On Vallée-Poussin type conditions for positivity of solutions to the Darboux problem for hyperbolic functional-differential equations

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On the rectangle $\mathbb{D} = [a, b] \times [c, d]$ we consider the equation

$$u_{tx}^{\prime\prime}(t,x)=p(t,x)uig(au(t,x),\mu(t,x)ig)+q(t,x)$$

 $Descript{p,q} \colon \mathbb{D} o \mathbb{R}$ integrable

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Solution: A function $u: \mathbb{D} \to \mathbb{R}$ absolutely continuous on \mathbb{D} in the sense of Carathéodory satisfying equality (1) almost everywhere in \mathbb{D} .

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Solution: A function $u \colon \mathbb{D} \to \mathbb{R}$ absolutely continuous on \mathbb{D} in the sense of Carathéodory satisfying equality (1) almost everywhere in \mathbb{D} .

We cannot pass between equation (1) and the wave equation

 $u_{tt}^{\prime\prime}(t,x)-u_{xx}^{\prime\prime}(t,x)= ilde{p}(t,x)uig(ilde{ au}(t,x), ilde{\mu}(t,x)ig)+ ilde{q}(t,x).$

Consider now, on the interval [a, b], the *n*-dimensional system of linear ordinary differential equations

$$v' = P(t)v + q(t), \tag{2}$$

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It is well known that every solution v to system (2) admits the representation

$$v(t)=C(t,t_0)v(t_0)+\int_{t_0}^t C(t,s)q(s)\mathrm{d}s \quad ext{for } t\in [a,b],$$

where $t_0 \in [a, b]$ and C is the Cauchy matrix of the system v' = P(t)v, i.e., for every $t_0 \in [a, b]$, $C(\cdot, t_0)$ is a fundamental matrix of the system v' = P(t)v, which satisfies $C(t_0, t_0) = E_n$.

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Remark. If n = 1 then, for every $t_0 \in [a, b]$, $C(\cdot, t_0)$ is a solution to the problem

$$v'=P(t)v, \qquad v(t_0)=1.$$

Indeed, we have

$$C(t,s) = \mathrm{e}^{\int_s^t P(\xi) \, \mathrm{d} \xi} \quad ext{ for } t,s \in [a,b]$$

in this case.

$$u_{tx}^{''} = p(t,x)u + q(t,x)$$
 (1')

Every solution u to the equation (1') admits the representation

$$egin{aligned} u(t,x) &= Z_{t,x}(a,c)u(a,c) + \int_a^t Z_{t,x}(s,c)u_s'(s,c)\mathrm{d}s + \ &+ \int_c^x Z_{t,x}(a,\eta)u_\eta'(a,\eta)\mathrm{d}\eta + \int_a^t \int_c^x Z_{t,x}(s,\eta)q(s,\eta)\mathrm{d}\eta\mathrm{d}s \end{aligned}$$

for $(t,x)\in\mathbb{D}$, where $Z_{t,x}$ are the Riemann functions of the equation $u_{tx}^{\prime\prime}=p(t,x)u.$

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for $(t,x)\in\mathbb{D}$, where $Z_{t,x}$ are the Riemann functions of the equation $u_{tx}^{\prime\prime}=p(t,x)u.$

The Riemann function Z_{t_0,x_0} is defined as a solution to the Darboux problem

$$u_{tx}^{\prime\prime}=p(t,x)u,$$
 $\iota(t,x_0)=1 ext{ for } t\in [a,b], ext{ } u(t_0,x)=1 ext{ for } x\in [c,d].$

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Proposition

Let

$$Z_{t,x}(s,\eta)\geq 0 \quad ext{for } a\leq s\leq t\leq b, \,\, c\leq \eta\leq x\leq d.$$

(*)

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Then the implication

holds.

$$u_{tx} = -ku$$
 $(k \ge 0)$ (3)
 $u(t, x_0) = 1$ for $t \in [a, b],$ $u(t_0, x) = 1$ for $x \in [c, d]$ (4)

