Heteroclinics for some third order differential equations

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This talk is based on this paper:

D. Bonheure, J. A. Cid, C. De Coster and L. Sanchez, Heteroclinics for some non autonomous third order differential equations, *to appear in Topol. Methods Nonlinear Anal.*

The existence of heteroclinic orbits for the third order problem

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arises for instance in the study of regularization of the Cauchy problem for the one-dimensional hyperbolic conservation law

$$u_t + g(u)_x = 0, \quad u(0, x) = \overline{u}(x).$$

It is known that the single shock wave joining the two states u_- (on the left) and u_+ (on the right)

$$u(t,x) := \begin{cases} u_- & \text{for } x < \lambda\, t, \\ u_+ & \text{for } x > \lambda\, t, \end{cases}$$

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$$g(u_+) - g(u_-) = \lambda\,(u_+ - u_-).$$

However weak solutions are in general not unique. A way to regularize the problem is to search for weak solutions which are limits as $\varepsilon \to 0^+$ of solutions of

$$u_t^\varepsilon + g(u^\varepsilon)_x = \varepsilon A(u^\varepsilon), \quad u^\varepsilon(0, x) = \overline{u}(x),$$

where A is a differential operator of higher order in x (the viscosity).

A choice of A is admissible, in the sense of Gelfand, if shock wave solutions can be obtained as limits of solutions of

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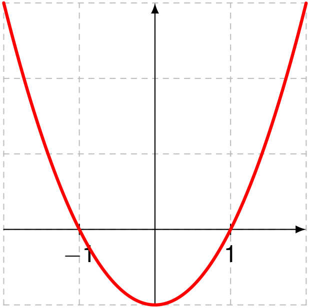
When A is a perfect derivative the admissibility is equivalent to the existence of a heteroclinic connection between u_- and u_+.

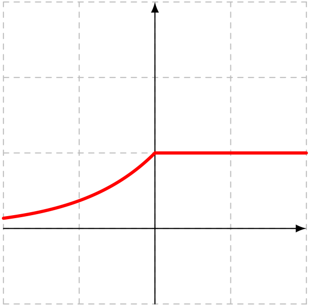
In particular, the question of the admissibility of operator $A(u) = -u_{xxxx}$ leads to problem

$$u''' = f(u), \quad u(-\infty) = u_-, \quad u(+\infty) = u_+,$$

$$u''' = f(u) + p(t)u', \quad u(-\infty) = -1, \quad u(+\infty) = 1.$$



$f(u)$



$0 \leq p(t) \leq M$

Solvability for

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C. Conley, *Isolated invariant sets and the Morse index*, C.B.M.S. **38**, Amer. Math. Soc., Providence 1978.

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J.F. Toland, *Existence and uniqueness of heteroclinic orbits for the equation* $\lambda u''' + u' = f(u)$, Proc. Royal Soc. Edinburgh **109A** (1988), 23-36.

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(p) p is continuous and $\exists M > 0$ such that $0 \leq p(t) \leq M$.

Proposition

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1. *If* u *is a solution in* $\mathbb{R}$, *bounded together with* pu', *then, for* $i \in \{1, 2, 3\}$, $u^{(i)}(\pm\infty) = 0$, $u(+\infty) = a^+$ *and* $u(-\infty) = a^-$ *with* $f(a^\pm) = 0$.

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2. *If in addition* u *is non constant and*

$$\forall x \in [-\|u\|_\infty, \|u\|_\infty] \setminus \{\pm 1\}, \quad f(x)(x^2 - 1) > 0,$$

then $u(-\infty) = -1$ *and* $u(+\infty) = 1$.

Solvability under symmetry

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($h1$) There exists $N_0>1$ such that

$$\forall u\in[0,N_0]\setminus\{1\},\ f(u)(u-1)>0 \quad \text{and} \quad \int_0^{N_0} f(u)du\ge 0.$$

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(s') p is even.

STRATEGY to solve the problem

$$u''' = f(u) + p(t)u', \quad u(-\infty) = -1, \quad u(+\infty) = 1$$

under SYMMETRY.

