential Equations 2014, Malá Morávka

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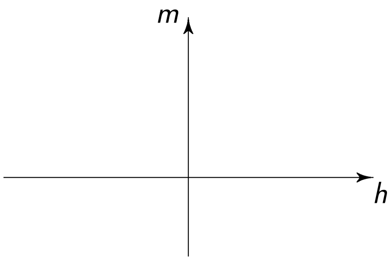
Difference between them:
Solids remember their shape, fluids have no memory.

Madelung's memory laws

- The shape of a monotone magnetization curve does not depend on the rate of change.
- The local shape of a curve starting from a turning point does not depend on the previous history.
- After second turn the curve returns back to its starting point.
- As soon as the minor loop is closed, the process continues as if no turn had taken place.



Erwin Madelung: Über Magnetisierung durch schnell verlaufende Ströme und die Wirkungsweise des Rutherford-Marconischen Magnetdetektors. *Ann. Phys.* **17** (1905), 861–890.

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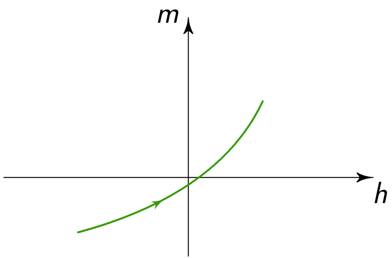

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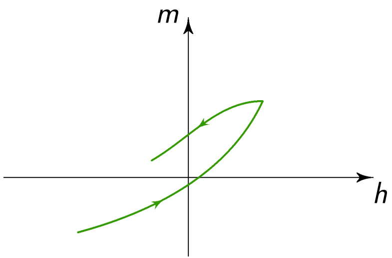

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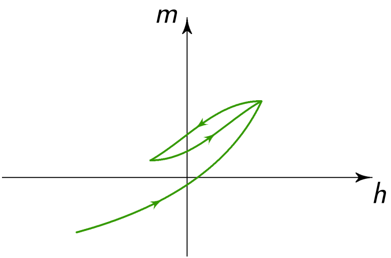

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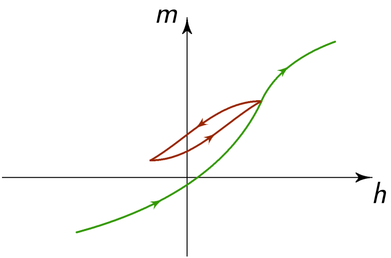

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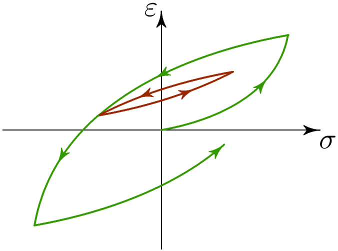
Prandtl's (Berliner's) elastoplastic experiment



σ ... stress

ε ... strain

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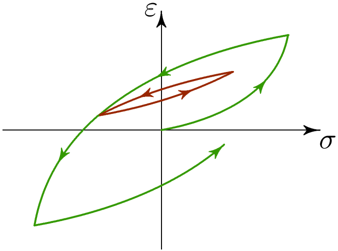

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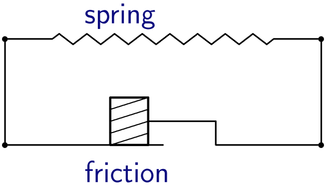
L. Prandtl: Ein Gedankenmodell zur kinetischen Theorie der festen Körper. *Z. Ang. Math. Mech.* **8** (1928), 85–106.

Analogical models



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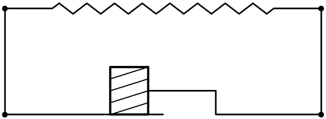
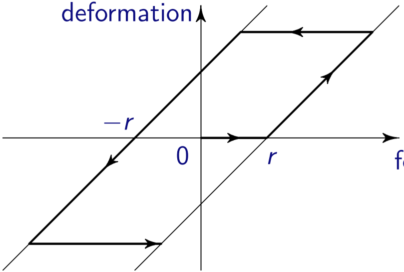


Phase diagram of a parallel combination of one elastic spring and one friction term with yield point r:



