EQUATIONS WITH INVOLUTIONS

Alberto Cabada and Adrián F. Tojo

University of Santiago de Compostela,
Santiago de Compostela, Galicia
SPAIN

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Devoted to our friend Milan Tvrdý

Parts of the Talk

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- Motivation and Historical Notes

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- Differential equations with Involutions

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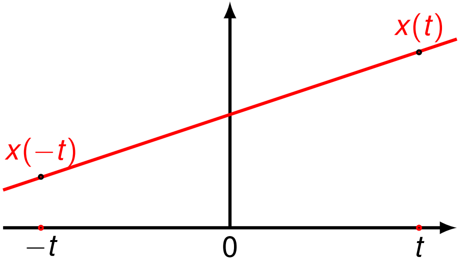
- Constant Sign Green's Functions

Part I

Motivation and Historical Notes

It is clear that, given $a, b \in \mathbb{R}$, the straight line $x(t) = a\,t + b$ satisfies the equation

$$x'(t) = \frac{x(t) - x(-t)}{2t}.$$



However we do not impose that the derivative must be constant. So our natural question is:

Are the straight lines the only solutions of this equation?

To answer this question we take into account the very well known result that any $f : \mathbb{R} \to \mathbb{R}$ can be expressed in a unique way as $f = f_e + f_o$, with

$$f_e(x) := \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) := \frac{f(x) - f(-x)}{2} \quad x \in \mathbb{R}.$$

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It is not difficult to verify the following properties:

1. $(f')_o=f' \iff f=f_e$,
2. $(f')_e=f' \iff f=f_o+c,\ c\in\mathbb{R}$.

Returning to our problem, we can rewrite it as

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Thus

$$(x_o)'(t) + 0 = x'(t) = (x_o)'(t) + (x_e)'(t).$$

or, which is the same:

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As consequence, $x_e(t) = c$, $x_o(t) = k\,t$ with $c, k \in \mathbb{R}$, i.e.,

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So we conclude that the set of solutions of this problem are the straight lines.

R. Figueroa and R. L. Pouso,
Minimal and maximal solutions to second-order boundary value problems with state-dependent deviating arguments,
Bull. Lond. Math. Soc., **43** (2011), 164–174.

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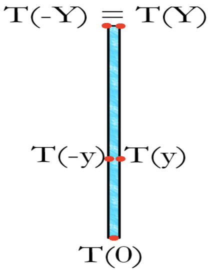
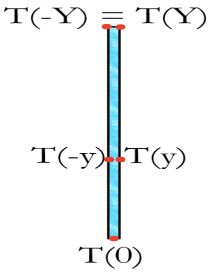
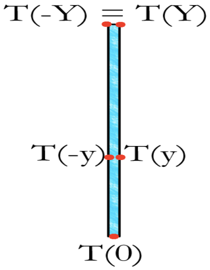
Consider a metal wire around a thin sheet of insulating material in a way that some parts overlap some others as in the figure.



Assuming that the position $Y = 0$ is the lowest of the wire, and the insulation goes up to the left at $-Y$ and to the right up to Y.

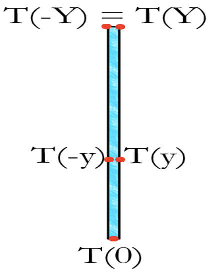




Assuming that the position $Y = 0$ is the lowest of the wire, and the insulation goes up to the left at $-Y$ and to the right up to Y. Traditional heat equation with respect to the wire is

$$\frac{\partial T}{\partial t}(t, y) = \alpha \frac{\partial^2 T}{\partial y^2}(t, y).$$

Problems Arising in Real Phenomena



Assuming that the position $Y = 0$ is the lowest of the wire, and the insulation goes up to the left at $-Y$ and to the right up to Y.
Traditional heat equation with respect to the wire is

$$\frac{\partial T}{\partial t}(t, y) = \alpha \frac{\partial^2 T}{\partial y^2}(t, y).$$

However, given the proximity of the other section of wire, we can add another term to affect the equation:

$$\frac{\partial T}{\partial t}(t, y) = \alpha \frac{\partial^2 T}{\partial y^2}(t, y) + \beta \frac{\partial^2 T}{\partial y^2}(t, -y).$$

After these two examples we are interested in to consider n^{th}-order differential equations that follows the expression

$$x^{(n)}(t) = f(t, x(t), x(-t)), \quad t \in [-T, T].$$

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- They are functional equations.

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$$x^{(n)}(t) = f(t, x(t), x(-t)), \quad t \in [-T, T].$$

It is important to note the following facts:

- They are functional equations.
- They are neither equations with delay nor advance.

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Definition

Let $A \subset \mathbb{R}$, a function $f : A \to A$ such that $f \circ f = \mathrm{Id}$ is called an involution.