Solve the following problem in the half-line

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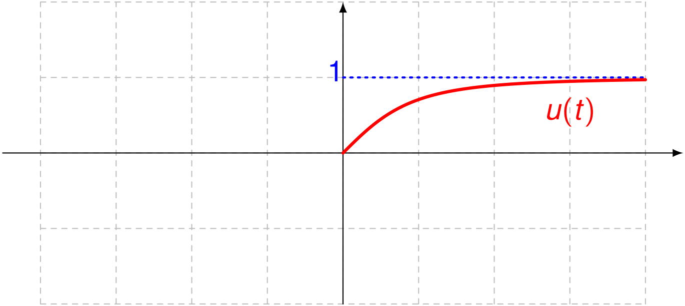

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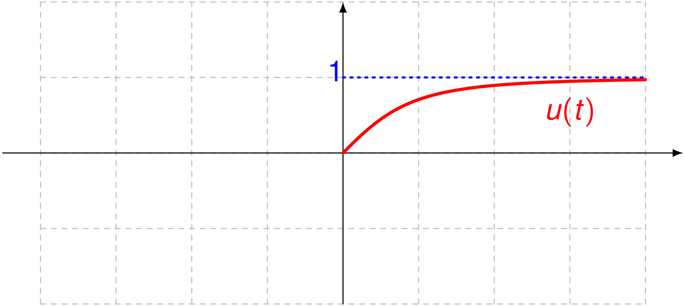

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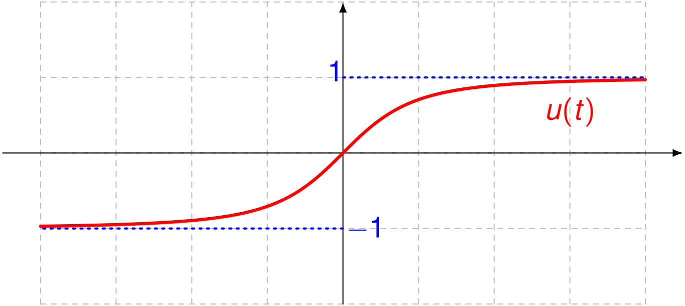


and take the ODD extension.

Theorem

Assume that hypotheses $(f1)$, (p), $(h1)$, (s) *and* (s') *hold. Then*

$$u''' = f(u) + p(t)u', \quad u(-\infty) = -1, \quad u(+\infty) = 1,$$

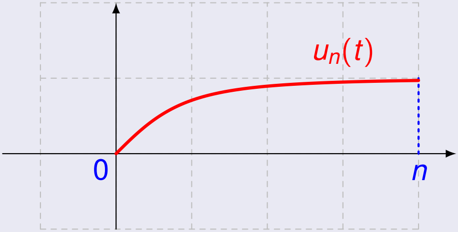
has a odd solution $u \in \mathcal{CB}^3(\mathbb{R})$ *which is nonnegative in* $]0, +\infty[$ *and satisfies*

$$u'(\pm\infty) = u''(\pm\infty) = u'''(\pm\infty) = 0.$$

Proof.

Claim 1. The BVP has a solution u_n for each $n \in \mathbb{N}$:

$$u''' = f(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u'(n) = 0.$$



Proof.

Step 1.- The modified problem.

We define the function $f^* : \mathbb{R} \to \mathbb{R}$ as

$$f^*(u) = \begin{cases} f(N_0), & \text{if } u > N_0, \\ f(u), & \text{if } u \in [0, N_0], \\ f(0), & \text{if } u < 0, \end{cases}$$

and consider the modified problem

$$u''' = f^*(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u'(n) = 0.$$

Proof.

Step 2.- Reduction to a fixed point problem.

For each $h \in \mathcal{C}([0,n])$, the linear problem

$$u''' - p(t)u' = h(t), \quad u(0) = u''(0) = 0, \quad u'(n) = 0, \qquad (1)$$

has a unique solution Ku.

Proof.

Step 2.- Reduction to a fixed point problem.

Then let $S : \mathcal{C}([0, n]) \to \mathcal{C}([0, n])$ *be given by*

$$Su = K(f^*(u)).$$

We consider the homotopy

$$u = K(\lambda f^*(u)), \quad \lambda \in [0, 1],$$

which is equivalent to the problem

$$u''' = \lambda f^*(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u'(n) = 0.$$

Proof.

