Landesman-Lazer condition for impulses and external forces in first and second order problems

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$$\begin{cases} u'(t) + \cos t \cdot u(t) = f(t), \quad t \in [0, 2\pi] \\ u(0) = u(2\pi) \end{cases}$$

$$\int_0^{2\pi} f(t) e^{\sin t} \, dt = 0$$

impuls condition

$$u(\pi_+) = u(\pi_-) - \arctan u(\pi_-)$$

$$-\frac{\pi}{2} < \int_0^{2\pi} f(t) e^{\sin t} \, dt < \frac{\pi}{2}$$

$\exists$ results for a wider class of forcing terms

$$\begin{cases} u'(t) + a(t)u(t) = f(t, u(t)), & t \in [0, T] \\ u(0) = u(T) \end{cases}$$

$$\int_0^T a(t)\,dt = 0 \quad (\text{resonance case})$$

$$t_i \in (0, T), \quad i = 1, \dots, p$$

$$u(t_j^+) = u(t_j^-) - I_j(u(t_j^-))$$

$$f_\pm(t) := \lim_{s \to \pm\infty} f(t, s) \qquad I_j(\pm\infty) := \lim_{s \to \pm\infty} I_j(s)$$

$$\int_0^T f_+(t) e^{\int_0^t a(\tau)d\tau} dt \quad \begin{matrix} < \\ > \end{matrix} \quad \sum_{j=1}^{p} I_j(+\infty) e^{\int_0^{t_j} a(\tau)d\tau}$$

$$\int_0^T f_-(t) e^{\int_0^t a(\tau)d\tau} dt \quad \begin{matrix} > \\ < \end{matrix} \quad \sum_{j=1}^{p} I_j(-\infty) e^{\int_0^{t_j} a(\tau)d\tau}$$

$$\begin{cases} u'(t) + a(t)u(t) = f(t), \quad t \in [0,T], \\ u(0) = u(T) \end{cases}$$

$$\sum_{j=1}^{p} I_j(-\infty)\, e^{\int_0^{t_j} a(\tau)\,d\tau} < \int_0^T f(t)\, e^{\int_0^t a(\tau)\,d\tau}\, dt < \sum_{j=1}^{p} I_j(+\infty)\, e^{\int_0^{t_j} a(\tau)\,d\tau}$$

If $\forall s \in \mathbb{R}$ $\forall j = 1, \dots, p$:

$$I_j(-\infty) < I_j(s) < I_j(+\infty)$$

then above condition is also necessary.

$$a \equiv 0 \Rightarrow \sum_{j=1}^{p} I_j(-\infty) < \int_0^T f(t)\,dt < \sum_{j=1}^{p} I_j(+\infty)$$

Functional-analytic approach:

$$\begin{cases} u'(t) + (a(t)+\delta)u(t) = \delta u(t) + f(t,u(t)) \\ u(0) = u(T),\ u(t_j^+) = u(t_j^-) - I_j(u(t_j^-)),\ j = 1,\dots,p \end{cases}$$

$$u(t) = \underbrace{\int_0^T G(t,s)\big(\delta u(s) + f(s,u(s))\big)\,ds - \sum_{j=1}^{p} G(t,t_j) I_j(u(t_j))}_{F(u)(t)\ :\ X \to X}$$

(compact)

Operator representation:

$$u = F(u)$$

("fixed point equation")

Schaeffer Theorem (degree theory) $\Rightarrow \exists$ sol'n

if

$\exists K>0 \quad \forall \lambda \in [0,1] \quad \forall u \in X: \quad u = \lambda F(u)$

$\Rightarrow \quad \|u\| < K$

proof via contradiction: $\exists u_n \in X, \lambda_n \in [0,1]$,
$\|u_n\| \geq n \quad : \quad u_n = \lambda_n F(u_n)$

$v_n := \dfrac{u_n}{\|u_n\|} \qquad v_n \longrightarrow v \neq 0, \ \lambda_n \longrightarrow \lambda$

$$\begin{cases} v'(t) + a(t) v(t) + \delta v(t) = \lambda \delta v(t), \\ v(0) = v(T) \end{cases}$$

