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Asymptotic Properties of Second Order Delay
Differential Equations without Damping Term:
Nonoscillation and Exponential Stability

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It is considered to be absolutely impossible to achieve exponential stability of second order delay differential equations without damping term. In this talk we try to refute this delusion. Nonoscillation for second order delay differential equations with non-tending to zero coefficients and without damping term are obtained. The exponential stability results for second order delay differential equations without damping term are proven. Our approach is based on solutions' representation formulas and positivity of fundamental functions (Cauchy functions) of functional differential equations. Theorems about positivity of the Cauchy functions are proposed in the form of assertions about differential inequalities. Corresponding choice of test functions leads to results on nonoscillation and exponential stability. On a basis of the obtained results, simple tests for stabilization by feedback delay control are proposed.

1. Introduction

Model for motion of a single mass point

$$X''(t) = f(t), \quad t \in [0, +\infty), \quad (1.1)$$

where $X(t) = \text{col} \{x_1(t), x_2(t), x_3(t)\}$, $f(t) = \text{col} \{f_1(t), f_2(t), f_3(t)\}$.

Instability of this system implies that small mistakes in f and in the values $X(t_0)$ and $X'(t_0)$ can imply very essential mistakes in calculation of $X(t_0 + \omega)$ and $X'(t_0 + \omega)$ for sufficiently large ω .

Let us assume that the trajectory $Y(t) = \text{col} \{y_1(t), y_2(t), y_3(t)\}$, which we want to "hold", is known, and we wish to hold our object "close" to this trajectory. Although we know this trajectory, it is impossible to "achieve" this proximity on semiaxis, because of instability of system (1.1). As a result, we have to make corrections permanently for localization of the single mass point in the space. Our purpose is to construct a control which makes this correction automatically.

1. Introduction (cont.)

A standard trick: to set an additional force which depends on the state and/or on the velocity of the single mass point as a control $u(t)$ in the right hand side. Wishing to stay in a linear case, we can choose, for example,

$$u(t) = -Q \{X'(t) - Y'(t)\} - P \{X(t) - Y(t)\}, \quad (1.2)$$

where P and Q are constant 3×3 matrices, and to analyze the exponential stability of the system

$$X''(t) + QX'(t) + PX(t) = g(t), \quad t \in [0, +\infty),$$

where $g(t) = f(t) + QY'(t) + PY(t)$ is a known vector function.

1. Introduction (cont.)

Remarks on Mathematical Model

1) Usually it is essentially technically easier to locate only the state $X(t)$ and not the velocity $X'(t)$, i.e. Q is the zero 3×3 matrix. In the simplest case of a diagonal matrix P we get three separate equations with pure imaginary roots of the characteristic equations. Thus each of these equations is Lyapunov's stable, but not exponentially stable and, as a result, there is no stability with respect to right hand side.

2) One of the basic properties of the feedback control is appearance of a delay τ in real systems in receiving signal and in reaction on this signal. Thus in real systems the control is chosen in the form

$$u(t) = - \sum_{i=1}^m P_i(t) \{X(t - \tau_i(t)) - Y(t - \tau_i(t))\}, \quad t \in [0, +\infty), \quad (1.3)$$

1. Introduction (cont.)

where $P(t)$ is an 3×3 matrix. Adding this control $u(t)$ into the equation of the motion, we get

$$X''(t) + \sum_{i=1}^m P_i(t)X(t - \tau_i(t)) = g(t), \quad t \in [0, +\infty), \quad (1.4)$$

where

$$g(t) = f(t) + \sum_{i=1}^m P_i(t)Y(t - \tau_i(t)), \quad t \in [0, +\infty),$$

is the known right hand side. We have to study the exponential stability of system (1.4).

1. Introduction (cont.)

In this talk we limit ourselves by the diagonal matrices P_i ($i = 1, \dots, m$) in order to be concentrated on the scalar second order equation

$$x''(t) + px(t - \tau) = 0, \quad t \in [0, +\infty), \quad (1.5)$$

and its natural generalizations

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \quad (1.6)$$

and

$$x''(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, +\infty) \quad (1.7)$$

where

$$x(\xi) = 0 \text{ for } \xi < 0. \quad (1.8)$$

1. Introduction (cont.)

The idea of the stabilization by feedback control was efficiently realized in various applications.

K.Pyragas. Control of chaos via an unstable delayed feedback controller, *Phys. Rev. Lett.*, vol.86, No 11, March 2001, pp.2265-2268.

E.Ott, C.Grebogi, J.A.Yorke. Controlling chaos. *Phys. Rev. Lett.* 64,1196 (1990).

J.E.S.Socolar, D.W.Sukow and D.J.Gauthier. Stabilizing unstable periodic orbits in fast dynamical systems. *Phys. Rev. E*50, 3245 (1994).

The finite spectrum assignment technique was originated, for example, in the work

Z. Artstein, Linear systems with delayed controls: a reduction, *IEEE Trans. Automat. Control* 27:4 (1982), 869–879.

1. Introduction (cont.)

Our approach to the study of stability is based on oscillation properties of solutions. Various aspects of oscillation and asymptotic behavior of second order delay equations were considered in the known monographs by

A. D. Myshkis. *Linear differential equations with delayed argument*. Moscow, Nauka, 1972.

S.B.Norkin, *Differential equations of the second order with retarded argument*. Transl. Math. Monographs, vol. 31, Amer. Math. Soc., Providence, 1972.

I.Gyori and G.Ladas. *Oscillation theory of delay differential equations*. Clarendon, Oxford, 1991.

R.G.Koplatadze. *On oscillatory properties of solutions of functional differential equations*. Memoirs on Differential Equations and Mathematical Physics, vol.3, Tbilisi, 1994.

L.N.Erbe, Q.Kong and B.G.Zhang. *Oscillation theory for functional differential equations*, Dekker, New York/Basel, 1995.

G.S.Ladde, V.Lakshmikantham and B.G.Zhang. *Oscillation theory of differential equations with deviating argument*. Dekker, New York/Basel, 1987.

1. Introduction (cont.)

Several specific properties of second order delay equations.

Solutions x of the delay equation

$$x''(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, +\infty) \quad (1.7)$$

for $p_i(t) \leq 0$ can change its sign. For example, the function $x(t) = (t - 1)(t - 2)$ is a solution of the equation

$$x''(t) - x(0) = 0, \quad t \in [0, +\infty), \quad (1.8)$$

changing the sign.