Step 3.- A priori estimates.

For all $\lambda \in [0, 1]$, *any solution of*

$$u''' = \lambda f^*(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u'(n) = 0.$$

is nonnegative on $[0, n]$.

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For any $n \in \mathbb{N}$*,* $\lambda \in [0,1]$ *and any solution* u *of*

$$u''' = \lambda f^*(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u'(n) = 0.$$

we have,

$$\text{for all } t \in [0,n], \quad |u(t)| \leq N_0.$$

Proof.

Step 4.- Conclusion.

By standard results of Leray-Schauder degree theory the equation has a solution for $\lambda = 1$*, that is, there exists a solution of*

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$$u''' = f^*(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u'(n) = 0.$$

Moreover we have that $0 \leq u \leq N_0$ *and hence it is also a solution of*

$$u''' = f(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u'(n) = 0,$$

□

Proof.

Claim 2. There exists a number $K > 0$ *with the property that, for all* $n \in \mathbb{N}$,

$$\|u_n\|_{\mathcal{C}^3([0,n])} \leq K.$$

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We first show that $\|u_n''\|_{L^2(0,n)}$ is bounded independently of n. Multiplying the equation by u_n' and integrating by parts between

$$\int_0^n u_n''^2(s)ds = -\int_0^n f(u_n(s))u_n'(s)ds - \int_0^n p(s)u_n'^2(s)\,ds$$

$$\leq -\min_{[0,N_0]} F.$$

Proof.

Let us extend u_n *to* $[0, +\infty[$ *with the constant value* $u_n(n)$ *in* $[n, +\infty[$*, and define* v_n *as the odd extension of* u_n *to* $\mathbb{R}$*.*

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Then $v_n \in \mathcal{C}^1(\mathbb{R})$ *and by the Gagliardo-Nirenberg's interpolation inequality, there is a constant* C *such that*

$$\|v_n'\|_{\mathcal{C}(\mathbb{R})} \leq C\|v_n''\|_{L^2(\mathbb{R})}^{2/3}\|v_n\|_{\mathcal{C}(\mathbb{R})}^{1/3}.$$

Proof.

Since

$$\|v_n'\|_{\mathcal{C}(\mathbb{R})} = \|u_n'\|_{\mathcal{C}([0,n])}, \quad \|v_n''\|_{L^2(\mathbb{R})} = 2\|u_n''\|_{L^2(0,n)},$$

$$\|v_n\|_{\mathcal{C}(\mathbb{R})} = \|u_n\|_{\mathcal{C}([0,n])},$$

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$$\|v_n\|_{\mathcal{C}(\mathbb{R})} = \|u_n\|_{\mathcal{C}([0,n])},$$

we infer

$$\sup_n \|u_n'\|_{\mathcal{C}([0,n])} < \infty$$

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$$\|v_n\|_{\mathcal{C}(\mathbb{R})} = \|u_n\|_{\mathcal{C}([0,n])},$$

we infer

$$\sup_n \|u_n'\|_{\mathcal{C}([0,n])} < \infty$$

and the differential equation yields

$$\sup_n \|u_n'''\|_{\mathcal{C}([0,n])} < \infty.$$

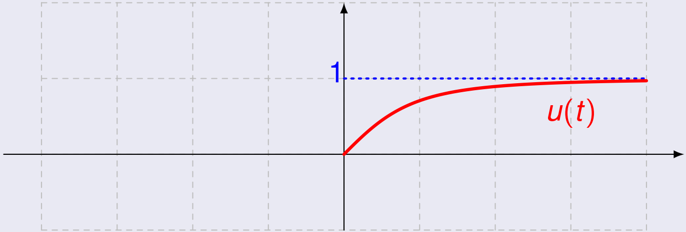
So, the claim follows from standard interpolation.

Proof.