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If u is looked for in the form u(t, x) = v(z), where $z = \sqrt{(t_0 - t)(x_0 - x)}$, then (3), (4) is reduced to

$$v''(z) + \frac{1}{z} v'(z) + 4kv(z) = 0, \qquad v(0) = 1$$
(5)

on the interval $\left[0,\sqrt{(t_0-a)(x_0-c)}
ight]$, which has a solution

 $v(z) = J_0(2\sqrt{k} z).$

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The Riemann function Z_{t_0,x_0} of the equation (3) satisfies

$$Z_{t_0,x_0}(t,x) = J_0\left(2\sqrt{k(t_0-t)(x_0-x)}
ight) \quad ext{ for } (t,x) \in [a,t_0] imes [c,x_0].$$

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Therefore, the condition (*) holds, i.e.,

$$Z_{t,x}(s,\eta) \ge 0 \quad ext{for } a \le s \le t \le b, \ c \le \eta \le x \le d,$$

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if and only if

$$2\sqrt{k(b-a)(d-c)}\leq j_0$$
 ,

i. e.,

$$k\leq rac{j_0^2}{4(b-a)(d-c)}$$
 .

$$u_{tx}^{\prime\prime}(t,x)=p(t,x)uig(au(t,x),\mu(t,x)ig)+q(t,x)$$

Definition

We say that a theorem on differential inequalities (maximum principle) holds for the equation (1) if the implication

 $\begin{array}{l} w \in AC(\mathbb{D}), \\ w_{tx}^{\prime\prime}(t,x) \ge p(t,x)w(\tau(t,x),\mu(t,x)) \text{ for a. e. } (t,x) \in \mathbb{D}, \\ w(a,c) \ge 0, \\ w_t^{\prime}(t,c) \ge 0 \text{ for a. e. } t \in [a,b], \\ w_x^{\prime\prime}(a,x) \ge 0 \text{ for a. e. } x \in [c,d] \end{array} \right\} \implies \begin{array}{l} w(t,x) \ge 0 \\ \text{ for } (t,x) \in \mathbb{D} \end{array}$

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In what follows we restrict our-self to the case where

$$p(t, x) \leq 0$$
 for a.e. $(t, x) \in \mathbb{D}$. (5)

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(1)

$$u_{tx}^{\prime\prime}(t,x)=p(t,x)uig(au(t,x),\mu(t,x)ig)+q(t,x)$$

Proposition

Let maximum principle holds for equation (1) with non-positive p then the inequalities

$$au(t,x) \leq t, \quad \mu(t,x) \leq x \quad ext{for a. e. } (t,x) \in \mathbb{D}.$$

hold, i. e., equation (1) is delayed in both arguments.

(6)

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hold, i. e., equation (1) is delayed in both arguments.

Theorem 1

Let conditions (5) and (6) hold. Then maximum principle holds for equation (1) provided that there exists a function $\gamma \in C(\mathbb{D}) \cap AC_{loc}([a, b[\times [c, d])$ satisfying the inequalities

$$egin{aligned} &\gamma(t,x)>0 \quad ext{for } (t,x)\in[a,b] imes[c,d]\,, \ &\gamma_{tx}^{\prime\prime\prime}(t,x)\leq p(t,x)\gammaig(au(t,x),\mu(t,x)ig) \quad ext{for a.e. } (t,x)\in\mathbb{D}, \ &\gamma_t^\prime(t,c)\leq 0 \quad ext{for a.e. } t\in[a,b], \ &\gamma_x^\prime(a,x)\leq 0 \quad ext{for a.e. } x\in[c,d]. \end{aligned}$$

(1)

$$u_{tx}^{\prime\prime}(t,x)=p(t,x)uig(au(t,x),\mu(t,x)ig)+q(t,x)ig)$$

(1)

Corollary 1

Let conditions (5) and (6) hold and

$$\displaystyle{ \iint_{\mathbb{D}} |p(t,x)| \mathrm{d}t \mathrm{d}x \leq 1. }$$

Then maximum principle holds for equation (1).