1. Introduction (cont.)

The equation

$$x''(t) - x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \quad (1.9)$$

with

$$\tau(t) = \begin{cases} t, & t \in [0, 4), \\ t - 2, & t \in (4, 8), \end{cases} \quad \tau(t + 8) = \tau(t),$$

possesses oscillating solution

$$x(t) = \begin{cases} (t - 1)(t - 3), & t \in [0, 4], \\ -(t - 4)^2 + 4(t - 4) + 3, & t \in (4, 8), \end{cases} \quad x(t + 8) = x(t),$$

Another group of solutions $y(t)$ of equation (1.9) satisfying the conditions

$$y(0) = 0, \quad y'(0) = \beta > 0,$$

possesses the property $y''(t) \geq 0$, $y'(t) > 0$ for $t \in [0, +\infty)$ that implies $y(t) > 0$ for $t \in [0, +\infty)$, and $y(t) \rightarrow +\infty$. We see that, in contrast with ODEs, delay equations can possess together oscillating and nonoscillating solutions.

1. Introduction (cont.)

Several possible types of solutions' behavior of this equation in case $p(t)$ and $\tau(t)$ are bounded functions on semiaxis and $\int_0^{\infty} |p(t)| dt = \infty$, can be only as following :

- a) $|x(t)| \rightarrow \infty$ for $t \rightarrow \infty$; b) $x(t)$ oscillates; c) $x(t) \rightarrow 0$, $x'(t) \rightarrow 0$ for $t \rightarrow \infty$.
(1.10)

Existence and uniqueness of solutions of each of these types were obtained in

A.L.Skubachevskii. Oscillating solutions of a second order homogeneous linear differential equation with time-lag. *Differentsialnye uravnenia*, 11, 462-469 (1975).

R.G.Koplatadze. *On oscillatory properties of solutions of functional differential equations*. Memoirs on Differential Equations and Mathematical Physics, vol.3, Tbilisi, 1994.

M.G.Shmul'yan. On the oscillating solutions of a linear second order differential equation with retarding argument. *Differentsial'nye uravnenia*, v.31, 1995, 622-629.

1. Introduction (cont.)

Assertions on existence of bounded solutions, their uniqueness and oscillation were obtained in the monograph

G.S.Ladde, V.Lakshmikantham and B.G.Zhang. *Oscillation theory of differential equations with deviating argument*. Dekker, New York/Basel, 1987.

Solutions tending to zero were considered in the paper

T.A.Burton and J.R.Haddock. On solution tending to zero for the equation $x''(t) + a(t)x(t - r(t)) = 0$. *Arch.Math. (Basel)*, 27 (1976), pp. 48-51.

1. Introduction (cont.)

For the case $p_i(t) \geq 0$ various results on oscillation of all solutions can be found in the book by Koplatadze, results on validity of Sturm's theorem for delay differential equations (between two adjacent zeros of every solution there are one and only one zero of each other non-proportional solution), estimates of distances between zeros from below and above were obtained in

N.V.Azbelev. About zeros of solutions of linear differential equations of the second order with delayed argument. *Differentsialnye Uravnenia*, 1971, V. 7, No. 7, pp. 1147–1157.

Y.Domshlak. Comparison theorems of Sturm type for first and second order differential equations with sign variable deviations of the argument. *Ukraine Mat.Zh.* 34, 158-163 (1982).

A.Domoshnitsky, Sturm's theorem for equation with delayed argument, *Georgian Math. Journal*, 1, No. 3, 267-276 (1994).

1. Introduction (cont.)

Note that delay equation

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \quad (1.6)$$

usually inherits oscillation properties of corresponding ordinary differential equation (ODE)

$$x''(t) + p(t)x(t) = 0. \quad (1.11)$$

For example, it was proven by J. J. A. M. Brands [11] that for every nonnegative $p(t)$ and bounded delay $\tau(t)$ equation (1.6) is oscillatory if and only if corresponding ordinary differential equation (1.11) is oscillatory.

The asymptotic behavior of ODE (1.11) is not inherited by

$$x''(t) + px(t - \tau) = 0, \quad t \in [0, +\infty), \quad (1.5)$$

A.D.Myshkis proved that there exists unbounded solution of equation (1.5) for each couple of positive constants p and τ .

1. Introduction (cont.)

The results on unboundedness of solutions in the case of variable coefficients and delays were obtained on the basis of growth of the Wronskians in

A.Domoshnitsky, Unboundedness of solutions and instability of differential equations of the second order with delayed argument, *Differential & Integral Equations*, **14** (2001), no.5, 559-576. MR 2002c:34117. Zbl 0884.34074

Theorem. *If there exists a positive constant ε such that $\tau(t) > \varepsilon$ and $p(t) > \varepsilon$, then there exist unbounded solution to equation*

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \quad (1.6)$$

1. Introduction (cont.)

If $\varepsilon = 0$, then all solutions of the equations

$$x''(t) + t^2 x(t) + t^{3/2} x\left(t - \frac{\varepsilon}{t}\right) = 0,$$

$$x''(t) + t^\alpha x\left(t - \frac{\varepsilon}{t^\beta}\right) = 0, \quad \alpha + 2 > 2\beta, \quad \alpha > 0,$$

$$x''(t) + x(t) + \frac{1}{\sqrt{t}} x\left(t - \frac{\varepsilon}{\sqrt{t}}\right) = 0,$$

$$x''(t) + e^t x(t - \varepsilon) = 0,$$

are bounded on $(1, +\infty)$. If $\varepsilon > 0$, then there exist unbounded solutions to these equations. Note that the delays in the first three equations tend to zero when $t \rightarrow +\infty$, but even these "very small" delays totally change the asymptotic behavior of solutions.

1. Introduction (cont.)

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \quad (1.6)$$

Theorem. All solutions of equation (1.6) with positive nondecreasing and bounded coefficient $p(t)$ and nondecreasing $h(t) \equiv t - \tau(t)$ are bounded if and only if

$$\int_0^{\infty} \tau(t) dt < \infty. \quad (1.13)$$

A study of advanced equations ($\tau(t) \leq 0$) can be found in the paper [27]

Z.Dosla and I.T.Kiguradze. On boundedness and stability of solutions of second order linear differential equations with advanced arguments. *Advances in Mathematical Sciences and Applications, Gakkotosho, Tokyo*, vol.9, No.1 , 1-24 (1999),

in which results on boundedness, stability and asymptotic representations of solutions are obtained.

1. Introduction (cont.)

Summarizing, we can conclude that it looks impossible to obtain results about exponential stability for one term equation (1.6). It means that the control of the form

$$u(t) = -P(t)\{X(t - \tau(t)) - Y(t - \tau(t))\}, \quad (1.15)$$

without damping term cannot help us in stabilization of system (1.1). It should be stressed that there are no results about the exponential stability of equation (1.7).

We demonstrate that the control of the form (1.3), where $m \geq 2$ can stabilize system (1.1) and achieve the exponential stability of equation (1.7). We obtain results explaining how to achieve nonoscillation of delay equations and positivity of their Cauchy (fundamental) functions. This will open a way to analysis of asymptotic behavior of nonlinear delay differential equations on the basis of the known schemes of upper and lower functions.