Variational inequalities

In a parallel combination of a spring with elasticity modulus E and a friction element with yield point r, the relation between a time dependent stress $\sigma(t)$ and time dependent strain $\varepsilon(t)$ is given by the variational inequality

$$\begin{aligned} &|\sigma(t) - E\varepsilon(t)| \le r && \forall t \in [0, T] \\ &\varepsilon(0) = \varepsilon^0 && \\ &\dot{\varepsilon}(t)(\sigma(t) - E\varepsilon(t) - y) \ge 0 && a.e.\ \forall |y| \le r \end{aligned}$$

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We thus can define a mapping $\mathfrak{p}_r$ called the Prandtl operator which with every input function σ and with every initial condition ε^0 associates the solution $E\varepsilon = \mathfrak{p}_r[\varepsilon^0, \sigma]$. It is Lipschitz continuous in the space of absolutely continuous functions and admits a Lipschitz continuous extension to the space of continuous functions.

Discontinuous processes

Prandtl's variational inequality can be solved also for discontinuous inputs σ, but it is necessary to extend the solution concept.

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A function σ is said to be regulated on $[0, T]$, if at each point $t \in [0, T]$ it admits both one-sided finite limits $\sigma(t+), \sigma(t-)$, with the convention $\sigma(0-) = \sigma(0)$, $\sigma(T+) = \sigma(T)$.

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Let σ be a regulated left continuous function on $[0, T]$. We say that ε is a Kurzweil solution of the Prandtl variational inequality if it has bounded variation on $[0, T]$, and we have

$$
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&|\sigma(t) - E\varepsilon(t)| \leq r \quad \forall t \in [0, T]\\
&\varepsilon(0) = \varepsilon^0\\
&\int_0^T (\sigma(t+) - E\varepsilon(t+) - y(t))\, \mathrm{d}\varepsilon(t) \geq 0\\
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the integral being understood in the Kurzweil sense.

Theorem. *There exists a unique solution* $E\varepsilon = \mathfrak{p}_r[\varepsilon^0, \sigma]$ *in* $BV_L(0, T)$.

Kurzweil integral: Partitions and gauges

For a given interval $[a, b] \subset \mathbb{R}$ we denote by $\mathcal{D}_{a,b}$ the set of *divisions* of the form

$$d = \{t_0, \ldots, t_m\}, \quad a = t_0 < t_1 < \cdots < t_m = b\,.$$

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With a division $d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b}$ we associate a *partition* D by adding *tags* τ_j, that is,

$$D = \{(\tau_j, [t_{j-1}, t_j]) ; j = 1, \ldots, m\}; \quad \tau_j \in [t_{j-1}, t_j] \quad \forall j = 1, \ldots, m.$$

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We define the set

$$\Gamma(a,b) \;:=\; \{\delta : [a,b] \to \mathbb{R}\,;\, \delta(t) > 0 \quad \text{for all} \quad t \in [a,b]\}\,.$$

An element $\delta \in \Gamma(a,b)$ is called a *gauge*. We say that a partition D is *δ-fine*, if for every $j = 1,\ldots,m$ we have

$$\boxed{\tau_j \;\in\; [t_{j-1},t_j] \;\subset\;]\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)[}$$

with $\tau_j = t_{j-1}$ only if $j = 1$ and $\tau_j = t_j$ only if $j = m$.

Kurzweil integral (*Kurzweil* 1957, *Henstock* 1963)

For given functions $f, g : [a, b] \to \mathbb{R}$ and a given partition D we define the Kurzweil sum $K_D(f, g)$ by the formula

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we say that $J \in \mathbb{R}$ is the *Kurzweil integral* of f with respect to g over $[a, b]$ and denote

$$J \;=\; \int_a^b f(t)\,\mathrm{d}g(t)\,,$$

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The Kurzweil integral is linear with respect to both f, g, and additive with respect to the integration domain.

General properties

(i) $\int_a^b f(t)\,\mathrm{d}g(t)$ exists if $f \in G(a,b)$ (= set of regulated functions) and $g \in BV(a,b)$ or vice versa.

