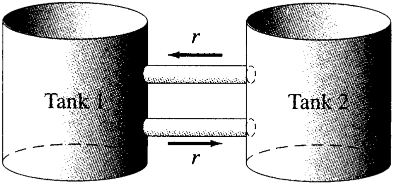
Mixing problems with many tanks

Antonín Slavík
Charles University, Prague

Workshop on Differential Equations
Malá Morávka 2012

A simple mixing problem

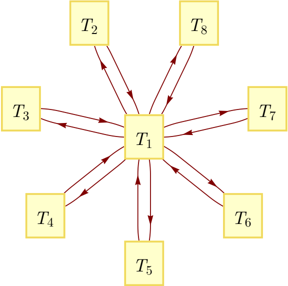
Consider two tanks filled with brine, which are connected by a pair of pipes. One pipe brings brine from the first tank to the second tank at a given rate, while the second pipe carries brine in the opposite direction at the same rate. Assuming that the initial concentrations in both tanks are known and that we have a perfect mixing in both tanks, find the concentrations in both tanks after a given period of time.



Mixing problems with many tanks

- Most authors restrict themselves to mixing problems involving two or three tanks arranged in various configurations (cascade with brine flowing in a single direction only, linear arrangement of tanks connected by pairs of pipes, cyclic arrangement of tanks, etc.).
- The problem leads to a linear system of differential equations for the unknown concentrations, which is solved by calculating the eigenvalues and eigenvectors of the corresponding matrix.
- With n tanks, there is a great variety of mixing problems. Can we still solve the corresponding DEs analytically?

Star arrangement of tanks (1)



- Flow through each pipe: f gallons per unit of time.
- The volume V in each tank remains constant.
- $x_i(t)$ = concentration of salt in tank T_i at time t.

$$
\begin{aligned}
x_1'(t) &= -(n-1)f\frac{x_1(t)}{V} + \sum_{i=2}^{n} f\frac{x_i(t)}{V} \\
x_i'(t) &= f\frac{x_1(t)}{V} - f\frac{x_i(t)}{V}, \qquad 2 \le i \le n.
\end{aligned}
$$

Without loss of generality, we may assume that $f = V$.
Then $x'(t) = Ax(t)$, where

$$A = \begin{pmatrix} -(n-1) & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 \end{pmatrix}.$$

$\det(A - \lambda I) = D_n$, where

$$D_k = \det \begin{pmatrix} -(n-1)-\lambda & 1 & \cdots & 1 \\ 1 & -1-\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1-\lambda \end{pmatrix} =$$

$$= -(-1-\lambda)^{k-2} + (-1-\lambda)D_{k-1}, \quad D_1 = -(n-1) - \lambda.$$

Star arrangement of tanks (3)

Solution:

$$D_k = (-1-\lambda)^{k-2}\lambda(k+\lambda)$$

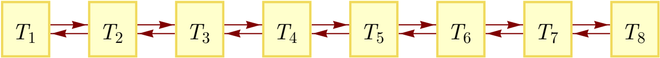
The eigenvalues of A:

- $\lambda = 0$ and $\lambda = -n$ (simple)
- $\lambda = -1$ (multiplicity $n-2$)

Eigenvectors are easy to find. In particular, the eigenspace corresponding to $\lambda = 0$ is spanned by $(1, \dots, 1)$.

Corollary: Every solution approaches the state where all tanks contain the same amount of salt (a globally asymptotically stable equilibrium).

Tanks in a row (1)



- Flow through each pipe: f gallons per unit of time.
- The volume V in each tank remains constant.
- $x_i(t)$ = concentration of salt in tank T_i at time t.

$$
\begin{aligned}
x_1'(t) &= -f\frac{x_1(t)}{V} + f\frac{x_2(t)}{V} \\
x_i'(t) &= f\frac{x_{i-1}(t)}{V} - 2f\frac{x_i(t)}{V} + f\frac{x_{i+1}(t)}{V}, \qquad 2 \le i \le n-1 \\
x_n'(t) &= -f\frac{x_{n-1}(t)}{V} - f\frac{x_n(t)}{V}
\end{aligned}
$$

Tanks in a row (2)

Without loss of generality, we may assume that $f = V$.
Then $x'(t) = Ax(t)$, where

$$A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

Preliminary information:
All eigenvalues are real and contained in $[-4, 0]$.

Tanks in a row (3)

$$\det(A-\lambda I)=(-1-\lambda)D_{n-1}-D_{n-2},$$

where

$$D_k=\det\begin{pmatrix} -2-\lambda & 1 & 0 & \cdots & 0 & 0\\ 1 & -2-\lambda & 1 & \cdots & 0 & 0\\ 0 & 1 & -2-\lambda & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -2-\lambda & 1\\ 0 & 0 & 0 & \cdots & 1 & -1-\lambda \end{pmatrix}=$$

$$=(-2-\lambda)D_{k-1}-D_{k-2},\quad D_1=-1-\lambda,\ D_0=1.$$

Solution:

$$D_k=\cos(k\gamma)+\cot(\gamma/2)\sin(k\gamma),$$

where $\gamma\in[0,\pi]$, $\cos\gamma=-(\lambda+2)/2$, $\sin\gamma=\frac{\sqrt{4-(\lambda+2)^2}}{2}$.

