Parametrization for non–linear problems with integral boundary conditions

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Abstract

We consider the integral boundary–value problem for a certain class of non–linear system of ordinary differential equations.
We give a new approach for studying this problem, namely by using an appropriate parametrization technique the given problem is reduced to the equivalent parametrized two–point boundary–value problem with linear boundary conditions without integral term.
To study the transformed problem we use a method based upon a special type of successive approximations, which are constructed analytically.

Problem Setting

We consider the nonlinear boundary–value problem subjected to the integral boundary conditions

$$\frac{dx(t)}{dt} = f(t, x(t)), t \in [0, T], \ x \in \mathbb{R}^n, \tag{1}$$

$$Ax(0) + \int_0^T B(s)x(s)ds + Cx(T) = d, \tag{2}$$

where A and $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{22} & O_{n-p} \end{pmatrix}$ are some given singular $n \times n$ matrixes, where C_{11} is a $p \times p$ matrix, $detC_{11} \neq 0$, C_{12} is a $p \times (n-p)$ matrix, C_{22} is a $(n-p) \times p$ matrix, O_{n-p} is a $(n-p) \times (n-p)$ zero–matrix and $B(\cdot)$ is a continuous $n \times n$ matrix.

Here, we suppose that the function

$$f : [0, T] \times D \to \mathbb{R}^n$$

is continuous, where $D \subset \mathbb{R}^n$ is a closed and bounded domain, and define $\Lambda \subset \mathbb{R}^n$ as

$$\Lambda := \left\{ \int_0^T B(s)x(s)ds : x \text{ has values in } D \right\}.$$

The problem is to find a continuously differentiable solution of the system of differential equations (1) satisfying the given integral boundary restrictions (2).

Parametrization of the Integral Boundary Conditions

To pass to the linear two–point boundary conditions from (2) we introduce the following parameters

$$
\begin{gathered}
z := x(0) = col\,(x_1(0), x_2(0), \ldots, x_n(0)) = col\,(z_1, z_2, \ldots, z_n)\,, \\
\lambda := \int_0^T B(s)x(s)ds = col\,(\lambda_1, \lambda_2, \ldots, \lambda_n)\,, \\
\eta := col\left(\underbrace{0, 0, \ldots, 0,}_{p} x_{p+1}(T), x_{p+2}(T), \ldots, x_n(T)\right) = \\
= col\left(\underbrace{0, 0, \ldots, 0,}_{p} \eta_{p+1}, \eta_{p+2}, \ldots, \eta_n\right).
\end{gathered} \tag{3}
$$

Using parametrization (3), the integral boundary restrictions (2) can be written as the linear ones:

$$Ax(0) + C_1 x(T) = d - \lambda + \eta, \tag{4}$$

where $C_1 = \begin{pmatrix} C_{11} & C_{12} \\ C_{22} & I_{n-p} \end{pmatrix}$, I_{n-p} is a $(n-p) \times (n-p)$ unit matrix, λ and η are parameters given by (3).

Let us put:

$$d(\lambda, \eta) := d - \lambda + \eta. \tag{5}$$

Taking into account (5) the parametrized boundary conditions (4) can be rewritten in the form:

$$Ax(0) + C_1 x(T) = d(\lambda, \eta). \tag{6}$$

Remark 1. The set of the solutions of the non–linear boundary–value problem with integral boundary conditions (1), (2) coincides with the set of the solutions of the parametrized problem (1) with linear boundary restrictions (6), satisfying additional conditions (3).

