GODEs: recent results and some applications

Rodolfo Collegari, Márcia Federson and Miguel Frasson

Universidade de São Paulo, Brazil

The Kurzweil integral

Example:

Consider $F : [0,1] \to \mathbb{R}$ given by

$$F(t) = \begin{cases} t^2 \sin \dfrac{1}{t^2}, & 0 < t \leq 1, \\ 0, & t = 0. \end{cases}$$

Assertion:

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- f **IS NOT Lebesgue integrable.**
- f **IS Kurzweil-Henstock integrable ($=$ Perron integrable)**

Tagged Divisions

A **tagged division** of $[a, b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, with

$$a = s_0 \leq s_1 \leq \ldots \leq s_k = b \quad \text{and} \quad \tau_i \in [s_{i-1}, s_i]\,,$$

for $i = 1, 2, \ldots, |D|$.

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Gauges

Given a function $\delta : [a, b] \to (0, +\infty)$ (called **gauge** of $[a, b]$), a tagged division $D = (\tau_i, [s_{i-1}, s_i])$ is δ-**fine**, whenever

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)),$$

for $i = 1, 2, \ldots, |D|$.

The Kurzweil Integral

A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is **Kurzweil integrable** over $[a, b]$, if $\exists!\ I \in X$ such that $\forall\ \varepsilon > 0$, $\exists$ a gauge δ of $[a, b]$ such that $\forall\ \delta$-fina tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$,

$$\left\| \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon.$$

In this case, $I = \int_a^b DU(\tau, t)$.

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Cousin Lemma

Given a gauge δ of $[a, b]$, there exists a δ-fine tagged division of $[a, b]$.

The Perron-Stieltjes integral

Let X be a Banach space and let $F\colon [a,b] \to L(X)$ and $g\colon [a,b] \to X$ be s.t.

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$$\int_a^b DU(\tau, t) = \int_a^b D[F(t)g(\tau)]$$

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can be rewritten as

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The Kurzweil-Cauchy Integral

A function $U\colon [a,b] \times [a,b] \to X$ is **Kurzweil-Cauchy integrable** over $[a,b]$, if $\exists\ I \in X$ s.t. $\forall\ \epsilon > 0$, $\exists$ a left continuous gauge δ of $[a,b]$ s.t. for every δ-fine tagged division $D = (\tau_i, [s_{i-1}, s_i])$ of $[a,b]$,

$$\left\| \sum_{i=1}^{|D|} [U(\tau_i, t_i) - U(\tau_i, t_{i-1})] - I \right\| < \epsilon.$$

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Cousin Lemma

Given a left continuous gauge of δ de $[a,b]$, there exists a δ-fine left tagged division of $[a,b]$.

Regulated Functions

A function $f\colon [a,b] \to X$ is **regulated**, whenever the lateral limits

$$\lim_{s\to t^-} f(s), \quad t \in (a,b], \quad \text{and} \quad \lim_{s\to t^+} f(s), \quad t \in [a,b)$$

exist. We write $f \in G([a,b],X)$.

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- Any $f \in G([a, b], X)$ can be approximated by step functions.
- $G^+([a, b], X) = \{f \in G([a, b], X);\ f \text{ is left continuous}\}$.

Semivariation

Let $F\colon [a,b] \to L(X)$, $D = \{t_0, t_1, \ldots, t_{|D|}\}$ be a division of $[a,b]$.

We define

$$SV(F,D) = \sup\left\{\left\|\sum_{i=1}^{|D|}[F(t_i) - F(t_{i-1})]x_i\right\|, x_i \in X, \|x_i\| \leq 1\right\}.$$

Then

$$\mathrm{(s)var}_{[a,b]}F = \sup_{D\in\mathcal{D}[a,b]} SV(F,D)$$

is the **semivariation** of F in $[a,b]$.

Functions of Bounded Semivariation

A function $F\colon [a,b] \to L(X)$ is of **bounded semivariation** in $[a,b]$, whenever

$$(\mathrm{s})\mathrm{var}_{[a,b]} F < \infty.$$

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We have:

- $SV([a, b], L(X))$ with the norm

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for $c \in [a, b]$, is a Banach space.

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  for $c \in [a,b]$, is a Banach space.
- $BV([a,b], L(X)) \subset SV([a,b], L(X))$.