A. Cabada, G. Infante and F. A. F. Tojo,
Nontrivial solutions of perturbed Hammerstein integral equations with deviated arguments and applications,
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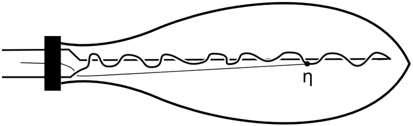

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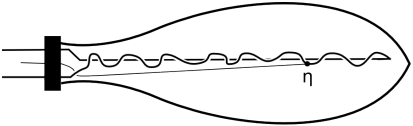

$$u''(t) + g(t)f(t, u(t), u(\varphi(t))) = 0, \ t \in (0,1), \qquad \varphi \circ \varphi = Id.$$

Definition

Let $A \subset \mathbb{R}$, $f : A \to A$, $k \in \mathbb{N}$, $k \geq 2$. We say f is an involution of order n if

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1. $f^n \equiv f \circ \overset{n}{\cdots} \circ f = \text{Id}$,
2. $f^k \neq \text{Id} \quad \forall k = 1, \dots, n-1.$

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1. $f:\mathbb{R}\to\mathbb{R},\ f(x)=-x$ is an involution known as reflection.
2. $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}\backslash\{0\},\ f(x)=1/x$ known as inversion.
3. Let $a,b,c\in\mathbb{R}$, $c\,b+a^2\neq 0$, $c\neq 0$,

$$f:\mathbb{R}\backslash\left\{\frac{a}{c}\right\}\to\mathbb{R}\backslash\left\{\frac{a}{c}\right\},\ f(x)=\frac{ax+b}{cx-a}$$

is a family of involutions known as bilinear involutions.

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- If $A \subset \mathbb{R}$ is connected and $f : A \to A$ is a continuous involution, then f is decreasing and has a unique fixed point.
- The only continuous involutions defined in connected subsets of $\mathbb{R}$ are of order 2.

Historical Notes

The study of functional differential equations with involutions can be traced back to the solution of the inversion equation $x'(t) = x(1/t)$ by Silberstein in 1940.

Silberstein, Ludwik.
Solution of the Equation $f'(x) = f(1/x)$.
Philos. Mag. 7:30 (1940), pp 185-186.

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Wiener proves that the solutions of the Silberstein equation solve the second order singular ordinary differential equation $t^2x''(t) + x(t) = 0$.

Wiener, Joseph.
Differential equations with involutions.
Differensial'nye Uravneniya, 5, (1969), 1131-1137.

On the other hand, by defining $y(t) = x(e^t)$, we conclude that x is a solution of the inversion Silberstein equation if and only if y solves the reflection equation$y'(t) = e^{-t}y(-t)$.

Šarkovskiĭ shows that they have some applications to the stability of differential – difference equations.

Šarkovskiĭ, Alexander N.
Functional-differential equations with a finite group of argument transformations. (Russian)
Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 157, (1978), 118-142.

Moreover this kind of equations has some interesting properties by itself.

Moreover this kind of equations has some interesting properties by itself. In fact it is not difficult to verify that the unique solution of the homogeneous harmonic oscillator

$$x''(t) + m^2\,x(t) = 0,$$

coupled with the initial conditions

$$x(0) = x_0, \quad x'(0) = -m\,x_0,$$

for any $x_0 \in \mathbb{R}$, is the unique solution of the first order equation with reflection

$$x'(t) + m\,x(-t) = 0, \quad x(0) = x_0$$

and vice-versa.

Historical Notes

Wiener and Watkins study the solution of the equation $x'(t) - a\,x(-t) = 0$ with initial conditions.

Wiener Joseph; Watkins, Will.
A Glimpse into the Wonderland of Involutions.
Missouri J. Math. Sci. 14 (2002), 3, 175-185.

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Missouri J. Math. Sci. 14 (2002), 3, 175-185.

Equation $x'(t) + a\,x(t) + b\,x(-t) = g(t)$ has been treated in

Piao, Daxiong
Pseudo almost periodic solutions for differential equations involving reflection of the argument.
J. Korean Math. Soc. 41 (2004), 4, 747-754.

Piao, Daxiong
Periodic and almost periodic solutions for differential equations with reflection of the argument.
Nonlinear Anal. 57 (2004), 4, 633-637.

In the following papers some results are introduced to transform this kind of problems with involutions and initial conditions into second order ordinary differential equations with initial conditions or first order two dimensional systems, granting that the solution of the last will be a solution to the first.

Kuller, Robert G.
On the differential equation $f' = f \circ g$, where $g \circ g = I$.
Math. Mag. 42 1969 195-200.

Shah, S. M.; Wiener, Joseph.
Reducible functional-differential equations.
Internat. J. Math. Math. Sci. 8 (1985), 1-27.

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Wiener, Joseph.
Generalized solutions of functional-differential equations.
World Scientific Publishing Co., Inc., River Edge, NJ, 1993.

Watkins, Will.
Modified Wiener Equations.
Int. J. Math. Math. Sci. 27:6 (2001), pp 347-356.

Second order boundary value problems have been considered for Dirichlet and Sturm-Liouville boundary value conditions in

Gupta, Chaitan P.
Existence and uniqueness theorems for boundary value problems involving reflection of the argument.
Nonlinear Anal. 11 (1987), 9, 1075-1083.

Gupta, Chaitan P.
Two-point boundary value problems involving reflection of the argument.
Internat. J. Math. Math. Sci. 10 (1987), 2, 361-371.

O'Regan, Donal; Zima, Miroslawa.
Leggett-Williams norm-type fixed point theorems for multivalued mappings.
Appl. Math. Comput. 187 (2007), 2, 1238-1249.

Higher order equations has been studied in

O'Regan, Donal.
Existence results for differential equations with reflection of the argument.
J. Austral. Math. Soc. Ser. A 57 (1994), 2, 237-260.

Part II

DIFFERENTIAL EQUATIONS WITH INVOLUTIONS

Despite all this progression of studies and to the best of our knowledge, the case of first order differential equations with involution and periodic boundary value conditions has been disregarded so far.

