Half-linear Euler type differential equation with periodic coefficients

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Malá Morávka, 29.3.2014

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Half-linear differential equations

Differential equation with the one-dimensional p-Laplacian

$$(HL) \qquad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x,\ p > 1,$$

r, c continuous functions, $r(t) > 0$,

Special case $p = 2$

$$(SL) \qquad (r(t)x')' + c(t)x = 0.$$

linear Sturm-Liouville differential equation

Why 1/2-linear equations

Motivation

- Partial differential equations with N-dimensional p-Laplacian

$$\Delta_p u + c(x)\Phi(u) = 0, \quad \Delta_p u = \operatorname{div}\left(||\nabla u||^{p-2}\nabla u\right)$$

with spherically symmetric potential c, i.e., $c(x) = c(\|x\|)$, can be reduced to (HL).

If $\|x\| = t$ then

$$\Delta_p u(x) = t^{1-N}\left(t^{N-1}\Phi(u'(t))\right)'$$

- Non-Newtonian fluid theory, models in glaceology,...

- Equation (HL) is the Euler-Lagrange equation of the *p*-degree functional

$$\mathcal{F}(y;a,b)=\int_a^b \left[r(t)|y'(t)|^p - c(t)|y(t)|^p\right] dt$$

- Extension of the results for the *linear* equation (special case $p=2$ in (HL))

$$(SL) \qquad (r(t)x')' + c(t)x = 0.$$

  to (HL).
- Pioneering works: I. Bihari (1963, 1968), Á. Elbert (1979), D. Mirzov (1976).

Differences between linear and half-linear

Differences between linear and half-linear.

- The solution space is only homogeneous, but not additive $\Longrightarrow$ half-linear equations.

▷ No half-linear analogue of the linear transformation identity

$$h\left[(rx')' + cx\right] \stackrel{x=hy}{\rightarrow} = (Ry')' + Cy,$$

$$R = rh^2, \quad C = h\left[(rh')' + ch\right]$$

- No reduction of order formula (D'Alembert formula) : $p = 2$ and $x(t) \neq 0$ is a solution of (SL) $\implies$

$$\tilde{x}(t) = x(t) \int^t \frac{ds}{r(s)x^2(s)}$$

  is also a solution of (SL).

- No problems with the existence, uniqueness and continuability of solutions, no singular solutions, in contrast to the Emden-Fowler differential equation (which has not homogeneity property!)

$$x'' + c(t)|x|^{p-2}x = 0,$$

  where no uniqueness and no continuability is guaranteed (singular solution of the first and second kind).

Half-linear trigonometric functions

Half-linear trigonometric functions.

- Denote by $S = S(t)$ the solution of

$$\left(\Phi(x')\right)' + (p-1)\Phi(x) = 0, \quad x(0) = 0,\ x'(0) = 1.$$

Multiplying by S' and using the initial condition

$$|S(t)|^p + |S'(t)|^p = 1 \quad \Longrightarrow \quad S' = \sqrt[p]{1 - S^p}$$

for $t > 0$ small. Further denote

$$\pi_p := 2\int_0^1 (1 - s^p)^{-\frac{1}{p}}\, ds = \frac{2}{p} B(1/p, 1/q) = \frac{2\pi}{p \sin \frac{\pi}{p}}$$

and let

$$\begin{aligned} \sin_p t &:= \text{“}2\pi_p \text{ periodic odd continuation of } S(t)\text{”}, \\ \cos_p t &:= (\sin_p t)'. \end{aligned}$$

Half-linear Prüfer transformation

- Half-linear Prüfer transformation

$$x(t) = \rho(t) \sin_p \varphi(t), \quad r^{q-1}(t) x'(t) = \rho(t) \cos_p \varphi(t),$$

$$\varphi' = r^{1-q}(t) |\cos_p \varphi|^p + \frac{c(t)}{p-1} |\sin_p \varphi|^p,$$

$$\rho' = \Phi(\sin_p \varphi(t)) \cos_p \varphi(t) \left[r^{1-q}(t) - \frac{c(t)}{p-1} \right] \rho.$$

- The right-hand side of the equation for φ is Lipschitzian with respect to φ (and does not contain ρ) $\Longrightarrow$ existence, uniqueness, and continuability for φ, ρ $\Longrightarrow$ existence, uniqueness and continuability for (HL).

Oscillation theory

- Linear Sturmian separation and comparison theory extends (almost) verbatim to (HL).