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$$\int_a^b f(t)\,\mathrm{d}g(t) + \int_a^b g(t)\,\mathrm{d}f(t) = f(b)\,g(b) - f(a)\,g(a)$$

$$+ \sum_{t\in[a,b]} \Big((f(t) - f(t-))(g(t) - g(t-)) - (f(t+) - f(t))(g(t+) - g(t))\Big).$$

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(iii) If $f_n \in G(a,b)$ and $g_n \in BV(a,b)$ satisfy the conditions $\|f_n - f\|_\infty \to 0\,,\ \|g_n - g\|_\infty \to 0$ for $n \to \infty$ and $\mathrm{Var}_{[a,b]}\, g_n \le C$ independently of n, then

$$\boxed{\lim_{n\to\infty} \int_a^b f_n(t)\,\mathrm{d}g_n(t) \;=\; \int_a^b f(t)\,\mathrm{d}g(t)\,.}$$

Method of construction of the Kurzweil solution to the Prandtl variational inequality

For a piecewise constant input $\sigma(t) = \sigma_k$ for $t \in (t_{k-1}, t_k]$ with an arbitrary choice of the division $\{t_0, \ldots, t_m\}$, the Kurzweil solution can be written explicitly and coincides with the classical Moreau time discrete approximation. In the interval $(t_{k-1}, t_k]$ is the value ε_k determined as the minimizer of the convex energy functional

$$\varepsilon \mapsto \frac{E}{2}|\varepsilon|^2 - \sigma_k\,\varepsilon + r|\varepsilon - \varepsilon_{k-1}|\,.$$

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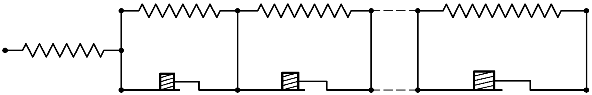
Theorem. *The Prandtl operator* $\mathfrak{p}_r$ *with a fixed initial condition maps the space* $G_L(a,b)$ *into* $BV_L(a,b)$*, and is Lipschitz continuous as mapping* $G_L(a,b) \to G_L(a,b)$*.*

Prandtl-Ishlinskii model



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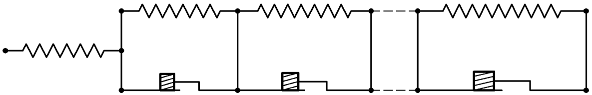

Consider a connection in series of Prandtl's parallel combinations of elastic springs with friction elements, with different yield points $r > 0$ and different elasticity moduli $E(r) > 0$. The total deformation ε is the given by the sum (integral) of individual deformations

$$\varepsilon_r(t) = \frac{1}{E(r)} \mathfrak{p}_r[\varepsilon_r^0, \sigma](t),$$

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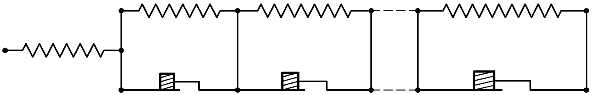

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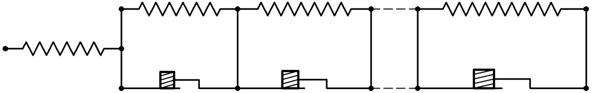

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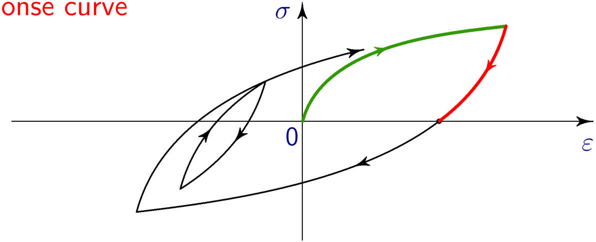
The variable r determines the level (depth) of memory.