Tanks in a row (4)

$$\det(A - \lambda I) = 2\cot\left(\frac{\gamma}{2}\right)\sin(n\gamma)$$

n distinct eigenvalues:

$$\lambda_k = -2\cos\frac{k\pi}{n} - 2, \quad k \in \{1, \dots, n\}$$

The eigenspace corresponding to $\lambda_n = 0$ is spanned by $(1, \dots, 1)$, the remaining eigenvalues are negative.

Tanks in a row (5)

Consider the BVP for the one-dimensional heat equation

$$\frac{\partial f}{\partial t}(t,x) = k\frac{\partial^2 f}{\partial x^2}(t,x), \quad x \in [a,b],$$

$$\frac{\partial f}{\partial x}(t,a) = \frac{\partial f}{\partial x}(t,b) = 0.$$

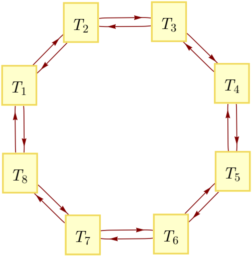
Discretize the spatial domain and replace second-order derivatives by the second-order central differences to get

$$\begin{aligned} y_0'(t) &\doteq \frac{k}{(\Delta x)^2}(-y_0(t) + y_1(t)), \\ y_i'(t) &\doteq \frac{k}{(\Delta x)^2}(y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)), \quad i \in \{1,\dots,n\}, \\ y_n'(t) &\doteq \frac{k}{(\Delta x)^2}(y_{n-1}(t) - y_n(t)), \end{aligned}$$

where $y_i(t) = f(t, a + i\Delta x)$.

$\Rightarrow$ mixing problem for n tanks in a row

Tanks in a circle (1)



$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

Tanks in a circle (2)

A is a circulant matrix with eigenvalues

$$\lambda_j = -2 + 2\cos(2\pi j/n), \quad j \in \{0, \dots, n-1\}.$$

The eigenspace corresponding to $\lambda_0 = 0$ is spanned by $(1, \dots, 1)$, the remaining eigenvalues are negative.

This type of mixing problem can be interpreted as the spatial discretization of the one-dimensional heat equation on a circle.

Qualitative analysis of a general mixing problem

For a general mixing problem, is it always true that every solution approaches the state with all tanks containing the same amount of salt?

1. All tanks hold the same constant volume V of brine; consequently, the total amount of brine flowing into a particular tank equals the total amount of brine flowing out of the tank.
2. Each pipe connecting a pair of tanks transports the same volume f of brine per unit of time. By a suitable choice of time units, we can assume that $f/V = 1$.
3. The mixing problem is irreducible in the following sense: The tanks cannot be divided into two disjoint nonempty groups such that the pipes always lead between tanks from the same group.

Graph representation

- The mixing problem can be represented by a directed graph, which is connected and balanced, i.e., the indegree $\deg^-(v)$ of an arbitrary vertex v equals its outdegree $\deg^+(v)$.
- Every balanced connected graph is strongly connected.

How does the corresponding matrix A look like? By the first condition, the sum of each row of A is zero. By the second condition, $a_{ij} = 1$ if there is a pipe transporting brine from the i-th tank to the j-th tank, a_{ii} equals minus the number of pipes originating in the i-th tank, and all remaining entries of A are zero.

Laplacian matrix

Given a directed graph $G = (V, E)$ with n vertices $v_1, \ldots, v_n$, the matrix $L = \{l_{ij}\}_{i,j=1}^n$ given by

$$l_{ij} = \begin{cases} \deg^+(v_i) & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } (v_i, v_j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

(where $\deg^+(v)$ stands for the outdegree of the vertex v) is known as the Laplacian matrix of G.

Observations:

- $L = -A$
- L has a zero eigenvalue with $(1, \ldots, 1)$ as the corresponding eigenvector.
- All remaining eigenvalues of L have positive real parts.

To finish the analysis, it is enough to show that the zero eigenvalue is a simple one.

Laplacian matrix eigenvalues

Lemma

If L is the Laplacian matrix of a balanced directed graph on n vertices and $x \in \mathbb{R}^n$ is an arbitrary vector, then

$$x^T L x = \frac{1}{2} \sum_{(v_i, v_j) \in E} (x_i - x_j)^2.$$

Theorem

The null space of the Laplacian matrix of a connected balanced directed graph has dimension 1.

Theorem

For the Laplacian matrix of a connected balanced directed graph, the zero eigenvalue is a simple one.