2. About Representation of Solutions for Second Order Delay Differential Equations

The main object of this talk is the second order delay differential equation

$$x''(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, +\infty), \quad (2.1)$$

$$x(\xi) = \varphi(\xi), \quad \text{for } \xi < 0, \quad (2.2)$$

with measurable essentially bounded f, p_i, φ, τ_i ($i = 0, 1, \dots, n - 1$), and

$\tau_i(t) \geq 0$ for $t \geq 0$.

$$x''(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, +\infty), \quad (2.3)$$

2. About Representation of Solutions for Second Order Delay Differential Equations (cont.)

We consider the zero initial functions

$$x(\xi) = 0, \quad \text{for } \xi < 0. \quad (2.4)$$

In this case the space of the solutions of the second order equation (2.3),(2.4) becomes two-dimensional and the key notions of the classical theory of ODEs can be used. The main concepts of these approach are presented in

N.V.Azbelev, V.P.Maksimov and L.F.Rakhmatullina. *Introduction to theory of functional-differential equations*, Nauka, Moscow, 1991.

2. About Representation of Solutions for Second Order Delay Differential Equations (cont.)

The general solution of equation (2.1),(2.4) can be represented in the form [4]

$$x(t) = \int_0^t C(t,s)f(s)ds + x_1(t)x(0) + x_2(t)x'(0), \quad (2.5)$$

where $x_1(t), x_2(t)$ are two solutions of homogeneous equation (2.3),(2.4) satisfying the conditions

$$x_1(0) = 1, \quad x_1'(0) = 0, \quad x_2(0) = 0, \quad x_2'(0) = 1, \quad (2.6)$$

and the kernel in this representation is called the Cauchy function $C(t,s)$ of equation (2.1).

2. About Representation of Solutions for Second Order Delay Differential Equations (cont.)

For every fixed s the function $C(t, s)$ as a function of the variable t satisfies the equation

$$x''(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [s, +\infty), \quad (2.7)$$

$$x(\xi) = 0, \quad \text{for } \xi < s, \quad (2.8)$$

and $C(s, s) = 0$, $C'_t(s, s) = 1$. Behavior of the fundamental system of solutions of (2.3),(2.4) determines existence and uniqueness of solutions of boundary value problems for this equation. Positivity of the Cauchy function $C(t, s)$ of equation (2.1),(2.4) is a basis of various theorems about differential inequalities (under corresponding conditions, solution of an inequality is greater than solution of an equation).

2. About Representation of Solutions for Second Order Delay Differential Equations (cont.)

Definition 2.1. Equation (2.3) is uniformly exponentially stable if there exist $N > 0$ and $\alpha > 0$, such that the solution of (2.3), (2.2), where

$$x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0, \quad x'(t_0) = x'_0, \quad (2.9)$$

satisfies the estimate

$$|x(t)| \leq Ne^{-\alpha(t-t_0)}, \quad 0 \leq t < +\infty, \quad (2.10)$$

where N and α do not depend on t_0 .

Definition 2.2. We say that the Cauchy function $C(t, s)$ of equation (2.1) satisfies the exponential estimate if there exist positive N and α such that

$$|C(t, s)| \leq Ne^{-\alpha(t-s)}, \quad 0 \leq s \leq t < +\infty. \quad (2.11)$$

It is known that for equation (2.1) with bounded delays these two definitions are equivalent [3].

3. Tests of Positivity of the Cauchy Functions and Stability

We will demonstrate that, although the ordinary differential equation

$$x''(t) + \left\{ \sum_{i=1}^m p_i(t) \right\} x(t) = 0, \quad t \in [0, +\infty), \quad (3.1)$$

can be oscillating and asymptotically unstable, the delay equation

$$x''(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, +\infty), \quad (2.3)$$

$$x(\xi) = 0, \quad \text{for } \xi < 0. \quad (2.4)$$

under corresponding conditions on the coefficients $p_i(t)$ and delays $\tau_i(t)$ is nonoscillating and exponentially stable.

The basic idea of our approach is to avoid the condition on nonnegativity of the coefficients $p_i(t)$ for all $i = 1, \dots, m$, and to allow terms with positive and terms with negative coefficients $p_i(t)$ to compensate each other.

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Let us consider the equation

$$x''(t) + a(t)x(t - \tau(t)) - b(t)x(t - \theta(t)) = 0, \quad t \in [0, +\infty), \quad (3.2)$$

$$x(\xi) = 0, \quad \text{for } \xi < 0, \quad (3.3)$$

where $a(t)$, $b(t)$, $\tau(t)$ and $\theta(t)$ are measurable essentially bounded nonnegative functions. Denote

$$q_* = \operatorname{ess\,inf}_{t \geq 0} q(t), \quad q^* = \operatorname{ess\,sup}_{t \geq 0} q(t). \quad (3.4)$$

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Theorem 3.1. Assume that $0 \leq \tau(t) \leq \theta(t)$, $0 \leq b(t) \leq a(t)$,

$$4 \{a(t) - b(t)\} \leq [b(\theta - \tau)]_*^2, \quad t \in [0, +\infty), \quad (3.5)$$

$$0 < [b(\theta - \tau)]^* \theta^* \leq \frac{1}{e}. \quad (3.6)$$

Then

- 1) the Cauchy function $C(t, s)$ of equation (3.2) is positive for $0 \leq s < t < +\infty$;
- 2) if there exists such positive ε that

$$a(t) - b(t) \geq \varepsilon, \quad (3.7)$$

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

then the Cauchy function $C(t, s)$ of equation (3.2) satisfies the exponential estimate, i.e. there exist positive N and α such that

$$|C(t, s)| \leq Ne^{-\alpha(t-s)}, \quad 0 \leq s \leq t < +\infty.$$

and the integral estimate

$$\sup_{t \geq 0} \int_0^t C(t, s) ds \leq \frac{1}{\varepsilon}; \quad (3.8)$$

3) if there exists $\lim_{t \rightarrow \infty} \{a(t) - b(t)\} = k$, with $k > 0$, then

$$\lim_{t \rightarrow \infty} \int_0^t C(t, s) ds = \frac{1}{k}. \quad (3.9)$$

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Corollary 3.1. *Assume that the delays $\tau(t) \equiv \tau$, $\theta(t) \equiv \theta$ are constants and*

$$0 < \varepsilon \leq 4 \{a(t) - b(t)\} \leq b_*^2(\theta - \tau)^2, \quad t \in [0, +\infty), \quad (3.10)$$

$$0 < b^*(\theta - \tau)\theta \leq \frac{1}{e}. \quad (3.11)$$

Then assertions of Theorem 3.1 are true.