▷ Half-linear Riccati equation for $w = r\Phi(x'/x)$:

$$(RE) \qquad w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad \frac{1}{p} + \frac{1}{q} = 1$$

▷ Associated energy functional

$$(F) \qquad \mathcal{F}(y) = \int_a^b [r(t)|y'|^p - c(t)|y|^p]\, dt.$$

Roundabout theorem

The following statements are equivalent:

- (HL) is disconjugate on $[a, b]$, i.e., every nontrivial solution has at most one zero in $[a, b]$.

- There exists a solution of the Riccati equation

$$w' + c(t) + (p - 1)r^{1-q}(t)|w|^q = 0$$

  defined on the whole interval $[a, b]$.

- The energy functional

$$\mathcal{F}(y; a, b) = \int_a^b \left[r(t)|y'(t)|^p - c(t)|y(t)|^p \right] dt$$

  is positive for every nontrivial $y \in W^{1,p}(a, b)$ with $y(a) = 0 = y(b)$.

Half-linear Euler differential equation

- The half-linear Euler equation

$$\left(\Phi(x')\right)' + \frac{\gamma}{t^p}\Phi(x) = 0$$

is oscillatory if $\gamma > \gamma_p := \left(\frac{p-1}{p}\right)^p$ and nonoscillatory in the opposite case.

- If $\gamma = \gamma_p$, then $x(t) = t^{\frac{p-1}{p}}$ is a solution
- The potential t^{-p} is a border line between oscillation and nonoscillation, Kneser type (non)oscillation criteria for the equation

$$\left(\Phi(x')\right)' + c(t)\Phi(x) = 0$$

$$\liminf_{t\to\infty} t^p c(t) > \gamma_p, \quad \limsup_{t\to\infty} t^p c(t) < \gamma_p.$$

Half-linear conditional oscillation

- The equation

$$(HL) \qquad \big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0$$

with positive c is conditionally oscillatory if there exists $\lambda_0 > 0$, the so-called oscillation constant, such that the equation

$$\big(r(t)\Phi(x')\big)' + \lambda c(t)\Phi(x) = 0$$

is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$.

- If $\int^\infty r^{1-q}(t)\,dt = \infty$ ($\frac{1}{p} + \frac{1}{q} = 1$), then the equation

$$\big(r(t)\Phi(x')\big)' + \frac{1}{r^{q-1}(t)\big(\int^t r^{1-q}(s)\,ds\big)^p}\Phi(x) = 0$$

is conditionally oscillatory with the oscillation constant $\lambda_0 = \gamma_p$.

Limiting case

What happens when

$$\lim_{t\to\infty} t^p c(t) = \gamma_p := \left(\frac{p-1}{p}\right)^p,$$

i.e., we have

$$c(t) = \frac{\gamma_p}{t^p} + d(t) \quad \Longrightarrow \quad (\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + d(t)\right]\Phi(x) = 0$$

with a “small” function d. This motivates the investigation of various perturbations of the “critical” half-linear Euler differential equation

$$\text{(EE)} \qquad (\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0.$$

- Transformation approach in the linear case

$$(*)\qquad x'' + \left[\frac{1}{4t^2} + d(t)\right] x = 0 \quad \left| x = \sqrt{t}y \right| \quad \left(ty'\right)' + td(t)y = 0.$$

- The change of independent variable $s = \log t$, $y(s) = x(t)$, in the last equation gives

$$\frac{d^2}{ds^2} y(s) + t^2 d(t) y(s) = 0$$

and from Euler equation we know that the limiting case is

$$t^2 d(t) = \frac{1}{4s^2} \qquad \Longrightarrow \qquad d(t) = \frac{1}{4t^2 \log^2 t}.$$

- If

$$\liminf_{t\to\infty} t^2 \log^2 t \, d(t) > \frac{1}{4}$$

then (*) is oscillatory, if $\limsup_{t\to\infty} t^2 \log^2 t \, d(t) < \frac{1}{4}$, then (*) is nonoscillatory.

Modified Riccati equation

In the linear case, the transformation $x = h(t)y$ transforms the equation

$$\big(r(t)x'\big)' + c(t)x = 0$$

into the equation

$$\big(R(t)y'\big)' + C(t)y = 0$$

with $R = rh^2, \quad C = h[(rh')' + ch]$.