In this talk we will present some of the results obtained in

F. A. F. Tojo, A. C.
Comparison results for first order linear operators with reflection and periodic boundary value conditions,
Nonlinear Anal., **78** (2013), 32–46.

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F. A. F. Tojo, A. C.
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F. A. F. Tojo, A. C.
Existence results for a linear equation with reflection, non-constant coefficient and periodic boundary conditions.
J. Math. Anal. Appl. **412** (2014), 529–546.

Let us consider the first order equation with involution

$$x'(t) = f(x(\varphi(t))), \quad x(c) = x_c, \tag{1}$$

and the second order ordinary differential equation

$$x''(t) = f'(f^{-1}(x'(t)))f(x(t))\varphi'(t), \quad x(c) = x_c, \; x'(c) = f(x_c). \tag{2}$$

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LEMMA

Let $f:\mathbb{R}\to\mathbb{R}$ be a diffeomorphism.

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Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism. Let $\varphi \in \mathcal{C}^1((a, b))$ be an involution and c be a fixed point of φ. Then x is a solution of (1) if and only if x is a solution of (2).

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Since differentiating (1) we get

$$x''(t) = f'(x(\varphi(t)))\, x'(\varphi(t))\, \varphi'(t)$$

and taking into account that $x'(\varphi(t)) = f(x(t))$ by (1), we obtain (2).

Proof.

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Integrating from c to t we have,

$$f^{-1}(x'(t)) - x_c = f^{-1}(x'(t)) - f^{-1}(x'(c)) = \int_c^t f(x(s))\varphi'(s)ds$$

Proof.

Defining $g(s) := f(x(\varphi(s))) - x'(s)$, we conclude that

$$\begin{aligned} x'(t) &= f\left(x_c + \int_c^t f(x(s))\varphi'(s)ds\right) \\ &= f\left(x(\varphi(t)) + \int_c^t (f(x(s)) - x'(\varphi(s)))\varphi'(s)ds\right) \end{aligned}$$

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One can verify, by Grönwall's Lemma, that $g(t) = 0$ and hence (1) is satisfied. □

Example

Notice that, as an immediate consequence of this result, we have that the unique solution of the equation

$$x''(t) = -\sqrt{1+(x'(t))^2}\,\sinh x(t), \quad x(0) = x_0,\ x'(0) = \sinh x_0,$$

coincide with the unique solution of

$$x'(t) = \sinh x(-t), \quad x(0) = x_0.$$

The second order ordinary differential equation (2) can be rewritten as the f^{-1} – laplacian equation

$$\frac{d}{d\,t}\left(f^{-1}(x'(t))\right) = \varphi'(t)\, f(x(t)), \quad x(c) = x_c,\ x'(c) = f(x_c).$$

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Recently, we have extended this result to a wider set of functions, not necessarily diffeomorphisms. In particular, the result is valid for the $p-$ Laplacian operator

$$f(x) = |x|^{p-2}\,x, \quad x \in \mathbb{R}, \qquad p > 1.$$

Previous Lemma can be extended, with a very similar proof, to the case with periodic boundary value conditions.

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Let us consider the problems

$$x'(t) = f(x(\varphi(t))), \quad x(a) = x(b) \tag{3}$$

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$$x''(t) = f'(f^{-1}(x'(t)))f(x(t))\varphi'(t), \quad x(a) = x(b) = f(x'(a)). \tag{4}$$

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Let $[a, b] \subset \mathbb{R}$ and let $f\mathbb{R} \to \mathbb{R}$ be a diffeomorphism. Let $\varphi \in \mathcal{C}^1([a, b])$ be an involution such that $\varphi([a, b]) = [a, b]$.

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Lemma

Let φ and ψ be two differentiable involutions on the intervals I_1 and I_2 respectively. Let t_0 and s_0 be the unique fixed points of φ and ψ respectively.

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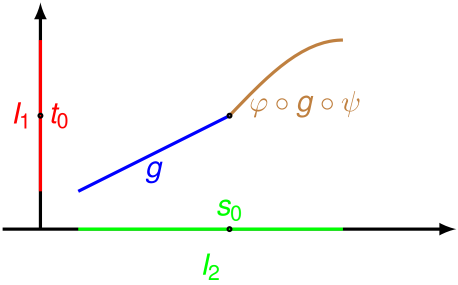
The f described in the lemma can be taken as follows:
Let $g : [\inf I_2, s_0] \to [\inf I_1, t_0]$ be an orientation preserving diffeomorphism, that is, $g(s_0) = t_0$.
Let us define

$$f(s) := \begin{cases} g(s) & \text{if } s \in [\inf I_2, s_0], \\ (\varphi \circ g \circ \psi)(s) & \text{if } s \in (s_0, \sup I_2]. \end{cases}$$

Correspondence of Involutions



COROLLARY

Under the hypothesis of previous Lemma, the problem

$$\begin{aligned} d(t)x'(t)+c(t)x'(\varphi(t))+b(t)x(t)+a(t)x(\varphi(t)) &= h(t), \\ x(\inf I_1) &= x(\sup I_1) \end{aligned}$$

is equivalent to

$$\begin{aligned} \frac{d(f(s))}{f'(s)}y'(s)+\frac{c(f(s))}{f'(\psi(s))}y'(\psi(s))+b(f(s))y(s)+a(f(s))y(\psi(s)) &= h(f(s)), \\ y(\inf I_2) &= y(\sup I_2). \end{aligned}$$

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This result is clear by making the change of variables $t=f(s)$ and $y(s):=x(t)=x(f(s))$.