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Consider the unstable equation

$$x''(t) + a(t)x(t - \tau) = f(t), \quad t \in [0, +\infty), \quad (3.12)$$

with chaos in solutions' behavior. To stabilize its solution to the given "trajectory" $y(t)$ satisfying this equation, we choose the control in the form

$$u(t) = b(t)[x(t - \theta) - y(t - \theta)]. \quad (3.13)$$

A possible algorithm to construct this stabilizing control is clear now: first of all to choose the delay θ close to τ such that condition (3.11) is fulfilled, then we choose $b(t)$ close to $a(t)$ such that condition (3.10) is fulfilled.

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Example 3.1. Stabilizing equation (3.12), where $a(t) \equiv a$, let us choose the control in the form (3.13) with constant coefficient $b(t) \equiv b$. We come to study of the exponential stability of the equation

$$x''(t) + ax(t - \tau) - bx(t - \theta) = g(t), \quad t \in [0, +\infty), \quad (3.14)$$

with constant coefficient and delays and $g(t) = f(t) + by(t - \theta)$. We can choose $\theta - \tau = \frac{1}{eb\theta}$ from (3.11), then from (3.10) we get the following condition of the exponential stability

$$0 < 4\{a - b\} \leq \frac{1}{e^{2\theta^2}}, \quad \tau < \theta. \quad (3.15)$$

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Example 3.2. The equation

$$x''(t) + a(t)x(t - \tau) = 0, \quad a(t) \rightarrow +\infty, \quad t \in [0, +\infty), \quad \tau = \text{const}, \quad (3.16)$$

where $a(t) \geq a_* > 0$ possesses oscillating solutions with amplitudes tending to infinity [21] that leads to the chaos in behavior of its solutions. This equation can be also stabilized by the control in form

$$u(t) = b(t)[x(t - \theta) - y(t - \theta)]. \quad (3.13)$$

Consider, for example, the equation

$$x''(t) + tx(t - \tau) = 0, \quad t \in [1, +\infty), \quad \tau = \text{const}, \quad (3.17)$$

if we choose $b(t) = t - \Delta$, $\theta(t) = \tau + \frac{\gamma}{t}$, then the stabilization can be achieved by the control (3.13) with the parameters satisfying the inequalities

$$0 < 2\sqrt{\Delta} < \gamma < \frac{1}{\tau e}, \quad (3.18)$$

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

In the following assertion we assume the smallness of the difference of the delays $\theta - \tau$ instead of the smallness of the delay θ .

Consider for simplicity the equation

$$x''(t) + a(t)x(t - \tau) - b(t)x(t - \theta) = 0, \quad t \in [0, +\infty), \quad (3.19)$$

$$x(\xi) = 0, \quad \text{for } \xi < 0, \quad (3.20)$$

with constant delays τ and θ and positive coefficients $a(t)$ and $b(t)$.

Theorem 3.2. *Assume that $0 < \tau < \theta \leq 2\tau$, there exists a positive ε such that*

$$\varepsilon \leq \{a(t) - b(t)\} \leq \frac{1}{4} [b(\theta - \tau)]_*^2, \quad t \in [0, +\infty), \quad (3.21)$$

and

$$\frac{1}{\sqrt{a^*} \exp \left\{ \frac{b_*(\theta - \tau)^2}{4} \right\}} \operatorname{arctg} \frac{b_*(\theta - \tau)}{2\sqrt{a^*} \exp \left\{ \frac{b_*(\theta - \tau)^2}{4} \right\}} > \theta - \tau \quad (3.22)$$

are fulfilled.

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Then

- 1) the Cauchy function $C(t, s)$ of equation (3.19) is positive for $0 \leq s < t < +\infty$;
- 2) the solutions $x_1(t), x_2(t)$ of equation (3.19), satisfying initial conditions (2.6), are positive for $0 < t < +\infty$.
- 3) the Cauchy function $C(t, s)$ of equation (3.19) satisfies the exponential estimate, i.e. there exist positive N and α such that

$$|C(t, s)| \leq Ne^{-\alpha(t-s)}, \quad 0 \leq s \leq t < +\infty,$$

and the integral estimate

$$\sup_{t \geq 0} \int_0^t C(t, s) ds \leq \frac{1}{\varepsilon}; \quad (3.8)$$

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

4) if there exists $\lim_{t \rightarrow \infty} \{a(t) - b(t)\} = k$, then

$$\lim_{t \rightarrow \infty} \int_0^t C(t, s) ds = \frac{1}{k}. \quad (3.9)$$

3. Tests of Positivity of the Cauchy Functions and Stability (cont.)

Example 3.3. Let us demonstrate that condition (3.6) is essential for positivity of the Cauchy function $C(t, s)$ and the solutions x_1 and x_2 . Consider the equation

$$x''(t) + x(t) - bx(t - \theta) = 0, \quad t \in [0, +\infty). \quad (3.23)$$

$$x(\xi) = 0, \quad \xi < 0, \quad (3.24)$$

where all other conditions of Theorem 3.1 are fulfilled. If $\pi < \theta$, then in the triangle $0 \leq s \leq t < \theta$, its Cauchy function is $C(t, s) = \sin(t - s)$ and changes the sign. If $\frac{\pi}{2} < \theta$, then the solution $x_1(t) = \cos t$ changes the sign on the interval $[0, \theta]$. It is clear that the conditions $0 < \tau < \theta$ and (3.22) are essential for positivity of $C(t, s)$ and the solutions x_1 and x_2 .

$$\frac{1}{\sqrt{a^*} \exp \left\{ \frac{b_*(\theta - \tau)^2}{4} \right\}} \operatorname{arctg} \frac{b_*(\theta - \tau)}{2\sqrt{a^*} \exp \left\{ \frac{b_*(\theta - \tau)^2}{4} \right\}} > \theta - \tau \quad (3.22)$$

5. Main Results

Let us consider the equation

$$(Mx)(t) \equiv x''(t) + \sum_{i=1}^{2m} p_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, +\infty), \quad (5.1)$$

and the corresponding homogeneous equation

$$x''(t) + \sum_{i=1}^{2m} p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, +\infty), \quad (5.2)$$

where

$$x(\xi) = 0 \text{ for } \xi < 0.$$

5. Main Results (cont.)

Theorem 5.1. *Let*

$(-1)^{i+1}p_i(t) \geq 0$, $p_{2i-1}(t) + p_{2i}(t) \geq 0$, $\tau_{2i-1}(t) \leq \tau_{2i}(t)$ for $i = 1, \dots, m$, $t \in [0, +\infty)$ and the Cauchy function of the first order equation

$$y'(t) + \sum_{i=1}^m |p_{2i}(t)| \int_{t-\tau_{2i}(t)}^{t-\tau_{2i-1}(t)} y(s) ds = 0, \quad t \in [0, +\infty), \quad (5.3)$$

where

$$y(\xi) = 0 \text{ for } \xi < 0,$$

is positive for $0 \leq s \leq t < +\infty$, then the following assertions are equivalent:

5. Main Results (cont.)

1) *there exists a bounded function v with absolutely continuous bounded derivative v' and essentially bounded derivative v'' such that*

$$v(t) > 0, \quad v'(t) \leq 0, \quad (Mv)(t) \leq 0, \quad t \in [0, +\infty); \quad (5.4)$$

2) *there exists a bounded absolutely continuous function u with essentially bounded derivative u' such that*

$$u(t) \geq 0, \quad u^2(t) - u'(t) + \sum_{i=1}^{2m} p_i(t) \chi(t - \tau_i(t), 0) \exp \left\{ \int_{t - \tau_i(t)}^t u(s) ds \right\} \leq 0, \quad t \in [0, +\infty), \quad (5.5)$$

where

$$\chi(t, s) = \begin{cases} 1, & t \geq s, \\ 0, & t < s. \end{cases}$$

3) *the Cauchy function $C(t, s)$ of equation (5.1) is non-negative for $0 \leq s < t < +\infty$, and solutions $x_1(t), x_2(t)$ of equation (5.2), satisfying initial conditions (2.6), $x_1(t) > 0$, $x_2(t) \geq 0$ for $0 < t < +\infty$*

5. Main Results (cont.)

Denote $H^* = \text{esssup}_{t \geq 0} \tau_{ij}(t)$.

Remark 5.1. The inequality

$$\int_{t-H^*}^t \sum_{i=1}^m |p_{2i}(s)| [\tau_{2i}(s) - \tau_{2i-1}(s)] ds \leq \frac{1}{e} \text{ for } t \geq 0, \quad (5.17)$$

where $p_{2i}(s) \equiv 0$ for $s < 0$.

implies the positivity of the Cauchy function of first order equation (5.3) (see Theorem 15.7, p. 358 in [1]).

Remark 5.2. In the intervals, where $t - \tau_i(t) < 0$, we set $v(t) = 0$ or $u(t) = 0$, that leads to a corresponding inconvenience in construction of the test functions. It is more convenient to construct the test functions $v(t)$ and $u(t)$ in the case $t - \tau_i(t) \geq 0$. In order to avoid this additional assumption, we can make the following trik.

5. Main Results (cont.)

Let us define the operator $B : C_{[-H^*, +\infty)} \rightarrow L_{[-H^*, +\infty)}^\infty$, where $C_{[-H^*, +\infty)}$ and $L_{[-H^*, +\infty)}^\infty$ the spaces of continuous and of essentially bounded functions respectively by the formula

$$(By)(t) = \begin{cases} \beta y(t), & -H^* \leq t \leq 0, \\ \sum_{i=1}^m p_{2i}(t) \int_{t-\tau_{2i}}^{t-\tau_{2i-1}} y(s) ds, & t \geq 0 \end{cases}, \quad (5.18)$$

where the parameter β will be defined below in the formulation of Theorem 5.2.

Consider the equation

$$(Mx)(t) \equiv x''(t) + (Bx')(t) + \sum_{i=1}^m \{p_{2i-1}(t) + p_{2i}(t)\} x(t - \tau_{2i-1}(t)) = f(t), \quad t \in [-H^*, +\infty), \quad (5.19)$$

Non-negativity of the Cauchy function of (5.19) implies non-negativity of the Cauchy function $C(t, s)$ of the given equation (5.1).

5. Main Results (cont.)

Theorem 5.2. *Assume that*

$(-1)^{i+1}p_i(t) \geq 0$, $p_{2i-1}(t) + p_{2i}(t) \geq 0$, $\tau_{2i-1}(t) \leq \tau_{2i}(t)$ for $t \in [0, +\infty)$ and there exists a real number α such that

- a) $\|B\| H^* \leq \frac{1}{e}$, where $\beta = \alpha$ in formula (5.18) defining the operator B ;*
- b) the inequality*

$$\alpha^2 + \sum_{i=1}^{2m} p_i(t) \exp \{ \alpha \tau_i(t) \} \leq 0, \quad t \in [0, +\infty). \quad (5.20)$$

is fulfilled.

Then

- 1) the Cauchy function $C(t, s)$ of equation (5.1) is non-negative for $0 \leq s < t < +\infty$;*

5. Main Results (cont.)

2) if in addition there exists a positive ε such that

$$\sum_{i=1}^{2m} p_i(t) \geq \varepsilon, \quad (5.21)$$

then the Cauchy function $C(t, s)$ of equation (5.1) satisfies the exponential estimate (2.11) and the integral estimate

$$\sup_{t \geq 0} \int_0^t C(t, s) ds \leq \frac{1}{\varepsilon}. \quad (5.22)$$

5. Main Results (cont.)

Corollary 5.1. *If under the conditions of Theorem 5.2 there exists a positive limit*

$$\lim_{t \rightarrow \infty} \sum_{i=1}^{2m} p_i(t) = k, \quad (5.44)$$

then

$$\lim_{t \rightarrow \infty} \int_0^t C(t, s) ds = \frac{1}{k}. \quad (5.45)$$

5. Main Results (cont.)

Remark 5.3. The positivity of ε in condition (5.21) is essential as the following example demonstrates. Consider the equation

$$x''(t) + x\left(t - \left|\sin \frac{1}{2}t\right|\right) - x\left(t - 2\left|\sin \frac{1}{2}t\right|\right) = 0, \quad t \in [0, +\infty), \quad (5.46)$$

Here $\varepsilon = 0$ and one of solutions is a constant and does not tend to zero when $t \rightarrow +\infty$.