Construction of Successive Approximations

Let us introduce the vector

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\right] \qquad (7)$$

and suppose that the original boundary–value problem (1), (2) is such that

$$D_{\frac{T}{2}\delta_D(f)} \neq \emptyset. \qquad (8)$$

where, for any β,

$$D_\beta := \left\{z \in D : B(z,\beta) \subset D\right\}$$

$D_{\frac{T}{2}\delta_D(f)}$ is the collection of points z belonging to D together with their $\frac{T}{2}\delta_D(f)$-neighbourghoods (in the sense of componentwise inequalities)

Assume that the function $f(t,x)$ in the right hand–side of (1) satisfies Lipschitz condition of the form

$$|f(t,u) - f(t,v)| \leq K\,|u - v|\,, \tag{9}$$

for all $t \in [0,T]$, $\{u,v\} \subset D$ with some non–negative constant matrix $K = \left(k_{ij}\right)_{i,j=1}^{n}$.
Moreover, we suppose that the spectral radius $r(K)$ of the matrix K satisfies the following inequality

$$r(K) < \frac{10}{3T}. \tag{10}$$

Let us connect with the parametrized boundary–value problem (1), (6) the sequence of functions:

$$x_m(t,z,\lambda,\eta) := z + \int_0^t f(s, x_{m-1}(s,z,\lambda,\eta))ds - \\ - \frac{t}{T}\left[\int_0^T f(s, x_{m-1}(s,z,\lambda,\eta))ds + \\ + C_1^{-1}\left[d(\lambda,\eta) - (A + C_1)\,z\right]\right], \quad (11)$$

where $m = 1, 2, 3, \ldots,$

$$x_0(t,z,\lambda,\eta) = z + \frac{t}{T}C_1^{-1}\left[d(\lambda,\eta) - (A + C_1)\,z\right] \in D_{\frac{T}{2}\delta_D(f)},$$

and z, λ, η are considered as parameters.

Convergence theorem I

Theorem 1. *Assume that the function $f : [0, T] \times D \to \mathbb{R}^n$ in the right hand–side of the system of differential equations (1) and the parametrized boundary restrictions (6) satisfy conditions (8)–(10).*
Then for all fixed $z \in D_{\frac{T}{2}\delta_D(f)}$, $\lambda \in \Lambda$, $\eta \in D$:

1. *The functions of the sequence (11) are continuously differentiable and satisfy the parametrized boundary conditions (6):*

$$Ax_m(0, z, \lambda, \eta) + C_1 x_m(T, z, \lambda, \eta) = d(\lambda, \eta),$$

m=1,2,3,. . .

Convergence theorem II

2. *The sequence of functions (11) for $t \in [0, T]$ converges uniformly as $m \to \infty$ to the limit function*

$$x^*(t, z, \lambda, \eta) = \lim_{m \to \infty} x_m(t, z, \lambda, \eta). \qquad (12)$$

3. *The limit function $x^*(t, z, \lambda, \eta)$ satisfies the parametrized linear two–point boundary conditions:*

$$Ax^*(0, z, \lambda, \eta) + C_1 x^*(T, z, \lambda, \eta) = d(\lambda, \eta).$$

Convergence theorem III

4. *The limit function (12) for all $t \in [0, T]$ is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_0^t f(s, x(s))ds - \frac{t}{T}\left[\int_0^T f(s, x(s))ds + \right.$$
$$\left. + C_1^{-1}\left[d(\lambda, \eta) - (A + C_1)\, z\right]\right],$$

Convergence theorem IV

i. e. it is the solution of the Cauchy problem for the modified system of differential equations:

$$\frac{dx}{dt} = f(t,x) + \Delta(z,\lambda,\eta),$$

$$x(0) = z,$$

where

$$\Delta(z,\lambda,\eta) := \frac{1}{T} C_1^{-1} \left[d(\lambda,\eta) - (A + C_1) z \right] - \\ - \frac{1}{T} \int_0^T f(s, x(s)) ds. \quad (13)$$

Convergence theorem V

5. *The following error estimation holds:*

$$|x^*(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)| \leq \\ \leq \frac{20}{9} t \left(1 - \frac{t}{T}\right) Q^m (I_n - Q)^{-1} \delta_D(f), \quad (14)$$

where

$$Q := \frac{3T}{10} K. \quad (15)$$

Control Parameter

Consider the Cauchy problem

$$\frac{dx}{dt} = f(t,x) + \mu, t \in [0,T] \tag{16}$$

$$x(0) = z, \tag{17}$$

where $\mu = col(\mu_1, \ldots, \mu_n)$ is a control parameter.