Remark. The number 1 in Corollary 1 is optimal, in general. A counter-example is constructed with

 $au(t,x)\equiv a, \qquad \mu(t,x)\equiv c.$

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Proposition

Let $\tau(t,x) \equiv t$, $\mu(t,x) \equiv x$, $p(t,x) \equiv k \le 0$. Then maximum principle holds for equation (1) if and only if

$$|k|(b-a)(d-c) \leq rac{j_0^2}{4} \sim 1.4458.$$

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(1)

$$u_{tx}^{\prime\prime}(t,x)=p(t,x)uig(au(t,x),\mu(t,x)ig)+q(t,x)$$

 $j_{
u}$... the first positive zero of the Bessel function $J_{
u}$ (
u > -1)

$$E_
u(z):=z^{-
u}J_
u(z)$$
 for $z\geq 0$

$$j_{
u}^{*} = rac{E_{
u+1}(j_{
u})}{E_{
u+1}(0)}$$

$$u_{tx}^{\prime\prime}(t,x)=p(t,x)uig(au(t,x),\mu(t,x)ig)+q(t,x)$$

Corollary 2

Let conditions (5) and (6) hold and there exist numbers $\alpha \in [0, 1[$, $\beta \in [0, \alpha]$ such that the inequalities

$$|p(t,x)|\leq rac{j_{-lpha}^2}{4(b-a)(d-c)}\,,$$

$$igg(E_{-lpha}ig(z[au(t,x),x]ig) - E_{-lpha}ig(z[t,x]ig) ig) |p(t,x)| \leq rac{eta}{2} \, rac{j^2_{-lpha}}{(b-a)(d-c)} \, E_{1-lpha}ig(z[au(t,x),x]ig),$$

$$igg(E_{-lpha}ig(z[t,x]ig)-E_{-lpha}ig(z[t,\mu(t,x)]ig)ig)|p(t,x)|\leq rac{lpha-eta}{2}\,rac{j_{-lpha}^2}{(b-a)(d-c)}\,E_{1-lpha}ig(z[au(t,x),x]ig)$$

are satisfied a.e. in \mathbb{D} , where

$$z[t,x]:=rac{j_{-lpha}}{2}\,\sqrt{rac{(t-a)(x-c)}{(b-a)(d-c)}}\quad ext{for}\,\,(t,x)\in\mathbb{D}.$$

Then maximum principle holds for equation (1).

(1)

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Corollary 3

Let conditions (5) and (6) hold and there exist numbers $\alpha \in [0, 1[$, $\beta \in [0, \alpha]$ such that the inequalities

$$egin{aligned} |p(t,x)| &\leq rac{j_{-lpha}^2}{4(b-a)(d-c)}\,, \ &(x-c)ig(t- au(t,x)ig)|p(t,x)| \leq eta\, j_{-lpha}^*, \ &(t-a)ig(x-\mu(t,x)ig)|p(t,x)| \leq (lpha-eta)\, j_{-lpha}^* \end{aligned}$$

are satisfied a.e. in \mathbb{D} . Then maximum principle holds for equation (1).

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are satisfied a.e. in \mathbb{D} . Then maximum principle holds for equation (1).

Remark. The number $\frac{j_{-\alpha}^2}{4(b-a)(d-c)}$ in Corollaries 2 and 3 is optimal, in general. A counter-example is constructed with

$$au(t,x)\equiv t, \qquad \mu(t,x)\equiv x, \qquad p(t,x)\equiv k\,\leq 0,$$

where we can choose $\alpha = \beta = 0$ and we know that the inequality

$$|k|\leq rac{j_0^2}{4(b-a)(d-c)}$$
 is such

is sufficient and necessary.