Consider the equation

$$(Mx)(t) \equiv x''(t) + \sum_{i=1}^{2m} p_i(t)x(t - \tau_i(t)) + \sum_{j=1}^n q_j(t)x(t - \theta_j(t)) = f(t), \quad t \in [0, +\infty), \quad (5.61)$$

5. Main Results (cont.)

and the corresponding homogeneous equation

$$x''(t) + \sum_{i=1}^{2m} p_i(t)x(t - \tau_i(t)) + \sum_{j=1}^n q_j(t)x(t - \theta_j(t)) = 0, \quad t \in [0, +\infty), \quad (5.62)$$

where

$$x(\xi) = 0 \text{ for } \xi < 0. \quad (5.63)$$

5. Main Results (cont.)

Theorem 5.3. *Let all assumption of Theorem 5.2 be fulfilled.*

Then

- 1) if $q_j(t) \leq 0$ for $t \in [0, +\infty)$, then the Cauchy function $C(t, s)$ of equation (5.61) is positive for $0 \leq s < t < +\infty$;*
- 2) if there exists positive ε_0 and ε such that*

$$\sum_{i=1}^{2m} p_i(t) \geq \varepsilon, \quad \varepsilon - \varepsilon_0 \geq \sum_{j=1}^n |q_j(t)|, \quad t \in [0, +\infty), \quad (5.64)$$

then the Cauchy function $C(t, s)$ of equation (5.61) satisfies the exponential estimate (2.11) and the integral estimate

$$\sup_{t \geq 0} \int_0^t |C(t, s)| ds \leq \frac{1}{\varepsilon_0}. \quad (5.65)$$

6. Open Problems

1. To obtain results on the exponential stability stability of the equation

$$x''(t) + \sum_{j=1}^n q_j(t)x'(t - \theta_j(t)) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, +\infty), \quad (6.1)$$

where $q_j(t) \leq 0$ for $t \in [0, +\infty)$. Results of this type were considered as impossible. It was assumed in previous works [5, 9, 13, 14, 20] that $q_j(t) > 0$, and in the presented paper $q_j(t) \equiv 0$ for $t \in [0, +\infty)$, $j = 1, \dots, m$.

6. Open Problems (cont.)

2. To obtain results about stabilization of the equation $x^{(n)}(t) = f(t)$, where $n > 2$, to the trajectory $y(t)$ by the control of the form

$$u(t) = - \sum_{i=1}^m p_i(t) \{x(t - \tau_i(t)) - y(t - \tau_i(t))\}, \quad t \in [0, +\infty), \quad (6.2)$$

without derivatives, i.e. to obtain results about the exponential stability of the equation

$$x^{(n)}(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, +\infty). \quad (6.3)$$

Results of this type were considered as impossible.

6. Open Problems (cont.)

3. To obtain results on oscillation/nonoscillation, existence of solutions tending to zero or tending to infinity for second order equation (2.3) without the assumption about nonnegativity of the coefficients.

Asymptotic properties of solutions of ODE (1.11) as well as delay equations

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \quad (1.6)$$

and (1.7) can be very different (concerning ODEs, see, for example, Chapter 6 of the known book by R.Bellman[8]).

The problem of similar asymptotic behavior of all solutions to linear second order equation has not yet been solved even with ordinary second order equation.

For example, it was discovered in Milloux [52] that if $p(t) \rightarrow +\infty$ for $t \rightarrow \infty$, then there exists solution of ODE (1.11) tending to zero when $t \rightarrow \infty$. There are also several examples of other solutions without tending to zero.

6. Open Problems (cont.)

The problem to find conditions under which all solutions tend to zero remains one of highlighted in the qualitative theory of differential equations (see, for example, the papers Elbert, Hatvani, Stacho [28, 36, 37]).

If $p(t) \rightarrow +\infty$ monotonically, then all solutions of ODE (1.11) are bounded [8].

If coefficient $p(t) \rightarrow 0$ for $t \rightarrow +\infty$, then there exist unbounded solutions of ODE (1.11) (see, the monograph by I.T.Kiguradze and T.A.Chanturia [42]). The equation

$$x''(t) + \frac{2}{t^2(t-1)}x(t) = 0, \quad t \in [2, +\infty), \quad (1.15)$$

is an example, when the second solution $x(t) = \frac{t-1}{t}$ is bounded.

6. Open Problems (cont.)

We can see a different asymptotic behavior also in the case of delay differential equations. A function $x = \sin t$ is one of solutions of the equation

$$x''(t) + x(t - \tau(t)) = 0, \quad t \in [0, +\infty),$$

where

$$\tau(t) = \begin{cases} 0, & 0 \leq t \leq \frac{\pi}{2}, \\ 2t - \pi, & \frac{\pi}{2} < t < \pi, \end{cases}$$
$$\tau(t + \pi) = \tau(t).$$

Other solutions are unbounded (see necessary and sufficient condition (1.13) of boundedness of all solutions).

6. Open Problems (cont.)

4. To obtain results about distance between adjacent zeros of oscillating solution of equation (2.3), (2.4) and Sturm's separation theorems without the assumption about nonnegativity of the coefficients, which could be analogs of the results obtained in [4, 18, 22, 24].

5. To obtain results on Lyapunov's zones of stability for equation (2.3),(2.4) without the assumption about nonnegativity of the coefficients which could be analogs of the classical assertions obtained in Krein, Yakubovich, Zhukovski [47, 69, 72]. The idea to connect oscillation and asymptotic properties of solutions of a second order ODE appeared in Lyapunov's investigation on stability. Note also that for ODE (1.11) with ω -periodic coefficient $p(t)$, the relation between nonoscillation of the interval $[0, \omega]$ and asymptotic properties of solutions is well known (see, the classical results by N.E.Zhukovskii [72], M.G.Krein [47], V.A.Yakubovich [69]): if the coefficient $p(t)$ is not zero and has nonnegative average on $[0, \omega]$ and for each t_0 equation (1.11) is

6. Open Problems (cont.)

nonoscillatory on $[t_0, t_0 + \omega]$, then all solutions are bounded on semiaxis $[0, +\infty)$. The classical Lyapunov's results claims that all solutions of second order ordinary differential equation (1.11), where

$$p(t) = p(t + \omega) \geq c > 0,$$

with ω - periodic coefficient are bounded on semiaxis if ω is less than distance between two adjacent zeros [72]. The classical estimate of distance between two adjacent zeros in this case

$$\int_0^{\omega} p(t) dt \leq \frac{4}{\omega}, \quad (1.16)$$

implies that all the solutions are bounded.

6. Open Problems (cont.)

It was obtained in [24] that in contrast with the ordinary differential equation all the solutions of the delay equation with ω -periodic coefficients $p(t)$ and $\tau(t)$ are unbounded if distance between zeros of solutions is different from 2ω . Coefficient tests based on this assertion were proposed in [24].

6. To obtain results about stabilization of equation (3.12), where $a(t) \rightarrow 0$, to the given "trajectory" $y(t)$ satisfying this equation, by the control in the form

$$u(t) = b(t)[x(t - \theta) - y(t - \theta)] - c(t)[x(t - r) - y(t - r)], \quad (6.4)$$

where $b(t) \geq 0$, $c(t) \geq 0$, $\theta > \tau$, $r > \tau$.

6. Open Problems (cont.)

7. To obtain results on stabilization of the system (1.1) to the trajectory $Y(t)$ by the control of the form







$$u(t) = - \sum_{i=1}^m P_i(t) \{X(t - \tau_i(t)) - Y(t - \tau_i(t))\}, \quad t \in [0, +\infty), \quad (6.5)$$

in the case of more general than diagonal matrices $P_i(t)$, i.e. to obtain results on the exponential stability of the system








$$X''(t) + \sum_{i=1}^m P_i(t) X(t - \tau_i(t)) = g(t), \quad t \in [0, +\infty). \quad (6.6)$$

8. To obtain results about the exponential stability of the system of functional differential equations of different orders.