$\Rightarrow \lambda = 1$ (otherwise only trivial sol'n)

$$\Rightarrow\quad v(t) = \pm \frac{e^{-\int_0^t a(\tau)\,d\tau}}{\max\limits_{s\in[0,T]} e^{-\int_0^s a(\tau)\,d\tau}} \quad \text{solves}$$

$$\begin{cases} v'(t) + a(t)v(t) = 0, & t\in(0,T),\\ v(0) = v(T) \end{cases}$$

If $v_n \longrightarrow v > 0 \Rightarrow \mu_n(t) \rightarrow +\infty$ on $[0,T]$ and after integration by parts $\mu_n = \lambda_n F(u_n)$ reads as follows:

$$\sum_{j=1}^{p} I_j(u_n(t_j))\, e^{\int_0^{t_j} a(\tau)\,d\tau} + \delta(1-\lambda_n)\int_0^T \mu_n(t)\, e^{\int_0^t a(\tau)\,d\tau}\,dt =$$

$$= \int_0^T f(t, u_n(t))\, e^{\int_0^t a(\tau)\,d\tau}$$

passing to the limit for $n \to \infty$:

$$\int_0^T f_+(t)\, e^{\int_0^t a(\tau)\,d\tau}\, dt \ge \sum_{j=1}^{p} I_j(+\infty)\, e^{\int_0^{t_j} a(\tau)\,d\tau}$$

↯

with opposite inequality in L.-L. type condition

If $v_n \to v < 0$ $\Rightarrow$ the other condition applies

$$\begin{cases} -u''(x) - u(x) = f(x), & x \in (0,\pi), \\ u(0) = u(\pi) = 0 \end{cases}$$

$$\int_0^{\pi} f(x) \sin x \, dx = 0$$

impuls condition

$$u'\left(\tfrac{\pi}{2}+\right) = u'\left(\tfrac{\pi}{2}-\right) + \arctan u\left(\tfrac{\pi}{2}-\right)$$

$$-\frac{\pi}{2} < \int_0^{\pi} f(x) \sin x \, dx < \frac{\pi}{2}$$

$\exists$ results for a wider class of forcing terms

$$\begin{cases} -u''(x) - u(x) + \arctan u(x) = f(x), \; x \in (0,\pi), \\ u(0) = u(\pi) = 0 \end{cases}$$

$$-\pi < \int_0^\pi f(x) \sin x \, dx < \pi$$

$$u'\left(\tfrac{\pi}{2}+\right) = u'\left(\tfrac{\pi}{2}-\right) \underset{(-)}{+} \arctan u\left(\tfrac{\pi}{2}-\right)$$

$$\underset{\left(-\frac{\pi}{2}\right)}{-\frac{3\pi}{2}} < \int_0^\pi f(x) \sin x \, dx < \underset{\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2}}$$

$$\begin{cases} -u''(x) - n^2 u(x) + g(u(x)) = f(x), \quad x \in (0,\pi), \\ u(0) = u(\pi) = 0 \end{cases}$$

$$t_j \in (0,\pi) \; : \quad u'(t_j^+) = u'(t_j^-) + I_j(u(t_j^-))$$

$$\lim_{s \to \pm\infty} g(s) = g(\pm\infty) \qquad \lim_{s \to \pm\infty} I_j(s) = I_j(\pm\infty)$$

$$g(-\infty)\int_0^\pi (\sin nx)^+ dx - g(+\infty)\int_0^\pi (\sin nx)^- dx + \sum_{j=1}^p I_j(-\infty)(\sin n t_j)^+$$

$$+ \sum_{j=1}^p I_j(+\infty)(\sin n t_j)^- < \int_0^\pi f(x) \sin nx \, dx <$$

$$g(+\infty)\int_0^\pi (\sin nx)^+ dx - g(-\infty)\int_0^\pi (\sin nx)^- dx + \sum_{j=1}^p I_j(+\infty)(\sin n t_j)^+$$

$$+ \sum_{j=1}^p I_j(-\infty)(\sin n t_j)^-$$

$$\begin{cases} -\left(|u'(x)|^{p-2}u'(x)\right)' - \lambda |u(x)|^{p-2}u(x) = f(x), \; x \in (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

① $\lambda \neq$ eigenvalue $\Rightarrow \forall f \; \exists$ sol'n

$\lambda \leq 0 \Rightarrow \forall f \; \exists!$ sol'n

$\lambda > 0, \; p \neq 2 \Rightarrow \exists f$ such that uniqueness fails!