Lemma

Let $\alpha \in SV_b([a,b], L(X))$, $\tau \in [a,b)$ and $x \in X$. Then

$$\int_a^b d[\alpha(t)]\chi_{[\tau,b)}(t)x = -\alpha(\tau)x,$$

where the integral is in the sense of **Kurzweil-Cauchy**.

Riesz Representation Theorem

An operator $F\colon G^+([a,b],X) \to X$ is a linear bounded operator if and only if $\exists\ \alpha \in SV_b([a,b],L(X))$ s.t. $\forall\ f \in G^+([a,b],X)$,

$$F(f) = \int_a^b d[\alpha(t)]f(t),$$

where the integral is in the sense of **Kurzweil-Cauchy**.

Generalized ODEs

Let X be a Banach space, $\mathcal{O} \subset X$ be open $[\alpha, \beta] \subset [a, +\infty)$ and $\Omega = \mathcal{O} \times [\alpha, \beta]$.

Definition

A function $x : [\alpha, \beta] \to X$ is **a solution** on $[\alpha, \beta]$ of the GODE

$$\frac{dx}{d\tau} = DF(x, t),$$

whenever $(x(t), t) \in \Omega \ \forall \ t \in [\alpha, \beta]$ and

$$x(v) = x(\gamma) + \int_{\gamma}^{v} DF(x(\tau), t), \qquad \gamma, \ v \in [\alpha, \beta].$$

Example

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$$\int_{s_1}^{s_2} DG(x(\tau),t) = \int_{s_1}^{s_2} Dr(t) = r(s_2) - r(s_1).$$

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Moreover, $x\colon [0,1] \to \mathbb{R}$ defined by

$$x(s) = r(s), \quad s \in [0,1]$$

is a solution of the GODE

$$\frac{dx}{d\tau} = DG(x,t) = Dr(t).$$

Linear GODEs

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- $A : [a, b] \to L(X)$ is BV in $[a, b]$.

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Linear GODEs

A function $x : [a, b] \to X$ is a solution of the linear GODE

$$\frac{dx}{d\tau} = DG(x, t) = D[A(t)x]$$

on $[a, b]$, with initial condition $x(t_0) = \widetilde{x}$, if

$$x(t) = \widetilde{x} + \int_{t_0}^{t} d[A(r)]x(r), \quad t \in [a, b].$$

Suppose

$$
\begin{aligned}
&(I + [A(t+) - A(t)])^{-1} = [I + \Delta^{+}A(t)]^{-1} \in L(X), \quad t \in [a, b) \\
&(I - [A(t) - A(t-)])^{-1} = [I - \Delta^{-}A(t)]^{-1} \in L(X), \quad t \in (a, b]
\end{aligned}
\tag{Δ}
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Existence and Uniqueness - Š. Schwabik

Let $A \in BV([a, b], L(X))$ fulfill (Δ). Then the initial value problem

$$\begin{cases} \dfrac{dx}{d\tau} = D[A(t)x] \\ x(t_0) = \widetilde{x} \end{cases}$$

admits a unique solution on $[a, b]$.

Fundamental Operator

Let $A \in BV([a,b], L(X))$ fulfill (Δ). Then $\exists$ a unique operator $U\colon [a,b] \times [a,b] \to L(X)$, called the fundamental operator, s.t.

$$U(t,s) = I + \int_s^t d[A(r)]U(r,s), \quad t,s \in [a,b],$$

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Then the linear GODE

$$\begin{cases} \dfrac{dx}{d\tau} = D[A(t)x] \\ x(s) = \widetilde{x} \end{cases}$$

has solution given by the relation

$$x(t) = U(t,s)\widetilde{x}, \quad t \in [a,b].$$

Dirichlet's Formula

Let $A, K \in BV([a,b], L(X))$, A fulfill (Δ) and $U\colon [a,b] \times [a,b] \to L(X)$ be the fundamental operator of the linear GODE

$$\begin{cases} \dfrac{dx}{d\tau} = DG(x,t) = D[A(t)x] \\ x(t_0) = \widetilde{x}. \end{cases}$$

Then

$$\int_{t_0}^{t} d[K(r)]\Big(\int_{t_0}^{r} d_s[U(r,s)]\varphi(s)\Big)$$

$$= \int_{t_0}^{t} d[K(s)]\varphi(s) + \int_{t_0}^{t} d_s\Big[\int_{s}^{t} d_r[K(r)]U(r,s)\Big]\varphi(s)$$

$\forall\ t_0, t \in [a,b]$ and $\forall\ \varphi \in G([a,b], X)$.