Bibliography

-  R.P.Agarwal, L.Berezansky, E.Braverman, A.Domoshnitsky, *Nonoscillation Theory of Functional Differential Equations with Applications*, Springer, New York, 2012.
-  Z. Artstein, Linear systems with delayed controls: a reduction, *IEEE Trans. Automat. Control* 27:4 (1982), 869–879.
-  N.V.Azbelev, V.P.Maksimov and L.F.Rakhmatullina. *Introduction to theory of functional-differential equations*, Nauka, Moscow, 1991.
-  N. V. Azbelev. About zeros of solutions of linear differential equations of the second order with delayed argument. *Differentsialnye Uravnenia*, 1971, V. 7, No. 7, pp. 1147–1157.
-  D.Bainov and A.Domoshnitsky, Stability of a second-order differential equations with retarded argument, *Dynamics and Stability of Systems*, vol 9, No. 2, 145-151 (1994).
-  D.Bainov and A.Domoshnitsky, Nonnegativity of the Cauchy matrix and exponential stability of a neutral type system of functional-differential equations, *Extracta Mathematicae*, vol. 8, N 1, 1993, 75-82.







Bibliography (cont.)

-  E.F.Beckenbach and R.Bellman. Inequalities, Springer-Verlag, New York, 1971.
-  R.Bellman, Stability theory of differential equations, McGraw-Hill, New York, 1953.
-  L.Berezansky, E.Braverman and A.Domoshnitsky, Stability of the second order delay differential equations with a damping term, *Differential Equations and Dynamical Systems*, vol. 16, No. 3&4, July & October 2008, 3-24.
-  J.S.Bradley, Oscillation theorems for a second-order delay equation, *J. Differential Equations*, 8 (1970), 397-403.
-  J. J. A. M. Brands. Oscillation theorems for second-order functional-differential equations. *Journal of Math. Analysis and Applications*, **63**, pp. 54–64, 1978.
-  T.A.Burton and J.R.Haddock. On solution tending to zero for the equation $x''(t) + a(t)x(t - r(t)) = 0$. *Arch.Math. (Basel)*, 27 (1976), pp. 48-51.
-  T.A.Burton, Stability and periodic solutions of ordinary differential and functional differential equations, Academic Press, Mathematics in Science and Engineering, Vol. 178, 1985.







Bibliography (cont.)

-  T.A.Burton, *Stability by fixed point theory for functional differential equations*, Dover Publications, Mineola, New York, 2006.
-  C.Corduneanu. *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.
-  G.A.Derfel and S.A.Molchanov, On T.Kato's problem regarding bounded solutions of differential-functional equations. *Funktsional'nyi Analiz i ego Prilozheniya*, vol. 24, 1990, pp. 67-69.
-  Yu.F.Dolgii and S.G.Nikolaev. Instability of a periodic delay system, *Differentsial'nye Uravnenija*, 1998, V. 34, pp.463-468.
-  A.Domoshnitsky, Extension of Sturm's theorem to apply to an equation with time-lag, *Differ. Uravn.*, 19, 1099-1105 (1983). Translation from *Differentsial'nye Uravnenia*, vol. 19(1983), pp.1475-1482.
-  A.Domoshnitsky and M.V.Sheina. Nonnegativity of Cauchy matrix and stability of systems with delay, *Differentsial'nye uravnenija*, 25 , 201-208 (1989).

Bibliography (cont.)

-  A.Domoshnitsky. About applicability of Chaplygin's theorem to one component of the solution vector, *Differentsial'nye uravnenija*, 26, 1699-1705 (1990).
-  A.Domoshnitsky, Unboundedness of solutions and instability of differential equations of the second order with delayed argument, *Differential & Integral Equations*, **14** (2001), no.5, 559-576. MR 2002c:34117. Zbl 0884.34074.
-  A.Domoshnitsky, Sturm's theorem for equation with delayed argument, *Georgian Math. Journal*, 1 , No. 3, 267-276 (1994).
-  A.Domoshnitsky, About asymptotic and oscillation properties of the Dirichlet problem for delay partial differential equations. *Georgian Mathematical Journal*, Vol. 10 , No. 3, 495-502, (2003)
-  A.Domoshnitsky, Wronskian of fundamental system of delay differential equations. *Functional Differential Equations*, vol. 7, 2002, 445-467.
-  A.Domoshnitsky, One approach to analysis of asymptotic and oscillation properties of delay and integral PDE. *Dynamics of Continuous, Discrete & Impulsive Systems. A: Mathematical Analysis*. 12, Number 2 , 263-274 (2005).

Bibliography (cont.)

-  Y.Domshlak. Comparison theorems of Sturm type for first and second order differential equations with sign variable deviations of the argument. *Ukraine Mat.Zh.* 34, 158-163 (1982).
-  Z.Dosla and I.T.Kiguradze. On boundedness and stability of solutions of second order linear differential equations with advanced arguments. *Advances in Mathematical Sciences and Applications, Gakkotosho, Tokyo*, vol. 9, No.1 , 1-24 (1999).
-  A.Elbert. An extension of Milloux's theorem to half-linear differential equations. *EJQTD, Proc. 6-th Coll.Qualitative Theory of Diff. Equations*, 8 , 1-10 (2000).
-  S.V.Eliason. Distance between zeros of certain differential equations having delayed arguments.- *Ann.Math. Pura Appl.*, 106, 273-291 (1975).
-  L.N.Erbe, Q.Kong and B.G.Zhang. *Oscillation theory for functional differential equations*, Dekker, New York/Basel, 1995.
-  L.Erbe, Oscillation criteria for second order nonlinear delay equations, *Canad. Math. Bull.* 16, 49-56 (1973).

Bibliography (cont.)



I.Gyori and F.Hartung, Fundamental solution and asymptotic stability of linear delay differential equations, *Dynamics of Continuous, Discrete and Impulsive Systems*, 13 (2006), No.2, 261-287.



I.Gyori and G.Ladas. *Oscillation theory of delay differential equations*. Clarendon, Oxford, 1991.



P.Hartman. *Ordinary differential equations*, Moscow, Nauka, 1970.



P.Hartman and A.Winter, On non-conservative linear oscillators of low frequency, *Amer.J.Math.*, 70, 529-539 (1948).








L.Hatvani. On stability properties of solutions of second order differential equations. *EJQTD*, Proc. 6-th Coll. Qualitative Theory of Diff. Equations, No. 11 (2000), pp.1-6.