In terms of the associated Riccati equations

$$w = \frac{rx'}{x} = \frac{r(h'y + hy')}{hy} = \frac{rh'}{h} + \frac{1}{h^2}\frac{rh^2y'}{y},$$

hence

$$v = h^2(w - w_h), \quad v = \frac{rh^2y'}{y}, \; w_h = \frac{rh'}{h}.$$

This motivates the tranformation of the Riccati equation associated with (HL)

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad \frac{1}{p} + \frac{1}{q} = 1$$

of the form

$$v(t) = h^p(t)(w(t) - w_h(t)), \quad w_h = \frac{r\Phi(h')}{h}.$$

By a direct computation v satisfies the Modified Riccati equation:

$$(MRE) \qquad v' + h^p(t)C(t) + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v) = 0,$$

$$H(t,v) := |v + G(t)|^q - qv\Phi^{-1}(G(t)) - |G(t)|^q,$$

$$G(t) := r(t)h(t)\Phi(h'(t)), \quad C = h[(r\Phi(h'))' + c\Phi(h)]$$

Special cases

- $p = 2 \implies$

$$H(t, v) = (v + G)^2 - 2Gv - G^2 = v^2,$$

hence modified RE is the equation corresponding to the transformed equation

$$(rh^2 y')' + h[(rh')' + ch]y = 0.$$

- If $r(t) = 1$, $h(t) = t^{\frac{p-1}{p}}$, then $G(t) \equiv \left(\frac{p-1}{p}\right)^{p-1} =: \Gamma_p$ and

$$H(v, G) = |v + \Gamma_p|^q - v + \gamma_p$$

and

$$C(t) = t^{p-1}\left(c(t) - \frac{\gamma_p}{t^p}\right).$$

Quadratization of the function H: Suppose that $h'(t) \neq 0$,

$$H(t,v) = |G|^q \left\{ \left| \frac{v}{G} + 1 \right|^q - q\frac{v}{G} - 1 \right\}$$
$$\sim \frac{q(q-1)}{2} |G|^q \left(\frac{v}{G} \right)^2 = \frac{q(q-1)}{2} |G|^{q-2} v^2 \quad \text{as } v \to 0.$$

We have

$$(p-1) r^{1-q} h^{-q} H(t,v) \sim \frac{q}{2} \frac{v^2}{R}, \quad R := r h^2 |h'|^{p-2}.$$

hence we obtain the Approximate Riccati equation

$$(ARE) \qquad v' + C(t) + \frac{q}{2} \frac{v^2}{R(t)} = 0.$$

(ARE) is the Riccati equation associated with the linear Sturm-Liouville equation

$$(R(t)y')' + \frac{q}{2} C(t) y = 0$$

Application of MRE

Elbert and Schneider, 2000. The half-linear Riemann-Weber equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right)\Phi(x) = 0, \quad \gamma_p = \left(\frac{p-1}{p}\right)^p,$$

is oscillatory iff

$$\mu > \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

Modified Riccati equation is (with $h(t) = t^{\frac{p-1}{p}}$)

$$v' + \frac{\mu}{t \log^2 t} + (p-1)t^{-1}\left[|v + \Gamma_p|^q - v + \gamma_p\right] = 0$$

and the Approximate Riccati equation is

$$z' + \frac{\mu}{t \log^2} + \frac{1}{2t\Gamma_p} z^2 = 0$$

The change of independent variable $s = \log t$ gives

$$z' + \frac{\mu}{s^2} + \frac{1}{2\Gamma_p} z^2 = 0.$$

The associated second order differential equation is

$$\left(2\Gamma_p y'\right)' + \frac{\mu}{s^2} y = 0$$

which the same as

$$y'' + \frac{\mu}{2\Gamma_p} \frac{1}{s^2} y = 0$$

and the last equation is nonoscillatory iff

$$\frac{\mu}{2\Gamma_p} \leq \frac{1}{4} \quad \iff \quad \mu \leq \frac{1}{2}\Gamma_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} =: \mu_p.$$

More general perurbation of the critical Euler equation

$$\left(\Phi(x')\right)' + \frac{1}{t^p}\left[\gamma_p + \frac{\mu_p}{\log^2 t} + \frac{\mu}{\log^2 t \log^2(\log t)}\right]\Phi(x) = 0$$

is using the same method nonoscillatory iff $\mu \le \mu_p$.
We consider a more general perturbation of critical Euler equation (EE):

$$\text{(EP)}\quad \left[\left(1 + \sum_{j=1}^{n} \frac{\alpha_j}{\mathrm{Log}_j^2(t)}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\beta_j}{t^p\, \mathrm{Log}_j^2(t)}\right]\Phi(x) = 0$$

with

$$\mathrm{Log}_k(t) = \prod_{j=1}^{k} \log_k(t), \quad \log_k(t) = \log_{k-1}(\log t),\ \log_1(t) = \log t.$$

(Non)oscillation of (EP)

Recall that

$$\gamma_p = \left(\frac{p-1}{p}\right)^p, \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^p.$$