Control parameter theorem

Theorem 2. *Let $z \in D_{\frac{T}{2}\delta_D(f)}$, $\lambda \in \Lambda$, $\eta \in D$ and $\mu \in \mathbb{R}^n$ — are some given vectors. Suppose that for the system of differential equations (1) all conditions of Theorem 1 are hold.*
Then in order the solution $x = x(t, z, \lambda, \eta, \mu)$ of the initial–value problem (16), (17) also satisfies parametrized boundary conditions (6) it is necessary and sufficient that the parameter μ was given by formula

$$\mu = \mu_{z,\lambda,\eta} = \frac{1}{T} C_1^{-1} [d(\lambda, \eta) - (A + C_1)z] - \\ - \frac{1}{T} \int_0^T f(s, x^*(s, z, \lambda, \eta)) ds. \quad (18)$$

The relation of the limit function to the solution of the original problem I

Theorem 3. *Let the conditions (8)–(10) are hold for the original boundary–value problem (1), (2). Then* $x^*(\cdot, z^*, \lambda^*, \eta^*)$ *is the solution of the parametrized boundary–value problem (1), (6) if and only if*

$$z^* = (z_1^*, z_2^*, \dots, z_n^*),$$
$$\eta^* = (\underbrace{0, 0, \dots, 0}_{p}, \eta_{p+1}^*, \eta_{p+2}^*, \dots, \eta_{p+n}^*),$$
$$\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$$

satisfy determining system of algebraic or transcendental equations

The relation of the limit function to the solution of the original problem II

$$\Delta(z,\lambda,\eta)=\frac{1}{T}\Bigg[C_1^{-1}\left[d(\lambda,\eta)-(A+C_1)z\right]-$$

$$-\int_0^T f(s,x^*(s,z,\lambda,\eta))ds\Bigg]=0, \quad (19)$$

$$\int_0^T B(s)x^*(s,z,\lambda,\eta)\,ds=\lambda, \quad (20)$$

$$x_i^*(T,z,\lambda,\eta)=\eta_i, \quad (21)$$

$i=\overline{p+1,n}.$

Lemma 1. *Let all conditions of Theorem 1 be satisfied. Furthermore there exist some vectors* $z \in D_{\frac{T}{2}\delta_D(f)}$, $\lambda \in \Lambda$ *and* $\eta \in D$ *that satisfy the system of determining equations (14)–(21). Then the non-linear boundary–value problem (1) with integral boundary conditions (2) has the solution* $x(\cdot)$ *such that*

$$x(0) = z,$$
$$\int_0^T B(s)x(s)ds = \lambda,$$
$$x_i(T) = \eta_i,$$

$i = \overline{p+1, n}$.

Moreover this solution is given by formula

$$x(t) = x^*(t, z, \lambda, \eta), \quad t = [0, T], \tag{22}$$

where $x^*(t, z, \lambda, \eta)$ *is the limit function of the sequence (11). And if the boundary–value problem (1), (2) has a solution* $x(\cdot)$*, then this solution is given by (22), and the system of determining equations (14)–(21) is satisfied when*

$$z = x(0),$$
$$\lambda = \int_0^T B(s)x(s)ds,$$
$$\eta_i = x_i(T),$$

$i = \overline{p+1, n}.$

Remarks I

Remark 2. The main difficulty of realization of this method is to find the limit function $x^*(\cdot, z, \lambda, \eta)$. But in most cases this problem can be solved using the properties of the approximate solution $x_m(\cdot, z, \lambda, \eta)$ built in an analytic form.
For $m \geq 1$ let us define the function $\Delta_m : D_{\frac{T}{2}\delta_D(f)} \times \Lambda \times D \to \mathbb{R}^n$ by formula

$$\Delta_m(z, \lambda, \eta) := \frac{1}{T}\Bigg[C_1^{-1}[d(\lambda, \eta) - (A + C_1)z] - \\ - \int_0^T f(s, x_m(s, z, \lambda, \eta))ds\Bigg], \quad m = 1,2,3,\ldots, \qquad (23)$$

where z, λ and η are given by the relation (3).