This correspondence allows us to study only one kind of involutions and adapt the obtained results to the other cases.

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So we will concentrate our attention on the reflection functional $\varphi(t) = -t$.

Part III

CONSTRUCTION OF THE GREEN'S FUNCTION

We will start by finding the solution of the simplest first order reflection equation $L_m\,x(t) = x'(t) + m\,x(-t) = h(t)$ with periodic boundary value conditions and then establish some properties of the solution.

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On the contrary to the majority of the previous mentioned papers, our approach consists on to study directly the first order functional equation and obtain the expression of the related Green's function.

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Sarah Post, Luc Vinet and Alexei Zhedanov,
Supersymmetric Quantum Mechanics with Reflections,
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$$\begin{aligned} x''(t) + m^2\, x(t) &= f(t), \quad t \in [-T, T] \equiv I, \\ x(T) - x(-T) &= 0, \\ x'(T) - x'(-T) &= \lambda \end{aligned}$$

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The solution is unique whenever $m^2 \neq (\frac{k\pi}{T})^2$, $k \in \mathbb{N}$. We will assume uniqueness conditions from now on.

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5. $G(T,s) = G(-T,s) \;\; \forall s \in I,$
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Consider the problem

$$\begin{aligned} &x'(t) + m\,x(-t) = h(t), \quad t \in [-T, T] \\ &x(T) - x(-T) = 0, \end{aligned} \tag{5}$$

where m is a real non-zero constant, $T > 0$ and $h \in L^1(I)$.

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If h is differentiable, by direct differentiation one can verify that any solution of the previous problem solves the second order ODE with boundary conditions

$$\begin{aligned} x''(t) + m^2\,x(t) &= h'(t) + m\,h(-t), \quad t \in I, \\ x(T) - x(-T) &= 0, \\ x'(T) - x'(-T) &= h(T) - h(-T) \end{aligned}$$

As consequence, we know that, under the regularity assumptions on h, the solutions of the first order reflection equation (5) are given by the following expression

$$x(t) = \int_{-T}^{T} G(t,s)(h'(s) + m\,h(-s))ds + G(t,-T)\,[h(T) - h(-T)]$$

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After integration by parts, by using the properties of the Green's function G and the density of the $\mathcal{C}^1(I)$ functions in $L^1(I)$, we arrive to the following expression for the Green's function related to the first order problem with reflection (5)

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THEOREM

Suppose that $m \neq k\,\pi/T$, $k \in \mathbb{Z}$. Then problem (5) has a unique solution given by the expression

$$u(t) := \int_{-T}^{T} \overline{G}(t,s)h(s)ds,$$

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where

$$\overline{G}(t,s) := m\,G(t,-s) - \frac{\partial G}{\partial s}(t,s)$$

is called the Green's function related to problem (5).

Example

It is not difficult to verify that the Green's function G related to the second order periodic boundary value problem

$$\begin{aligned} x''(t) + m^2\, x(-t) &= h(t), \quad t \in [-T, T] \\ x(T) - x(-T) &= 0, \\ x'(T) - x'(-T) &= 0, \end{aligned}$$

follows the expression

$$2m\sin(mT)G(t,s) = \begin{cases} \cos m(T+s-t) & \text{if} \quad s \le t, \\ \cos m(T-s+t) & \text{if} \quad s > t. \end{cases}$$

Example

Therefore,

$$2\sin(mT)\overline{G}(t,s)=\begin{cases}\cos m(T-s-t)+\sin m(T+s-t) & \text{if} \quad t>|s|,\\ \cos m(T-s-t)-\sin m(T-s+t) & \text{if} \quad |t|<s,\\ \cos m(T+s+t)+\sin m(T+s-t) & \text{if} \quad -|t|>s,\\ \cos m(T+s+t)-\sin m(T-s+t) & \text{if} \quad t<-|s|.\end{cases}$$

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Corollary

Suppose that $m \neq k\pi/T$, $k \in \mathbb{Z}$. Then the problem

$$x'(t) + m\,x(-t) = h(t), \quad t \in I, \qquad x(-T) - x(T) = \lambda,$$

with $\lambda \in \mathbb{R}$ has a unique solution given by the expression

$$u(t) := \int_{-T}^{T} \overline{G}(t,s)h(s)ds + \lambda \overline{G}(t,-T).$$

If we consider the problem with constant coefficients

$$x'(t) + a\,x(-t) + b\,x(t) = h(t), t \in I; \quad x(-T) = x(T), \qquad (6)$$

where $a, b \in \mathbb{R}$, $a \neq 0$.

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where $a, b \in \mathbb{R}$, $a \neq 0$.

Considering the homogeneous case ($h \equiv 0$) we can reduce it, by differentiating and making substitutions, to the second order ODE problem,

$$x''(t) + (a^2 - b^2)x(t) = 0, \quad x(T) = x(-T), \quad x'(T) = x'(-T). \tag{7}$$

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The Green's function for problem (7) satisfies, changing $\pm\omega^2$ by $a^2 - b^2$, the same properties as G for $a \neq \pm b$.

Theorem

Suppose that $a^2 - b^2 \neq n^2\,(\pi/T)^2$, $n = 0, 1, \ldots$ *Then problem* (6) *has a unique solution given by the expression*

$$u(t) := \int_{-T}^{T} \overline{G}(t,s)h(s)ds,$$

where

$$\overline{G}(t,s) := a\,G(t,-s) - b\,G(t,s) + \frac{\partial G}{\partial t}(t,s) \tag{8}$$

*is called the **Green's function** related to problem* (6).