$$u_{tx}^{\prime\prime}(t,x)=p(t,x)uig(au(t,x),\mu(t,x)ig)+q(t,x)$$

Corollary 2

Let conditions (5) and (6) hold and there exist numbers $\alpha \in [0, 1[$, $\beta \in [0, \alpha]$ such that the inequalities

$$egin{aligned} |p(t,x)| &\leq rac{j_{-lpha}^2}{4(b-a)(d-c)}\,, \ &iggl(E_{-lpha}iggl(z[au(t,x),x]iggr) - E_{-lpha}iggl(z[t,x]iggr) iggr) |p(t,x)| &\leq rac{eta}{2}\,rac{j_{-lpha}^2}{(b-a)(d-c)}\,E_{1-lpha}iggl(z[t,x]iggr) , \ &E_{-lpha}iggl(z[t,x]iggr) - E_{-lpha}iggl(z[t,\mu(t,x)]iggr) iggr) |p(t,x)| &\leq rac{lpha-eta}{2}\,rac{j_{-lpha}^2}{(b-a)(d-c)}\,E_{1-lpha}iggl(z[t,x]iggr) . \end{aligned}$$

are satisfied a.e. in \mathbb{D} , where

$$z[t,x]:=rac{j_{-lpha}}{2}\,\sqrt{rac{(t-a)(x-c)}{(b-a)(d-c)}}\quad ext{for}\,\,(t,x)\in\mathbb{D}.$$

Then maximum principle holds for equation (1).

(1)

We put

$$egin{aligned} \gamma(t,x) &= E_{-lpha}ig(z[t,x]ig) & ext{for } (t,x)\in\mathbb{D} \ \end{bmatrix} \ z[t,x] &:= rac{j_{-lpha}}{2} \sqrt{rac{(t-a)(x-c)}{(b-a)(d-c)}} \end{aligned}$$

and apply

Theorem 1

Let conditions (5) and (6) hold. Then maximum principle holds for equation (1) provided that there exists a function $\gamma \in C(\mathbb{D}) \cap AC_{loc}([a, b] \times [c, d])$ satisfying the inequalities

$$egin{aligned} &\gamma(t,x)>0 \quad ext{for}~(t,x)\in [a,b[imes[c,d[\,,&\ \gamma_{tx}^{\prime\prime}(t,x)\leq p(t,x)\gammaig(au(t,x),\mu(t,x)ig) \quad ext{for}~\mathsf{a.e.}~(t,x)\in\mathbb{D},\ &\gamma_t^\prime(t,c)\leq 0 \quad ext{for}~\mathsf{a.e.}~t\in [a,b],\ &\gamma_x^\prime(a,x)<0 \quad ext{for}~\mathsf{a.e.}~x\in [c,d]. \end{aligned}$$

We put

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We put

$$\begin{split} \gamma(t,x) &= E_{-\alpha} \left(z[t,x] \right) \quad \text{for } (t,x) \in \mathbb{D} \\ z[t,x] &:= \frac{j_{-\alpha}}{2} \sqrt{\frac{(t-a)(x-c)}{(b-a)(d-c)}} \\ J_{-\alpha}''(s) &+ \frac{1}{s} J_{-\alpha}'(s) + \left(1 - \frac{\alpha^2}{s^2}\right) J_{-\alpha}(s) = 0 \end{split}$$

$$E_{-lpha}^{\prime\prime}(s)+rac{1-2lpha}{s}\,E_{-lpha}^{\prime}(s)+E_{-lpha}(s)=0$$

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$$egin{aligned} \gamma(t,x) &= E_{-lpha}ig(z[t,x]ig) & ext{for } (t,x)\in\mathbb{D} \ \end{bmatrix} \ z[t,x] &:= rac{j_{-lpha}}{2} \sqrt{rac{(t-a)(x-c)}{(b-a)(d-c)}} \ J_{-lpha}^{\prime\prime}(s) + rac{1}{s} J_{-lpha}^{\prime}(s) + ig(1-rac{lpha^2}{s^2}ig) J_{-lpha}(s) = 0 \end{aligned}$$

$$E_{-lpha}^{\prime\prime}(s)+rac{1-2lpha}{s}\,E_{-lpha}^{\prime}(s)+E_{-lpha}(s)=0$$

$$\gamma_{tx}^{\prime\prime}(t,x)=-rac{j_{-lpha}^2}{4(b-a)(d-c)}\,\gamma(t,x)+rac{eta}{x-c}\,\gamma_t^\prime(t,x)+rac{eta}{t-a}\,\gamma_x^\prime(t,x)$$