Primary response curve



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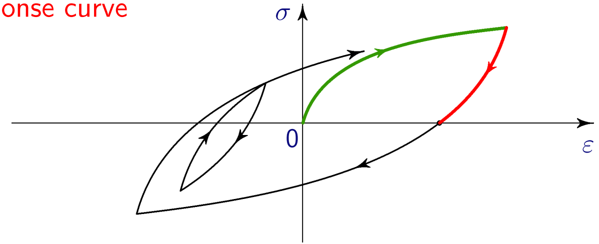

We define the primary response curve

$$g(\sigma) = \frac{r}{E(0)} + \int_0^\sigma (\sigma - \varrho) \frac{1}{E(\varrho)} \, \mathrm{d}\varrho$$

and rewrite the Prandtl-Ishlinskii constitutive law in the form

$$\varepsilon(t) = -\int_0^\infty \frac{\partial}{\partial r} \mathfrak{p}_r[\varepsilon_r^0, \sigma](t) \, \mathrm{d}g(r) \, .$$

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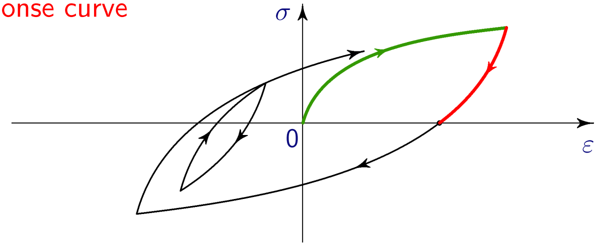

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$$\boxed{\varepsilon(t) = -\int_0^{\infty} \frac{\partial}{\partial r} \mathfrak{p}_r[\varepsilon_r^0, \sigma](t) \, \mathrm{d}g(r) \, .}$$

All secondary branches have the form $g^*(\sigma) = \sigma^* \pm 2g(\frac{1}{2}|\varepsilon - \varepsilon^*|)$.

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The function $r \mapsto \frac{\partial}{\partial r}\mathfrak{p}_r[\varepsilon_r^0, \sigma](t)$ is regulated on every interval $[a, b] \subset (0, \infty)$. The formula

$$\varepsilon(t) = -\int_0^\infty \frac{\partial}{\partial r}\mathfrak{p}_r[\varepsilon_r^0, \sigma](t)\,\mathrm{d}g(r)$$

is therefore meaningful for arbitrary nondecreasing primary response curve g if the integral is interpreted in the Kurzweil sense!

A Substitution Theorem

Let $f : [0, b] \to \mathbb{R}$ be a bounded function and let $f\big|_{[a,b]} \in G(a, b)$ for all $a \in (0, b)$. Let $\varphi : [0, b] \to [0, B]$ be a nondecreasing function, $\varphi(0) = 0$, $\varphi(b) = B$, and let $\psi : [0, B] \to \mathbb{R}$ be a right continuous function with bounded variation. For $s \in [0, B]$ put

$$\varphi^{-1}(s) = \inf\{t \in [0, b] : s \le \varphi(t)\}.$$

Then for all $a \in [0, b)$ we have

$$\int_a^b f(t)\,\mathrm{d}(\psi \circ \varphi)(t) = \int_{\varphi(a)}^{\varphi(b)} f(\varphi^{-1}(s))\,\mathrm{d}\psi(s)\,.$$

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This result enables us to consider inversions and superpositions of Prandtl-Ishlinskii operators also in case of discontinuous primary response curves.

Financial markets

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Consider trading in a time interval $t \in [0, T]$ with a particular commodity. We denote by $p(t) > 0$ a basic price at time t for one unit in a referential currency. It may depend on changing production costs, traffic problems, political situation, etc.

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Let A be the set of *traders* who buy and sell the product. The traders are divided into classes $A_r \subset A$ according to their *trading strategy* parameterized by a number $0 < r < 1$ characterizing their *risk susceptibility*.

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We say that a trader $\alpha \in A$ belongs to the class A_r, if his market strategy is the following:

(a) If α buys the product at time t_0 for the price $q(t_0)$, he keeps it until the relative price decrease with respect to the maximum at times $t > t_0$ attains the value r. The selling time t_1 is thus defined by

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(b) If α sells the product at time t_1 for the price $q(t_1)$, he does not buy it back until the relative price increase with respect to the minimum at times $t > t_1$ attains the value r. The buying time t_2 is thus defined by

$$t_2 = \min\left\{t > t_1 : \frac{q(t)}{\min\{q(\tau) : t_1 \le \tau \le t\}} \ge 1 + r\right\}.$$

Logarithmic prices

We define the *log-prices* $v(t) = \log p(t)$, $w(t) = \log q(t)$ and *logarithmic market sentiment* $\sigma(t) = \log \varrho(t)$. The are related through the equation

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Market sentiment

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All traders from A_r have the same strategy. Hence, all of them simultaneously are or are not in possession of the product. The state of possession will be described by a function $S_r(t)$ which can only take values 1 (traders from A_r possess the product at time t) or 0 (do not possess).