L.Hatvani and L.Stacho. On small solutions of second order differential equations with random coefficients. *Archivum Mathematicum (Brno)*, Tomus 34 (1998), pp. 119-126.







Bibliography (cont.)

-  J.Hofbauer and J. W.-H. So, Diagonal dominance and harmless off-diagonal delays, *Proc.Amer.Math.Soc.*, 128 (2000), 2675-2682.
-  D. V. Izjumova. About boundedness and stability of solutions of nonlinear functional-differential equations of the second order. *Proceedings of the Georgian Academy of Science*, V. **100**, No. 2, 1980, pp. 285–288 (in Russian).
-  G.A.Kamenskii. About asymptotic behavior of solutions of linear differential equations of a second order with delayed argument. *Uchenye Zapiski of Moscow State University*, 165, Mathematics, 7 (1954), pp.195-204.
-  G.A.Kamenskii. About solutions of homogeneous second order differential equations of unstable type with delayed argument. *Trudy of seminars on theory of differential equations with deviating argument*, Moscow, the Patrice Lumumba University, 2 (1963), pp.82-93.
-  I.T.Kiguradze and T.A.Chanturia. *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.








Bibliography (cont.)

-  M.A. Krasnosel'skii, G.M. Vainikko, P.P. Zabreiko, Ja.B. Rutitskii and V.Ja. Stezenko, *Approximate methods for solving operator equations*, Moscow, Nauka, 1969 (in Russian).
-  Yu.V.Komlenko, Sufficient conditions of regularity of periodic problem for Hill's equation with deviating argument. In book: *Mathematical Physics*, Kiev, 1977, pp.5-12.
-  R.G.Koplatadze. *On oscillatory properties of solutions of functional differential equations*. *Memoirs on Differential Equations and Mathematical Physics*, vol.3, Tbilisi, 1994, 180 pp.
-  R.G.Koplatadze, Criteria for the oscillation of solutions of differential inequalities and second-order equations with retarded argument, *Tbiliss.Gos. Univ. Prikl. Mat. Trudy*, 17(1986), 104-121.
-  M.G.Krein. About some problems on maximum and minimum for the characteristic numbers and Lyapunov's zones of stability, *Applied Mathematics and Mechanics*, 15 (1951), 323-348.








Bibliography (cont.)

-  S.M.Labovskii. About properties of fundamental system of solutions of a second order differential equations with delayed argument. *Trudy TIHM*, vol. 6, 1971, pp. 49-52 (in Russian).
-  S.M.Labovskii. Condition of nonvanishing of Wronskian of fundamental system of linear equation with delayed argument. *Differentsial'nye Uravnenia*, 1974, V. 10, No. 3, pp. 426–430..
-  G.S.Ladde, V.Lakshmikantham and B.G.Zhang. *Oscillation theory of differential equations with deviating argument*. Dekker, New York/Basel, 1987.
-  A. Z. Manitius and A. W. Olbrot, Finite spectrum assignment problem for systems with delays, *IEEE Trans. Automat. Control* 24:4 (1979), 541–553.
-  H.Milloux. Sur l'equation differentielle $x'' + A(t)x = 0$, *Prace Mat. – Fiz.*, 41(1934), pp. 39 – 54.
-  A. D. Myshkis. *Linear differential equations with delayed argument*. Moscow, Nauka, 1972, 352 p.

Bibliography (cont.)

-  S.B.Norkin, *Differential equations of the second order with retarded argument*. Transl. Math. Monographs, vol. 31, Amer. Math. Soc., Providence, 1972.
-  E.Ott, C.Grebogi, J.A.Yorke. Controlling chaos. *Phys. Rev. Lett.* 64,1196 (1990).
-  D.V.Paatashvili, About unbounded solutions of linear differential equations of second order with delay argument, *Differentsial'nye Uravnenia*, **25** (1989), 774-780. MR 90d:34141. Zbl 0695.34069.
-  P.Parmananda, R.Madrigal M.Rivera, L.Nyikos, Z.Kissand V.Gaspar. Stabilization of unstable steady states and periodic orbits in an electrochemical system using delayed-feedback control. *Phys. Rev. E* 59, 5266 (1999).
-  M.Pinto. Nonlinear delay-differential equations with small lag. *Internat.J.Math. & Math. Sci.*, Vol. 20, 1997, pp. 137-146.
-  K.Pyragas. Control of chaos via an unstable delayed feedback controller, *Phys. Rev. Lett.*, vol.86, No 11, March 2001, pp.2265-2268.
-  K.Pyragas and Tamasevicius. Experimental control of chaos by delayed self-controlling feedback. *Phys. Lett. A* 180, 99 (1993).

Bibliography (cont.)

-  E.Yu.Romanenko and A.N.Sharkovskii, *Asymptotic behavior of solutions of linear functional-differential equations*. Akad. Nauk of Ukraina, Inst.Math.,Kiev, 1978.
-  N. V. Shevelo. *Oscillation of solutions of differential equations with retarded argument*. Kiev, Naukova dumka, 1978.
-  M.G.Shmul'yan. On the oscillating solutions of a linear second order differential equation with retarding argument. *Differentsial'nye uravnenia*, v.31, 1995, 622-629.
-  A.L.Skubachevskii. Oscillating solutions of a second order homogeneous linear differential equation with time-lag. *Differentsialnye uravnenia*, 11, 462-469 (1975).
-  J.E.S.Socular, D.W.Sukow and D.J.Gauthier. Stabilizing unstable periodic orbits in fast dynamical systems. *Phys. Rev. E*50, 3245 (1994).
-  P.Waltman, A note on an oscillation criterion for an equation with functional argument, *Canad. Math. Bull.* 11 , 593-595 (1968).
-  J.J.Wei, Oscillation of second order delay differential equation, *Ann. Differential Equations*, 4 , No. 4, 473-478 (1988).

Bibliography (cont.)



J.S.W.Wong, Second order oscillation with regarded arguments, Proc. Conf. Math. Res.Center, Naval Res. Lab., Washington, D.C. (1971), 581-596, Academic Press, New York, 1972.



V.A.Yakubovich. About boundedness of solutions of the equation $y'' + p(t)y = 0$, $p(t + \omega) = p(t)$. DAN 74, No. 5,(1950), 901-903.



V.A. Yakubovich and V.M. Starzhinski .Lineinye differentsialnye uravneniya s periodicheskimi koeffitsientami i ikh prilozheniya, *Izdat.Nauka* , Moscow, 1972.



J.Yan, Oscillatory property for second order linear delay differential equations, *JMAA* 122 , No. 2, 380-384 (1987).



N.E.Zhukovskii. *Complete collected works, Common Mechanics*, Moscow-Leningrad, 1937, 636 pp.(in Russian).