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$$\gamma_{tx}^{\prime\prime}(t,x)=-rac{j_{-lpha}^2}{4(b-a)(d-c)}\,\gamma(t,x)+rac{eta}{x-c}\,\gamma_t^\prime(t,x)+rac{eta}{t-a}\,\gamma_x^\prime(t,x)$$

We need to show that

$$\gamma_{tx}^{\prime\prime}(t,x) \leq p(t,x) \gammaig(au(t,x),\mu(t,x)ig)$$

$$\gamma_{tx}^{\prime\prime}(t,x)=-rac{j_{-lpha}^2}{4(b-a)(d-c)}\,\gamma(t,x)+rac{eta}{x-c}\,\gamma_t^\prime(t,x)+rac{eta}{t-a}\,\gamma_x^\prime(t,x)$$

$$\gamma_{tx}^{\prime\prime}(t,x) \leq p(t,x) \gammaig(au(t,x),\mu(t,x)ig)$$

 $\gamma(t,x)>0 \quad ext{for every } (t,x)\in \mathbb{D}, (t,x)
eq (b,d),$

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- $\gamma_t(t,x) \leq 0 \quad ext{for every } (t,x) \in]a,b] imes [c,d],$
- $\gamma_x(t,x) \leq 0 \quad ext{for every } (t,x) \in [a,b] imes]c,d],$
- $\gamma_{tx}^{\prime\prime}(t,x)\leq 0 \hspace{1em} ext{for every} \hspace{1em} (t,x)\in]a,b[imes]c,d[$

$$\begin{split} \gamma_{tx}^{\prime\prime}(t,x) &= -\frac{j_{-\alpha}^2}{4(b-a)(d-c)} \,\gamma(t,x) + \frac{\beta}{x-c} \,\gamma_t^\prime(t,x) + \frac{\beta}{t-a} \,\gamma_x^\prime(t,x) \\ &\qquad \gamma_{tx}^{\prime\prime}(t,x) \leq p(t,x) \gamma\left(\tau(t,x),\mu(t,x)\right) \\ &\qquad \gamma(t,x) > 0 \quad \text{for every } (t,x) \in \mathbb{D}, (t,x) \neq (b,d), \\ &\qquad \gamma_t(t,x) \leq 0 \quad \text{for every } (t,x) \in]a,b] \times [c,d], \\ &\qquad \gamma_x(t,x) \leq 0 \quad \text{for every } (t,x) \in]a,b[\times]c,d], \\ &\qquad \gamma_{tx}^{\prime\prime}(t,x) \leq 0 \quad \text{for every } (t,x) \in]a,b[\times]c,d[\\ &\qquad \gamma\left(\tau(t,x),\mu(t,x)\right) = \gamma(t,x) - \int_{\tau(t,x)}^t \gamma_s^\prime(s,x) \mathrm{d}s - \int_{\mu(t,x)}^x \gamma_{\eta}^\prime(t,\eta) \mathrm{d}\eta + \int_{\tau(t,x)}^t \int_{\mu(t,x)}^x \gamma_{s\eta}^{\prime\prime}(s,\eta) \mathrm{d}\eta \mathrm{d}s \\ &\leq \gamma(t,x) - \int_{\tau(t,x)}^t \gamma_s^\prime(s,x) \mathrm{d}s - \int_{\mu(t,x)}^x \gamma_{\eta}^\prime(t,\eta) \mathrm{d}\eta \end{split}$$