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In other words, we assume that there exists a nonnegative measurable function $\mu(r)$ characterizing the *relative weight of the opinion* of the traders from A_r, and such that

$$\sigma(t) = \int_0^1 \mu(r) S_r(t) \, \mathrm{d}r.$$

Typically, μ has a small compact support in the interval $(0,1)$, $\int_0^1 \mu(r) \, \mathrm{d}r = M$.

Financial markets and Prandtl's variational inequality

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Let p_0 be some smallest admissible trading amount in referential currency unit, and let $v_0 = \log p_0$.
The initial condition is chosen such as if all traders from A_r had sold their assets at some time prior to $t = 0$ for the log-price v_0, that is, $S_r(0-) = 0$, with the next buying log-price $v_0 + r$ for each $r > 0$.

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Let $\mathfrak{p}_r[\lambda(r), \cdot]$ be the solution operator of the Prandtl variational inequality with initial condition $\lambda(r)$. We have the following crucial result:

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Theorem. *The initial memory state $\lambda(r)$ can be chosen in such a way that for each regulated log-price $w(t)$, the possession function $S_r(t)$ defined by the trading strategy (a'), (b') is represented by the formula*

$$S_r(t) = \frac{1}{2}\Big(1 - \frac{\partial}{\partial r}\mathfrak{p}_r[\lambda(r), 2w](t)\Big).$$

Mathematical consequences

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Theorem *The logarithmic market sentiment* $\sigma(t)$ *can be rewritten in terms of the Prandtl-Ishlinskii operator* $\mathcal{P}$ *with primary response curve*

$$\phi(x) = \int_0^x \mu(\xi)\,\mathrm{d}\xi\,.$$

as the sum $\sigma(t) = \frac{1}{2}(M + \mathcal{P}[2w](t))$.

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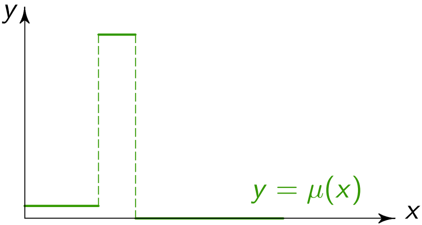

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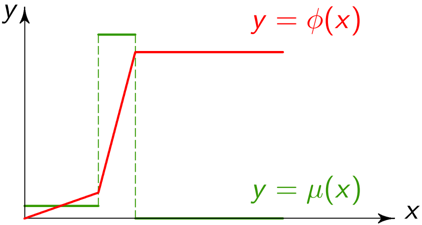

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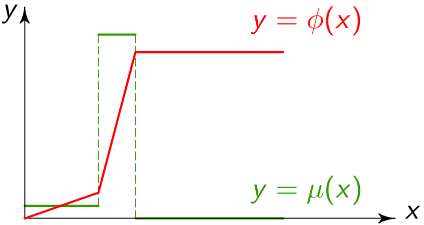

The market log-price $w(t)$ is thus the solution of the equation

$$w(t) = \frac{1}{2}(M + \mathcal{P}[2w](t)) + \kappa v(t).$$

Inversion of Prandtl-Ishlinskii operators

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For each continuous/regulated function $v(t)$, the equation

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admits a unique continuous/regulated solution $w(t) = \frac{1}{2}(I - \mathcal{P})^{-1}[M + 2\kappa v](t)$ if and only if the primary response curve $x - \phi(x)$ of the operator $I - \mathcal{P}$ is increasing, that is,

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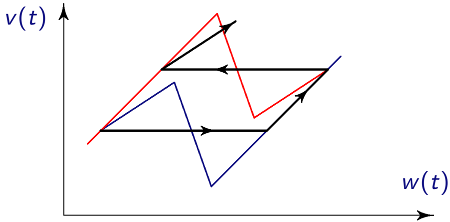