We center our attention on the first order equation with non constant coefficients

$$d(t)x'(t)+c(t)x'(-t)+b(t)x(t)+a(t)x(-t)=h(t),\; x(-T)=x(T). \tag{9}$$

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Then, the solutions of equation (9) satisfy

$$\Lambda\begin{pmatrix}x_o'\\ x_e'\end{pmatrix}=\begin{pmatrix}a_o-b_o & -a_e-b_e\\ a_e-b_e & -a_o-b_o\end{pmatrix}\begin{pmatrix}x_o\\ x_e\end{pmatrix}+\begin{pmatrix}h_e\\ h_o\end{pmatrix},$$

where

$$\Lambda=\begin{pmatrix}c_e+d_e & d_o-c_o\\ c_o+d_o & d_e-c_e\end{pmatrix}.$$

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So we can assume that $\Lambda=\mathrm{Id}$, that is, $d\equiv 1$ and $c\equiv 0$. Hence, the equation to study is

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and the system consists on

$$\begin{pmatrix} x_o' \\ x_e' \end{pmatrix} = \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix} \begin{pmatrix} x_o \\ x_e \end{pmatrix} + \begin{pmatrix} h_e \\ h_o \end{pmatrix}.$$

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It is a well known result that, if we have a system of linear ODE defined by a matrix M which commutes with its integral, then the solution of the system is given by the exponential of the integral of M.

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We can try to compute the solution of the problem as an exponential, but, under which circumstances can we do this?

Definition

Let $S \subset \mathbb{R}$ be an interval. Define $\mathcal{M} \subset \mathcal{C}^1(\mathbb{R}, \mathcal{M}_{n\times n}(\mathbb{R}))$ such that for every $M \in \mathcal{M}$,

- there exists $P \in \mathcal{C}^1(\mathbb{R}, \mathcal{M}_{n\times n}(\mathbb{R}))$ such that $M(t) = P^{-1}(t)J(t)P(t)$ for every $t \in S$ where $P^{-1}(t)J(t)P(t)$ is a Jordan decomposition of $M(t)$;
- the superdiagonal elements of J are independent of t, as well as the dimensions of the Jordan boxes associated to the different eigenvalues of M;
- two different Jordan boxes of J correspond to different eigenvalues;
- if two eigenvalues of M are ever equal, they are identical in the whole interval S.

Theorem (Kotin and Irving, 1982)

Let $M \in \mathcal{M}$. Then, the following statements are equivalent.

- *M commutes with its derivative.*
- *M commutes with its integral.*
- *M commutes functionally, that is $M(t)M(s) = M(s)M(t)$ for all $t, s \in S$.*
- *$M = \sum_{k=0}^{r} \gamma_k(t) C^k$ For some $C \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $\gamma_k \in \mathcal{C}^1(S, \mathbb{R})$, $k = 1, \dots, r$.*

Furthermore, any of the last properties imply that $M(t)$ has a set of constant eigenvectors, i.e. a Jordan decomposition $P^{-1}J(t)P$ where P is constant.

Let us check when

$$M := \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix}$$

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$$0 = M(t)M(s) - M(s)M(t)$$

$$= 2 \begin{pmatrix} a_e(t)b_e(s) - a_e(s)b_e(t) & a_o(s)[a_e(t) + b_e(t)] - a_o(t)[a_e(s) + b_e(s)] \\ a_o(t)[a_e(s) + b_e(s)] - a_o(s)[a_e(t) + b_e(t)] & a_e(s)b_e(t) - a_e(t)b_e(s) \end{pmatrix}.$$

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We note that if (C1)–(C4) hold, with $k \neq 0$ in case (C1), we deduce that a must be even.

In order to obtain the Green's function for our problem

$$x'(t) + a(t)x(-t) + b(t)x(t) = h(t), \; x(-T) = x(T),$$

when one of the conditions $(C1) - (C5)$ holds.

Let us denote $A(t) := \int_0^t a(s)ds$ and $B(t) := \int_0^t b(s)ds$.

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3. Remember that there is no sign assumptions on a and b.
4. In particular we cannot ensure that A or B are monotone.

Considering the cases, with a even

(C1) $b_e = k\,a,\ k \in \mathbb{R},\ |k| < 1.$

(C2) $b_e = k\,a,\ k \in \mathbb{R},\ |k| > 1.$

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We refer to the constant coefficients problem

$$x'(t)+x(-t)+k\,x(t) = h(t),\quad t \in [-|A(T)|, |A(T)|],\ x(A(T)) = x(-A(T)).$$

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This problem has been completely studied before and so we know that is is uniquely solvable if and only if

(C1*) (C1) is satisfied, $(1-k^2)A(T)^2 \neq (n\pi)^2$ for all $n = 0, 1, \dots$ and $\cos\left(\sqrt{1-k^2}A(T)\right) \neq 0.$

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(C2*) (C2) is satisfied and $(1-k^2)A(T)^2 \neq (n\pi)^2$ for all $n = 0, 1, \dots$

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Assume one of $(C1^*)$–$(C3^*)$. The Green's function G_2 for the constant coefficients problem can be expressed as

$$G_2(t,s) := \begin{cases} k_1(t,s), & t > |s|, \\ k_2(t,s), & s > |t|, \\ k_3(t,s), & -t > |s|, \\ k_4(t,s), & -s > |t|. \end{cases}$$

where the k_i are analytic functions.