Corollary

Let $A \in BV([a,b], L(X))$ fulfill (Δ) and $U\colon [a,b] \times [a,b] \to L(X)$ be the fundamental operator of the linear GODE

$$\begin{cases} \dfrac{dx}{d\tau} = DG(x,t) = D[A(t)x] \\ x(t_0) = \widetilde{x}. \end{cases}$$

Then

$$\int_{t_0}^{t} d[A(r)]\Big(\int_{t_0}^{r} d_s[U(r,s)]\varphi(s)\Big) = \int_{t_0}^{t} d[A(s)]\varphi(s) + \int_{t_0}^{t} d_s[U(t,s)]\varphi(s)$$

$\forall\ t_0, t \in [a,b]$ and $\forall\ \varphi \in G([a,b], X)$.

Variation of Constants Formula

Let $A \in BV([a,b], L(X))$, $F\colon X \times [a,b] \to L(X)$, $[\alpha,\beta] \subseteq [a,b]$, $t_0 \in [\alpha,\beta]$. If $x \in G([\alpha,\beta], X)$ is a solution of the perturbed problem

$$\begin{cases} \dfrac{dx}{d\tau} = D[A(t)x + F(x,t)] \\ x(t_0) = \widetilde{x} \end{cases}$$

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then x can be rewritten as

$$x(t) = U(t,t_0)\widetilde{x} + \int_{t_0}^{t} DF(x(\tau),s) - \int_{t_0}^{t} d_\sigma[U(t,\sigma)] \left(\int_{t_0}^{\sigma} DF(x(\tau),s) \right),$$

$\forall\ t \in [\alpha,\beta]$,

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$\forall\ t \in [\alpha,\beta]$, where $U\colon [a,b] \times [a,b] \to L(X)$ is the fundamental operator of the linear GODE

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Linear FDEs as Linear GODEs

Let $r, \sigma > 0$ and $t_0 \in \mathbb{R}$. Given $y\colon \mathbb{R} \to \mathbb{R}^n$, let $y_t\colon [-r, 0] \to \mathbb{R}^n$ be given by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r, 0], \quad t \in \mathbb{R}.$$

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Linear FDEs

$$\begin{cases} \dot{y} = \mathcal{L}(t) y_t \\ y_{t_0} = \phi \end{cases}$$

where $\phi \in G([-r, 0], \mathbb{R}^n)$ and $\mathcal{L}(t) \colon G([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is linear and bounded $\forall\ t \in [t_0, t_0 + \sigma]$.

Suppose

(Int) $\forall\ y \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, the mapping

$$t \mapsto \mathcal{L}(t) y_t$$

is Kurzweil integrable over $[t_0, t_0 + \sigma]$.

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(Lip) $\exists\ M\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}$ Lebesgue integrable s.t.

$$\left| \int_{s_1}^{s_2} \mathcal{L}(s)(y_s - z_s)ds \right| \leq \int_{s_1}^{s_2} M(s)\|y_s - z_s\|ds$$

$\forall\ s_1, s_2 \in [t_0 - r, t_0 + \sigma]$ and $y, z \in G([t_0, t_0 + \sigma], \mathbb{R}^n)$.

Consider

$$[A(t)y](\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \displaystyle\int_{t_0}^{\vartheta} \mathcal{L}(s) y_s \, ds, & t_0 \le \vartheta \le t \le t_0 + \sigma, \\ \displaystyle\int_{t_0}^{t} \mathcal{L}(s) y_s \, ds, & t_0 \le t \le \vartheta \le t_0 + \sigma. \end{cases}$$

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and

$$\widetilde{x}(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \le \vartheta \le t_0, \\ \phi(0) = x(t_0)(t_0), & t_0 \le \vartheta \le t_0 + \sigma. \end{cases}$$

Assume that, $\forall\ t \in [t_0, t_0 + \sigma]$ and $\forall\ y \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$,

- $A(t)y$ is continuous on $[t_0 - r, t_0 + \sigma]$.
- $A(t)$ is a linear operator.
- $\|A(t)y\| \leq \|y\| \int_{t_0}^{t} M(s)ds$.