$$\begin{split} \gamma_{tx}^{\prime\prime}(t,x) &= -\frac{j_{-\alpha}^2}{4(b-a)(d-c)} \,\gamma(t,x) + \frac{\beta}{x-c} \,\gamma_t^\prime(t,x) + \frac{\beta}{t-a} \,\gamma_x^\prime(t,x) \\ \gamma_{tx}^{\prime\prime}(t,x) &\leq p(t,x) \gamma\left(\tau(t,x),\mu(t,x)\right) \\ \gamma(t,x) &> 0 \quad \text{for every } (t,x) \in \mathbb{D}, (t,x) \neq (b,d), \\ \gamma_t(t,x) &\leq 0 \quad \text{for every } (t,x) \in [a,b] \times [c,d], \\ \gamma_x(t,x) &\leq 0 \quad \text{for every } (t,x) \in [a,b] \times]c,d], \\ \gamma_{tx}^{\prime\prime}(t,x) &\leq 0 \quad \text{for every } (t,x) \in [a,b] \times]c,d] \\ \gamma\left(\tau(t,x),\mu(t,x)\right) &= \gamma(t,x) - \int_{\tau(t,x)}^t \gamma_s^\prime(s,x) \mathrm{d}s - \int_{\mu(t,x)}^\infty \gamma_{\eta}^\prime(t,\eta) \mathrm{d}\eta + \int_{\tau(t,x)}^t \int_{\mu(t,x)}^\infty \gamma_{s\eta}^{\prime\prime}(s,\eta) \mathrm{d}\eta \mathrm{d}s \\ &\leq \gamma(t,x) - \int_{\tau(t,x)}^t \gamma_s^\prime(s,x) \mathrm{d}s - \int_{\mu(t,x)}^\infty \gamma_{\eta}^\prime(t,\eta) \mathrm{d}\eta \\ &= \gamma(t,x) - \varphi(t,x) \,\gamma_t^\prime(t,x) \int_{\tau(t,x)}^t \frac{\mathrm{d}s}{\varphi(s,x)} - \psi(t,x) \,\gamma_x^\prime(t,x) \int_{\mu(t,x)}^\infty \frac{\mathrm{d}\eta}{\psi(t,\eta)} \end{split}$$

 $\gamma_{tx}^{\prime\prime}(t,x) \leq p(t,x) \gammaig(au(t,x),\mu(t,x)ig)$

$$igwedge \ |p(t,x)| \leq rac{j^2_{-lpha}}{4(b-a)(d-c)} \ |p(t,x)| \, arphi(t,x) \int_{ au(t,x)}^t rac{\mathrm{d}s}{arphi(s,x)} \leq rac{eta}{x-c} \ |p(t,x)| \, \psi(t,x) \int_{\mu(t,x)}^x rac{\mathrm{d}\eta}{\psi(t,\eta)} \leq rac{lpha-eta}{t-a} \ \end{cases}$$

$$arphi(t,x) := rac{(t-a)^{1-lpha}}{ig(z[t,x]ig)^{1-lpha} J_{1-lpha}ig(z[t,x]ig)}, \qquad \psi(t,x) := rac{(x-c)^{1-lpha}}{ig(z[t,x]ig)^{1-lpha} J_{1-lpha}ig(z[t,x]ig)}$$

 $\gamma_{tx}^{\prime\prime}(t,x) \leq p(t,x) \gammaig(au(t,x),\mu(t,x)ig)$

$$igcap_{\mu(t,x)} ert \leq rac{j_{-lpha}^2}{4(b-a)(d-c)} ert p(t,x) ert arphi(t,x) ert arphi(t,x) \int_{ au(t,x)}^t rac{\mathrm{d}s}{arphi(s,x)} \leq rac{eta}{x-c} ert p(t,x) ert \psi(t,x) \int_{\mu(t,x)}^x rac{\mathrm{d}\eta}{\psi(t,\eta)} \leq rac{lpha - eta}{t-a}$$

$$arphi(t,x) := rac{(t-a)^{1-lpha}}{ig(z[t,x]ig)^{1-lpha} J_{1-lpha}ig(z[t,x]ig)}, \qquad \psi(t,x) := rac{(x-c)^{1-lpha}}{ig(z[t,x]ig)^{1-lpha} J_{1-lpha}ig(z[t,x]ig)}$$

$$\int_{ au(t,x)}^t rac{\mathrm{d}s}{arphi(s,x)} = rac{const.}{(x-c)^lpha} igg(E_{-lpha}igg(z[au(t,x),x]igg) - E_{-lpha}igg(z[t,x]igg)igg)$$

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