Let us define

$$G_1(t,s) := e^{B_e(s)-B_e(t)}\begin{cases} k_1(A(t),A(s)), & t>|s|,\\ k_2(A(t),A(s)), & s>|t|,\\ k_3(A(t),A(s)), & -t>|s|,\\ k_4(A(t),A(s)), & -s>|t|.\end{cases}$$

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Here $B_e(t) := \int_0^t b_e(t)dt$.

Theorem

Assume one of $(C1^*)$–$(C3^*)$. *If* $G_1(t,\cdot)h(\cdot)\in L^1(I)$ *for every* $t\in I$, *then problem*

$$x'(t)+a(t)\,x(-t)+b(t)\,x(t)=h(t),\quad \text{for a. e. } t\in I,$$
$$x(-T)=x(T),$$

has a unique solution given by

$$u(t)=\int_{-T}^{T}G_1(t,s)h(s)ds.$$

Example

Consider the problem

$$x'(t)+\cos(\pi t)x(-t)+\sinh(t)x(t)=\cos(\pi t)+\sinh(t),$$
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Since $b_e(t) = 0$, we have that $b_e = k\,a$, for $k = 0$.

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So, we are in the case (C1).

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If we compute the Green's function, we obtain

$$2\sin(\sin(\pi T))G_1(t,s) = e^{\cosh(s)-\cosh(t)}H(t,s),$$

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$$H(t,s)=\begin{cases}\sin\left(\frac{\sin(\pi s)}{\pi}-\frac{\sin(\pi t)}{\pi}-\frac{\sin(\pi T)}{\pi}\right)+\cos\left(\frac{\sin(\pi s)}{\pi}+\frac{\sin(\pi t)}{\pi}-\frac{\sin(\pi T)}{\pi}\right), & |t|<s,\\ \sin\left(\frac{\sin(\pi s)}{\pi}-\frac{\sin(\pi t)}{\pi}+\frac{\sin(\pi T)}{\pi}\right)+\cos\left(\frac{\sin(\pi s)}{\pi}+\frac{\sin(\pi t)}{\pi}+\frac{\sin(\pi T)}{\pi}\right), & |t|<-s,\\ \sin\left(\frac{\sin(\pi s)}{\pi}-\frac{\sin(\pi t)}{\pi}+\frac{\sin(\pi T)}{\pi}\right)+\cos\left(\frac{\sin(\pi s)}{\pi}+\frac{\sin(\pi t)}{\pi}-\frac{\sin(\pi T)}{\pi}\right), & |s|<t,\\ \sin\left(\frac{\sin(\pi s)}{\pi}-\frac{\sin(\pi t)}{\pi}-\frac{\sin(\pi T)}{\pi}\right)+\cos\left(\frac{\sin(\pi s)}{\pi}+\frac{\sin(\pi t)}{\pi}+\frac{\sin(\pi T)}{\pi}\right), & |s|<-t.\end{cases}$$

Example

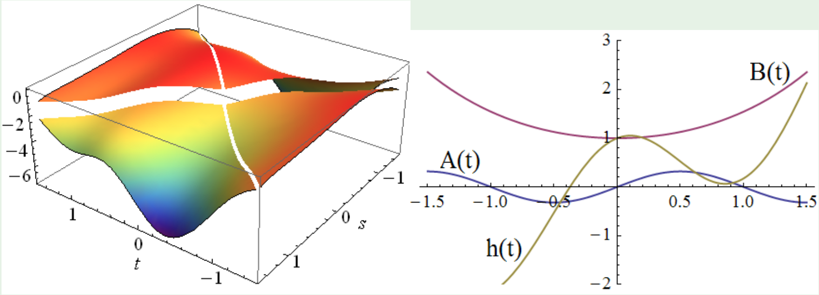

Figure: Graphs of the kernel *(left)* and of the functions involved in the problem *(right)* for $T = 3/2$.

Theorem

If condition (C4) ($b_e = -a$) holds, then problem

$$x'(t)+a(t)\,x(-t)+b(t)\,x(t) = h(t),\ \text{for a. e. } t \in I, \quad x(-T) = x(T),$$

has solution if and only if

$$\int_0^T e^{B_e(s)} h_e(s)ds = 0,$$

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and in that case the solutions are given by

$$u_c(t) = e^{-B_e(t)} \left\{ c + \int_0^t \left(e^{B_e(s)} h(s) + 2a_e(s) \int_0^s e^{B_e(r)} h_e(r)dr \right) ds \right\}$$

for $c \in \mathbb{R}$.

Theorem

If condition (C5) $b_e = a_e = 0$) holds, then problem

$$x'(t)+a(t)\,x(-t)+b(t)\,x(t) = h(t), \text{ for a. e. } t \in I, \quad x(-T) = x(T),$$

has solution if and only if

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If condition (C5) $b_e = a_e = 0$) holds, then problem

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and in that case the solutions are given by

$$u_c(t) = e^{A(t)} \int_0^t e^{-A(s)} h_e(s)ds + e^{-A(t)} \left\{ c + \int_0^t e^{A(s)} h_o(s)ds \right\}$$

for $c \in \mathbb{R}$.

When we are not on the cases (C1)-(C5), since the fundamental matrix of M is not given by its exponential matrix, it is more difficult to precise when our problem has a solution.