O. D. H. Funková (2012):

- If $\beta_1 + \gamma_p\alpha_1 < \mu_p$ then (EP) is nonoscillatory and if $\beta_1 + \gamma_p\alpha_1 > \mu_p$ then it is oscillatory.
- Let $\beta_1 + \gamma_p\alpha = \mu_p$, if $\beta_2 + \gamma_p\alpha_2 < \mu_p$ then (EP) is nonoscillaotry and if $\beta_2 + \gamma_p\alpha_2 > \mu_p$ then it is oscillatory.

$$\vdots$$

- Let

  $$\beta_k + \gamma_p\alpha_k = \mu_p, \quad k = 1, \dots, n-1.$$

  Then (EP) is nonoscillatory if and only if

  $$\beta_n + \gamma_p\alpha_n \leq \mu_p.$$

Periodic perturbations

We consider the equation

$$(HLP) \qquad \big(r(t)\Phi(x')\big)' + \frac{\lambda c(t)}{t^p}\Phi(x) = 0$$

with positive α-periodic functions r, c

- If $r(t) \equiv 1$, $c(t) \equiv 1$, then (HLP) reduces to the half-linear Euler equation and its oscillation constant is $\lambda_0 = \gamma_p = \left(\frac{p-1}{p}\right)^p$.
- **Theorem** (P. Hasil, 2009). Let

  $$\bar{r} = \frac{1}{\alpha}\int_0^\alpha \frac{ds}{r^{q-1}(s)}, \qquad \bar{c} = \frac{1}{\alpha}\int_0^\alpha c(s)\, ds.$$

  Then (HLP) is oscillatory if

  $$\lambda > \gamma_{r,c} := \frac{\gamma_p}{(\bar{r})^{p-1}\bar{c}}.$$

  and nonoscillatory if $\lambda < \gamma_{r,c}$.

The limiting case $\lambda = \gamma_{r,c}$

Consider the equation

$$\text{(RW)} \qquad \big(r(t)\Phi(x')\big)' + \left[\frac{\gamma_{r,c}c(t)}{t^p} + \frac{\mu d(t)}{t^p \log^2 t}\right]\Phi(x) = 0.$$

Theorem (O. D., P. Hasil, 2011). Let $\overline{r}, \overline{c}$ be as before and

$$\overline{d} = \frac{1}{\alpha}\int_0^\alpha d(s)\, ds.$$

Then (RW) is oscillatory if

$$\mu > \mu_{r,d} := \frac{\mu_p}{(\overline{r})^{p-1}\overline{d}}$$

and nonoscillatory if $\mu < \mu_{r,d}$. In particular, equation (HLP) is nonoscillatory also in the limiting case $\lambda = \gamma_{r,c}$.

Consider the equation

(EPP)

$$\left[\left(r(t)+\sum_{j=1}^{n}\frac{\alpha_j(t)}{\mathrm{Log}_j^2(t)}\right)^{1-p}\Phi(x')\right]'+\left[\frac{c(t)}{t^p}+\sum_{j=1}^{n}\frac{\beta_j(t)}{t^p\,\mathrm{Log}_j^2(t)}\right]\Phi(x)=0.$$

with T periodic functions α_j, β_j, $j=1,\dots,n$. Denote by $\bar r, \bar c, \bar\alpha_j, \bar\beta_j$ $j=1,\dots,n$, their mean values over the period T, i.e.,

$$\bar r=\frac{1}{T}\int_0^T r(t)\,dt,\quad \bar c=\frac{1}{T}\int_0^T c(t)\,dt,$$

$$\bar\alpha_j=\frac{1}{T}\int_0^T \alpha_j(t)\,dt,\quad \bar\beta_j=\frac{1}{T}\int_0^T \beta_j(t)\,dt.$$

Theorem. (O. D. H.Funková, 2013). Let $\bar{c}\bar{r}^{p-1} = \gamma_p$. If there exists $k \in \{1, \dots, n\}$ such that

$$\bar{\beta}_j \bar{r}^{p-1} + (p-1)\gamma_p \bar{\alpha}_j \bar{r}^{-1} = \mu_p, \quad j = 1, \dots, k-1,$$

and $\bar{\beta}_k \bar{r}^{p-1} + (p-1)\gamma_p \bar{\alpha}_k \bar{r}^{-1} \neq \mu_p$, then (EPP) is oscillatory if

$$\bar{\beta}_k \bar{r}^{p-1} + (p-1)\gamma_p \bar{\alpha}_k \bar{r}^{-1} > \mu_p$$

and nonoscillatory if

$$\bar{\beta}_k \bar{r}^{p-1} + (p-1)\gamma_p \bar{\alpha}_k \bar{r}^{-1} < \mu_p.$$

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