Claim 3. Passing to the limit, the boundary value problem

$$u''' = f(u) + p(t)u', \quad u(0) = u''(0) = 0, \quad u(+\infty) = 1$$

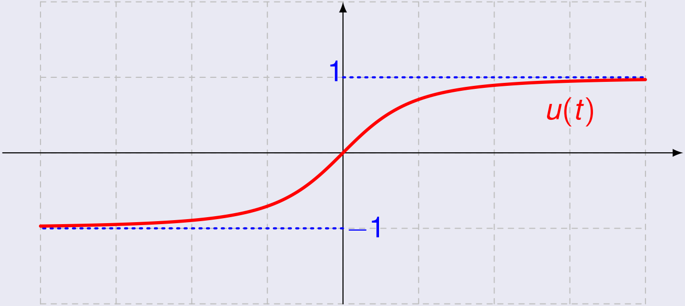
has a solution $u \in \mathcal{C}^3([0, +\infty[)$ *which is nonnegative on* $[0, +\infty[$ *and such that* u', u'' *and* u''' *are bounded in* $\mathbb{R}^+$.



Proof.

Claim 4. Extending the solution by symmetry we get an odd solution $u \in \mathcal{CB}^3(\mathbb{R})$ *which is nonnegative in* $]0, +\infty[$ *and satisfies*

$$u'(\pm\infty) = u''(\pm\infty) = u'''(\pm\infty) = 0.$$



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$(h2)$ There exist $\alpha < -1$ and $\beta > 1$ such that,

$$\forall u \in [\alpha, \beta] \setminus \{-1, 1\}, \; f(u)(u^2 - 1) > 0,$$
$$F(\beta) = F(-1) \quad \text{and} \quad F(\alpha) = F(1);$$

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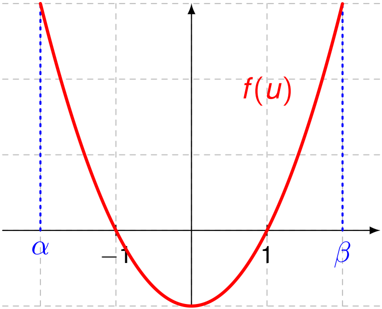

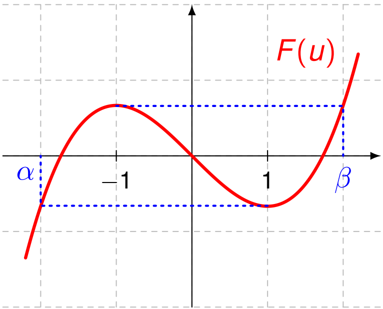

Theorem

Assume that hypotheses $(f1)$, (p) *and* $(h2)$ *hold. Then*

$$u''' = f(u) + p(t)u', \quad u(-\infty) = -1, \quad u(+\infty) = 1,$$

has a solution $u \in \mathcal{CB}^3(\mathbb{R})$ *which satisfies*

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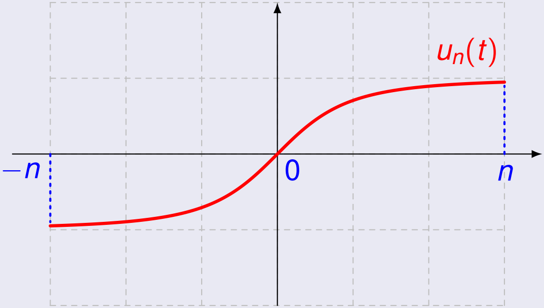

Proof. (Step 1.1. The modified problem)

We define the functions f_+, $f_- : [-n, n] \times \mathbb{R} \to \mathbb{R}$ *by*

$$f_+(u) = \begin{cases} f(\beta), & \text{if } u > \beta, \\ f(u), & \text{if } u \in [-1, \beta], \\ 0, & \text{if } u < -1, \end{cases}$$

Proof. (Step 1.1. The modified problem)

We define the functions f_+, f_- : $[-n, n] \times \mathbb{R} \to \mathbb{R}$ *by*

$$f_-(u) = \begin{cases} 0, & \text{if } u > 1, \\ f(u), & \text{if } u \in [\alpha, 1], \\ f(\alpha), & \text{if } u < \alpha. \end{cases}$$

Proof. (Step 1.1. The modified problem)

Then we set

$$f^*(t,u) = \begin{cases} f_+(u), & \text{if } t \geq 0, \\ f_-(u), & \text{if } t < 0. \end{cases}$$

and we consider then the modified problem

$$u''' = f^*(t,u) + p(t)u', \quad u'(-n) = 0, \quad u(0) = 0, \quad u'(n) = 0.$$

Proof. (Step 1.2. Reduction to a fixed point problem)

For each $h \in \mathcal{C}([-n, n])$, *the linear problem*

$$u''' - p(t)u' = h(t), \quad u'(-n) = 0, \quad u(0) = 0, \quad u'(n) = 0,$$

has a unique solution K(h).