Theorem

Define $\upsilon = a + b$. *Let h, a, b in problem*

$$x'(t)+a(t)\,x(-t)+b(t)\,x(t) = h(t), \ \text{for a. e. } t \in I, \ x(-T) = x(T),$$

be in $L^1(I)$ *and assume* $\int_{-T}^{T} \upsilon(t)dt \neq 0$.

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Denoting $1/p + 1/p^* = 1$. *If*

$$\frac{e^{\|\upsilon\|_1}}{\left|e^{\|\upsilon^+\|_1} - e^{\|\upsilon^-\|_1}\right|}\|a\|_1 \inf_{p\in[1,+\infty]}\left\{\{(2T)^{\frac{1}{p}}(\|a\|_{p^*} + \|b\|_{p^*})\}\right\} < 1.$$

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then the problem has a unique solution.

The proof follows from the Banach Contraction Theorem.

Part IV

CONSTANT SIGN GREEN'S FUNCTIONS

Now we are interested in to obtain the set of functions $a(t)$ and $b(t)$ for which the Green's function $\overline{G}$ has constant sign on $I \times I$.

We start with the constant coefficient equation

$$x'(t) + m\,x(-t) = h(t), \quad t \in [-T, T]$$
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$$\begin{aligned} x'(t) + m\,x(-t) &= h(t), \quad t \in [-T, T] \\ x(T) - x(-T) &= 0, \end{aligned}$$

We have proved that there is the Green's function if and only if

$$m \neq \pm \frac{k\pi}{T}, \quad k = 0, 1, \ldots.$$

To this end, denote by $\alpha := m\,T$ and $\overline{G}_\alpha$ be the related Green's function for a particular value of the parameter α.

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Lemma

$$\overline{G}_\alpha(t,s) = -\overline{G}_{-\alpha}(-t,-s) \ \ \forall t,s \in I.$$

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LEMMA

$$\overline{G}_\alpha(t,s) = -\overline{G}_{-\alpha}(-t,-s) \;\; \forall t,s \in I.$$

COROLLARY

$\overline{G}_\alpha$ is positive if and only if $\overline{G}_{-\alpha}$ is negative on $I \times I$.

Remember that

$$2\sin(mT)\overline{G}(t,s) = \begin{cases} \cos m(T-s-t) + \sin m(T+s-t) & \text{if} \quad t > |s|, \\ \cos m(T-s-t) - \sin m(T-s+t) & \text{if} \quad |t| < s, \\ \cos m(T+s+t) + \sin m(T+s-t) & \text{if} \quad -|t| > s, \\ \cos m(T+s+t) - \sin m(T-s+t) & \text{if} \quad t < -|s|. \end{cases}$$

After some manipulation, the change of variables $t = Tz$, $s = Ty$ and using the trigonometric identity

$$\cos(a-b) \pm \sin(a+b) = (\cos a \pm \sin a)(\cos b \pm \sin b),$$

we get

$$2\sin(\alpha)\overline{G}(z,y) =$$

$$\begin{cases} [\cos\alpha(1-z)+\sin\alpha(1-z)][\sin\alpha y+\cos\alpha y] & \text{if} \quad z>|y|, \\ [\cos\alpha z-\sin\alpha z][\sin\alpha(y-1)+\cos\alpha(y-1)] & \text{if} \quad |z|<y, \\ [\cos\alpha(1+y)+\sin\alpha(1+y)][\cos\alpha z-\sin\alpha z] & \text{if} \quad -|z|>y, \\ [\cos\alpha y+\sin\alpha y][\cos\alpha(z+1)-\sin\alpha(z+1)] & \text{if} \quad z<-|y|. \end{cases}$$

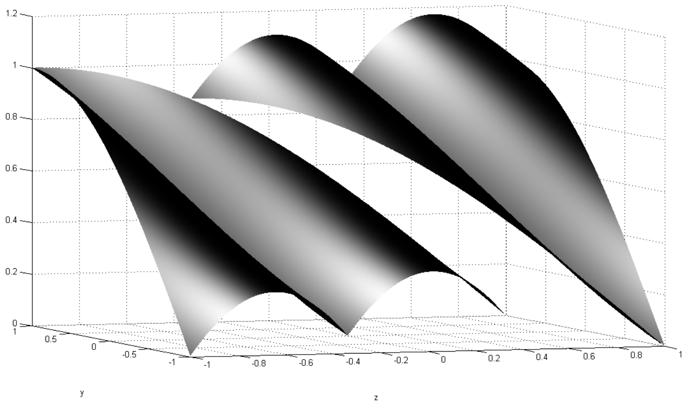

FIGURE: Plot of the function $\overline{G}(z, y)$ for $\alpha = \frac{\pi}{4}$

Theorem

1. *If* $\alpha \in (0, \frac{\pi}{4})$ *then* $\overline{G}$ *is strictly positive on* $I \times I$.

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2. *If* $\alpha \in (-\frac{\pi}{4}, 0)$ *then* $\overline{G}$ *is strictly negative on* $I \times I$.
3. *If* $\alpha = \frac{\pi}{4}$ *then* $\overline{G}$ *vanishes on* $P := \{(-T, -T), (0, 0), (T, T), (T, -T)\}$ *and is strictly positive on* $(I \times I) \backslash P$.

THEOREM

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5. *If* $\alpha \in \mathbb{R}\backslash[-\frac{\pi}{4}, \frac{\pi}{4}]$ *then* $\overline{G}$ *is not positive nor negative on* $I \times I$.