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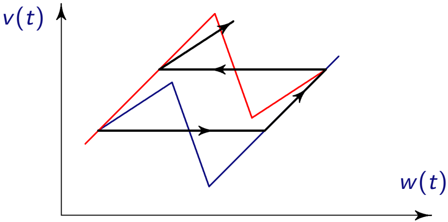

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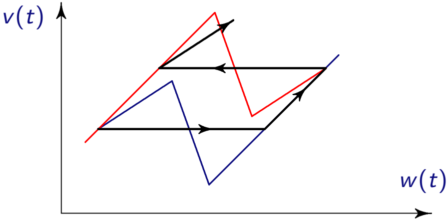

Backward jump
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Consequence. Financial bubbles may arise if a small group of strong traders has a dominant influence on the market sentiment.

More complex models

More complex models

Consider the case that the market price is not the same for all traders. Different traders may have different market sentiments. Let, for example, $[0, 1]$ be divided into disjoint subsets $R_1, \dots, R_n$, let $\sigma_i(t)$ be the market sentiment of the traders from A_r for $r \in R_i$, and let $w_i(t)$ be their log-price. We have

$$\sigma_i(t) = \int_{R_i} \mu_i(r) S_r(t) \, \mathrm{d}r =: M_i + \mathcal{P}_i[w_i] \, .$$

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If w_i is influenced also by market sentiments of other traders, we have

$$w_i(t) = \sum_{j=1}^{n} a_{ij} \sigma_j(t) + \kappa_i v(t)$$

with interaction matrix $\mathbf{A} = (a_{ij})$. Log-prices $w_i(t)$ satisfy

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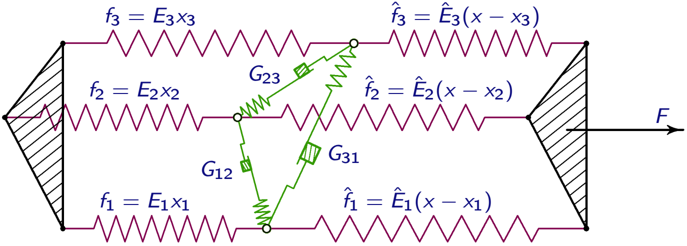
In such a way, we can model even more complex behavior of the market: local discontinuities, loss of memory, etc.

Mechanical three-trader model



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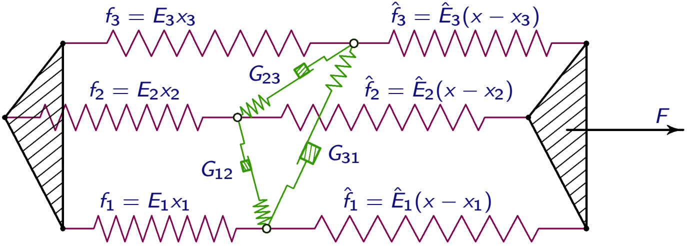

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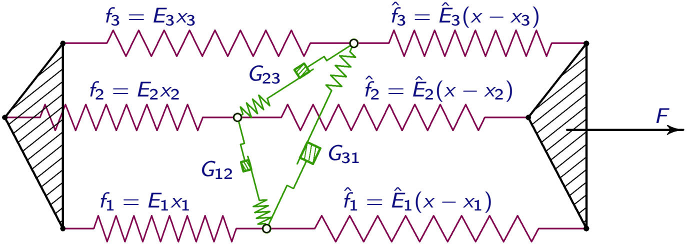

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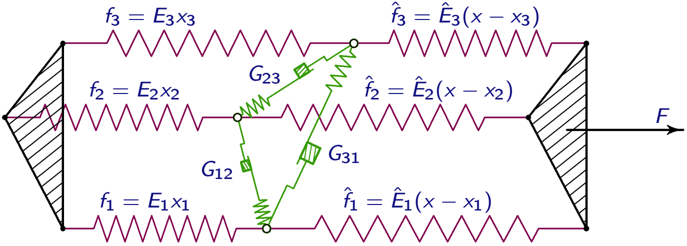

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Amazingly, this trivial construction can model different types of singularities on the global market!

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- The concept of the Kurzweil-Stieltjes integration with respect to both the time and the memory variable plays a crucial role.