Proof. (Step 1.2. Reduction to a fixed point problem)

Define the open and bounded set

$$\Omega = \{u \in \mathcal{C}([-n,n]) \mid u(-n) < 1 \textit{ and } u(n) > -1 \\ \textit{and } \; \alpha < u(t) < \beta \quad \forall t \in [-n,n]\}.$$

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and let $S : \overline{\Omega} \to \mathcal{C}([-n,n])$ *be given by*

$$Su = K(f^*(t,u)).$$

Proof. (Step 1.2. Reduction to a fixed point problem)

In order to obtain a fixed point we consider the homotopy

$$u = K(\lambda f^*(t,u)), \quad \lambda \in [0,1],$$

which is equivalent to the problem

$$u''' = \lambda f^*(t,u) + p(t)u', \quad u'(-n) = 0, \quad u(0) = 0, \quad u'(n) = 0.$$

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1. *For* $\lambda = 0$*, the solution* u *is in* Ω*.*
2. *For* $\lambda \in]0,1]$ *and* u *a solution with* $u(-n) < 1$ *and* $u(n) > -1$*, we have,* $\forall t \in [0,n]$*,* $-1 < u(t) < \beta$ *and,* $\forall t \in [-n,0]$*,* $\alpha < u(t) < 1$*.*
3. *For* $\lambda \in]0,1]$*, there is no solution on* $\partial\Omega$*.*

Proof.

Step 2. There exists a number $K > 0$ *with the property that, for all* $n \in \mathbb{N}$ *the solution* u_n *of*

$$u''' = f(u) + p(t)u', \quad u'(-n) = 0, \quad u(0) = 0, \quad u'(n) = 0,$$

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$$\|u_n\|_{\mathcal{C}^3([-n,n])} \leq K.$$

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Step 3. Passing to the limit, the boundary value problem

$$u''' = f(u) + p(t)u', \quad u(-\infty) = -1, \quad u(+\infty) = 1,$$

has a solution $u \in \mathcal{C}^3(\mathbb{R})$ and such that u', u'' and u''' are bounded in $\mathbb{R}$.

($f1$) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(-1) = f(1) = 0$.

(p) p is continuous and $\exists M > 0$ such that $0 \leq p(t) \leq M$.

($h2$) There exist $\alpha < -1$ and $\beta > 1$ such that,

$$\forall u \in [\alpha, \beta] \setminus \{-1, 1\},\ f(u)(u^2 - 1) > 0,$$
$$F(\beta) = F(-1) \quad \text{and} \quad F(\alpha) = F(1);$$

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($h2$) There exist $\alpha<-1$ and $\beta>1$ such that,

$$\forall u\in[\alpha,\beta]\setminus\{-1,1\},\ f(u)(u^2-1)>0,$$
$$F(\beta)=F(-1)\quad\text{and}\quad F(\alpha)=F(1);$$

($h3$) f is nondecreasing on $[0,\beta]$ and nonincreasing on $[\alpha,0]$;

($h4$) f satisfies

$$\int_\alpha^0 F(s)\,ds>0\qquad\text{and}\qquad\int_0^\beta F(s)\,ds<0.$$

Theorem

Suppose that $f : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz on* $[\alpha, \beta]$ *and satisfies* $(f1)$, $(h2)$, $(h3)$ *and* $(h4)$*. In addition assume* p *is a nonnegative constant. Then*

$$u''' = f(u) + p\,u', \quad u(-\infty) = -1, \quad u(+\infty) = 1,$$

has a unique (up to translations) solution $u \in \mathcal{CB}^3(\mathbb{R})$*. Moreover* u *has a unique simple zero and*

$$u'(\pm\infty) = u''(\pm\infty) = u'''(\pm\infty) = 0.$$

MANY THANKS FOR YOUR ATTENTION...

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… AND CONGRATULATIONS MILAN!