Denoting as $x \succ 0$ and $x \prec 0$ for $x \neq 0$ and $x \geq 0$ and $x \leq 0$ a.e. respectively, we arrive at the following definition

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The next corollary establishes maximum and anti-maximum principles.

Corollary

The operator $R_m : \mathcal{F}_\lambda(I) \to \mathcal{F}_\lambda(I)$ defined as $R_m(x(t)) = x'(t) + m\,x(-t)$, with $m \in \mathbb{R}\backslash\{0\}$, satisfies

1. *R_m is strongly inverse positive if and only if $m \in (0, \frac{\pi}{4T}]$ and $\lambda \geq 0$,*

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With these results we get the following corollary for non constant coefficients.

COROLLARY

If $b = 0$, a is nonnegative on I and $A(T) \neq n\pi$ then the following assertions are fulfilled:

- *If $A(T) \in (0, \frac{\pi}{4})$ then G_1 is strictly positive on $I \times I$.*
- *If $A(T) \in (-\frac{\pi}{4}, 0)$ then G_1 is strictly negative on $I \times I$.*
- *If $A(T) = \frac{\pi}{4}$ then G_1 vanishes on $P := \{(-A(T), -A(T)), (0,0), (A(T), A(T)), (A(T), -A(T))\}$ and is strictly positive on $(I \times I) \backslash P$.*
- *If $A(T) = -\frac{\pi}{4}$ then G_1 vanishes on P and is strictly negative on $(I \times I) \backslash P$.*
- *If $A(T) \in \mathbb{R} \backslash [-\frac{\pi}{4}, \frac{\pi}{4}]$ then G_1 is not positive nor negative on $I \times I$.*

Corollary

If $b = 0$, a is nonnegative on I and $A(T) \neq n\pi$ the operator $R_a : \mathcal{F}_\lambda(I) \to L^1(I)$ defined as $R_a(x(t)) = x'(t) + a(t)\,x(-t)$ satisfies

- *R_a is strongly inverse positive if and only if $A(T) \in (0, \frac{\pi}{4T}]$ and $\lambda \geq 0$,*
- *R_a is strongly inverse negative if and only if $A(T) \in [-\frac{\pi}{4T}, 0)$ and $\lambda \geq 0$.*

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Consider the homogeneous initial value problem

$$x'(t) + a(t)\,x(-t) + b(t)\,x(t) = 0,\ \ t \in I;\ \ x(t_0) = 0.$$

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If this problem has a unique solution ($x \equiv 0$) on I for all $t_0 \in I$ then, if the Green's function for

$$x'(t) + a(t)\,x(-t) + b(t)\,x(t) = h(t), a.\,e.\ t \in I,\ \ x(-T) = x(T),$$

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What is more, if we further assume $a + b$ has constant sign, the Green's function has the same sign as $a + b$.

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Let g be the $2T$-periodic extension of $G_1(\cdot, s_1)$.Let f be the restriction of g to $(s_1, s_1 + 2T)$. f is absolutely continuous and satisfies

$$f'(t) + a(t)\, f(-t) + b(t)\, f(t) = 0, \;\; t \in I; \;\; f(t_1) = 0.$$

hence, $f \equiv 0$. A contradiction. □

Second Part of the Proof.

Realize now that $x \equiv 1$ satisfies

$$x'(t) + a(t)x(-t) + b(t)x(t) = a(t) + b(t), \; x(-T) = x(T).$$

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Since both G_1 and $a + b$ have constant sign, they have the same sign. □

Corollary

Assume a has constant sign.

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Assume a has constant sign. Under the assumption

(C1*) $b_e = k\,a,\ k \in \mathbb{R},\ |k| < 1$ *and,* $(1-k^2)A(T)^2 \neq (n\pi)^2$ *for all* $n = 0, 1, \dots$ *and* $\cos\left(\sqrt{1-k^2}A(T)\right) \neq 0.$

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In order to prove this Corollary, we use (by means of the change of variable to a constant coefficients equation) that the general solution of equation

$$x'(t) + a(t)\,x(-t) + b(t)\,x(t) = 0, \;\; t \in I.$$

is given by

$$x(t) = \alpha e^{-B_e(t)} \left\{ \cos\left(\sqrt{1-k^2}A(t)\right) - \frac{1+k}{\sqrt{1-k^2}} \sin\left(\sqrt{1-k^2}A(t)\right) \right\}$$

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Now, we study the values of $A(t)$ for which any solution satisfying $x(t_0) = 0$ for some $t_0 \in I$ must be equals to zero.

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Assume a has constant sign. Under the assumption

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In this case, we use (by means of the change of variable to a constant coefficients equation) that the general solution of equation

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the Green's function G_1 *has constant sign if* $|A(T)| < \frac{1}{2}$.

If we consider σ defined piecewise as in previous Corollaries we get

$$\sigma(k) := \begin{cases} \frac{\arccos(k)}{2\sqrt{1-k^2}} & \text{if } k \in (-1,1) \\ \frac{1}{2} & \text{if } k = 1 \\ -\frac{\ln(k-\sqrt{k^2-1})}{2\sqrt{k^2-1}} & \text{if } k > 1 \end{cases}$$

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As consequence if $|A(T)| < \sigma(k)$ the Green's function has constant sign in $I \times I$.

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Some particular case in which the Green's function changes its sign has been also studied.

THANKS FOR YOUR ATTENTION