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Assume further that

- $\|A(s_2) - A(s_1)\| \leq \int_{s_1}^{s_2} M(s)ds$, $\forall\ s_1, s_2 \in [t_0, t_0 + \sigma]$, $s_1 \leq s_2$.

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Hence

$$A\colon [t_0, t_0+\sigma] \to L(G([t_0 - r, t_0+\sigma], \mathbb{R}^n))$$

is BV in $[t_0, t_0+\sigma]$ and

$$\operatorname{var}_{t_0}^{t_0+\sigma}(A) \leq \int_{t_0}^{t_0+\sigma} M(s)ds.$$

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where

$$\widetilde{x}(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \leq \vartheta \leq t_0, \\ \phi(0) = x(t_0)(t_0), & t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

Correspondence of Equations

Let $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ be a solution on $[t_0, t_0 + \sigma]$ of the linear FDE

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Then $x\colon [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ is a solution of the linear GODE

$$\begin{cases} \dfrac{dx}{d\tau} = DG(x, t) = D[A(t)x] \\ x(t_0) = \widetilde{x}. \end{cases}$$

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Variation of Constants Formula for FDEs

Solution Operator for Linear FDEs

Let $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ be a solution of the linear FDE

$$\dot{y} = \mathcal{L}(t)y_t$$

with initial condition $y_s = \phi$.

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with initial condition $y_s = \phi$. For $t, s \in [t_0, t_0 + \sigma]$, $t \geq s$, the operator $T(t,s)\colon G([-r,0], \mathbb{R}^n) \to G([-r,0], \mathbb{R}^n)$ defined by

$$T(t,s)\phi = y_t, \quad t,\, s \in [t_0, t_0 + \sigma],\ t \geq s,$$

is called **solution operator** of the linear FDE .

Let $g\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ be a given function. For $s \in [t_0, t_0 + \sigma]$, let $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ be a solution of the linear FDE

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with initial condition $y_s = g_s$.

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with initial condition $y_s = g_s$. For $t \in [t_0, t_0 + \sigma]$, $t \geq s$, define

$$T(t,s)\colon G([t_0 - r, t_0 + \sigma], \mathbb{R}^n) \to G([-r, 0], \mathbb{R}^n)$$

by

$$T(t,s)g = y_t.$$

For the next two lemmas and the main theorem, consider

$$h(w)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(y_u, u)du, & t_0 \leq \vartheta \leq w, \\ \displaystyle\int_{t_0}^{w} f(y_u, u)du, & w \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

where $\vartheta \in [t_0 - r, t_0 + \sigma]$, $w \in [t_0, t_0 + \sigma]$.

Lemma

Let y and x be, respectively, the corresponding solutions of the perturbed problems

$$\begin{cases} \dot{y} = \mathcal{L}(t)y_t + f(y_t, t), \\ y_{t_0} = \phi, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{d\tau} = D[A(t)x + F(x,t)], \\ x(t_0) = \widetilde{x}, \end{cases}$$

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Let $T(t,s)$ and $U(t,s)$ be, respectively, the solution and fundamental operators of

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Then, for $t_0 \leq w \leq s \leq t \leq t_0 + \sigma$, we have

$$U(t,s)\left(\int_{t_0}^{w} DF(x(\tau),u)\right)(t) = T(t,s)(h(w)_s)(0).$$

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Then, for $t_0 \leq t \leq t_0 + \sigma$, we have

$$\int_{t_0}^{t} d_s[U(t,s)] \left(\int_{t_0}^{s} DF(x(\tau), u) \right)(t) = \int_{t_0}^{t} d_s[T(t,s)]h(s)(0),$$

where the integrals are in the sense of Kurzweil-Cauchy.

VCF for linear FDEs

Let y be a solution of the perturbed problem

$$\begin{cases} \dot{y} = \mathcal{L}(t)y_t + f(y_t, t) \\ y_{t_0} = \phi \end{cases}$$

where $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ and $\mathcal{L}(t)\colon G([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ satisfy (Int) and (Lip) with integrals in the sense of Kurzweil-Cauchy.

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$$y(t) = T(t,t_0)\phi(0) + \int_{t_0}^{t} f(y_u,u)du - \int_{t_0}^{t} d_s[T(t,s)]h(s)(0).$$

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Prague, 2010

Prague, 2010

São Carlos, 2011

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Happy birthday, Milan!

Thanks for your attention!