② $\lambda =$ eigenvalue

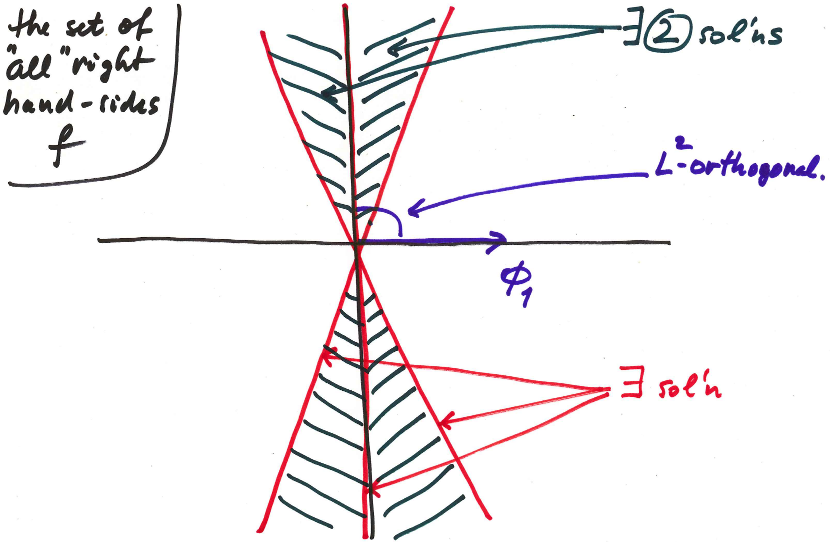
only $\lambda = \lambda_1$ is well understood if $p \neq 2$

$\lambda_1 = (p-1)\pi_p^p$

$\phi_1(x) = \sin_p \pi_p x$

$\int_0^1 f(x)\phi_1(x)\,dx = 0$

sufficient but not necessary condition for the existence


the set of "all" right hand-sides f
∃ ② sol'ns
L^2-orthogonal.
Φ_1
∃ sol'n

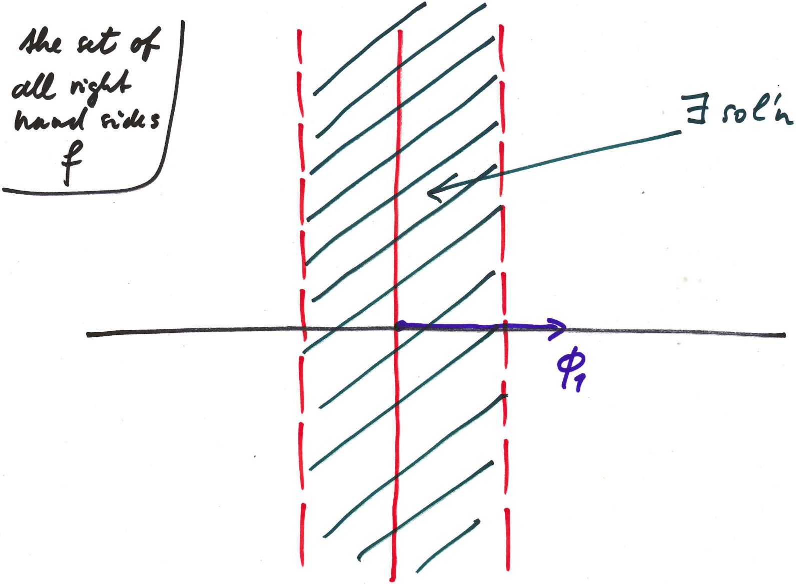
$$\begin{cases} -\left(|u'(x)|^{p-2}u'(x)\right)' - \lambda_1 |u(x)|^{p-2}u(x) = f(x), \ x \in (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

impuls condition:

$$|u'(\tfrac{1}{2}+)|^{p-2}u'(\tfrac{1}{2}+) = |u'(\tfrac{1}{2}-)|^{p-2}u'(\tfrac{1}{2}-) + \arctan u(\tfrac{1}{2}-)$$

$$-\frac{\pi}{2(p-1)} < \int_0^1 f(x)\,\phi_1(x)\,dx < \frac{\pi}{2(p-1)}$$

$$\left(\sin_p \frac{\pi_p}{2} = \frac{1}{p-1} \right)$$


the set of all right hand sides f
∃ sol'n
φ1

$$\begin{cases} -\left(|u'(x)|^{p-2}u'(x)\right)' - \lambda_n |u(x)|^{p-2}u(x) = f(x), \; x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$$t_i \in (0,1) \; : \; |u'(t_j^+)|^{p-2}u'(t_j^+) = |u'(t_j^-)|^{p-2}u'(t_j^-) + I_j(u(t_j^-))$$

$$\sum_{j=1}^{p} I_j(-\infty)\,\phi_n^+(t_j) + \sum_{j=1}^{p} I_j(+\infty)\,\phi_n^-(t_j) <$$

$$< \int_0^1 f(x)\,\phi_n(x)\,dx < \sum_{j=1}^{p} I_j(+\infty)\,\phi_n^+(t_j) + \sum_{j=1}^{p} I_j(-\infty)\,\phi_n^-(t_j)$$

Děkuji za pozornost!