Remarks II

Remark 3. To investigate the solvability of the parametrized boundary–value problem (1), (6) we observe an approximate determining system of algebraic or transcendental equations of the form

$$\Delta_m(z,\lambda,\eta) = \frac{1}{T}\Bigg[C_1^{-1}\left[d(\lambda,\eta) - (A + C_1)z\right] - \\ - \int_0^T f(s, x_m(s,z,\lambda,\eta))ds\Bigg] = 0, \quad (24)$$

$$\int_0^T B(s)x_m\,(s,z,\lambda,\eta)\,ds = \lambda, \quad (25)$$

$$x_{m,i}\,(T,z,\lambda,\eta) = \eta_i, \quad (26)$$

$i = \overline{p+1,n}.$

Existence Of The Solutions Of The Integral Boundary–Value Problem

Lemma 2. *Let conditions of Theorem 1 be satisfied. Then for arbitrary $m \geq 1$ and z, λ, η of the form (3) for exact and approximate determining functions*

$$\Delta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$\Delta_m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

from (13) and (23), the estimate

$$|\Delta(z,\lambda) - \Delta_m(z,\lambda)| \leq \frac{10T}{27} K Q^m (I_n - Q)^{-1} \delta_D(f), \tag{27}$$

is true, where K, Q, $\delta_D(f)$ are given, respectively, by (9), (15), and (7).

Lemma 3. *Let conditions of Theorem 1 be satisfied. Then for arbitrary $m \geq 1$ and z, λ, η of the form (3) for the functions $x^*(t,z,\lambda,\eta)$ and $x_m(t,z,\lambda,\eta)$ correspondingly of the form (12) and (11) the following estimate*

$$\left| \int_0^T B(s)x^*(s,z,\lambda,\eta)\,ds - \int_0^T B(s)x_m(s,z,\lambda,\eta)\,ds \right| \leq$$
$$\leq \frac{10}{9}\bar{B}Q^m (I_n - Q)^{-1}\,\delta_D(f) \quad (28)$$

is true, where Q, $\delta_D(f)$ are given correspondingly by (15), (7) and

$$\bar{B} = \int_0^T |B(s)|\alpha_1(s)ds.$$

On the base of equations (14)–(21) and (24)–(26) let us introduce the mappings:

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{3n},$$
$$\Phi_m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{3n},$$

by setting for all z, λ, η of form (3)

$$\Phi(z,\lambda,\eta) := \begin{pmatrix} \frac{1}{T} C_1^{-1} \left[d(\lambda,\eta) - (A + C_1) z \right] - \\ -\frac{1}{T} \int_0^T f\left(s, x^*(s,z,\lambda,\eta)\right) ds \\ \int_0^T B(s) x^*\left(s,z,\lambda,\eta\right) ds - \lambda \\ x_i^*(T,z,\lambda,\eta) - \eta_i \end{pmatrix}, \tag{29}$$

$$\Phi_m(z,\lambda,\eta) := \begin{pmatrix} \frac{1}{T} \left[C_1^{-1} d(\lambda,\eta) - (A + C_1) z \right] - \\ -\frac{1}{T} \int_0^T f\left(s, x_m(s,z,\lambda,\eta)\right) ds \\ \int_0^T B(s) x_m\left(s,z,\lambda,\eta\right) ds - \lambda \\ x_{m,i}(T,z,\lambda,\eta) - \eta_i \end{pmatrix}, \tag{30}$$

$i = \overline{p+1, n}$.

Deffinition 1. *Let $H \subset \mathbb{R}^{3n}$ be an arbitrary non–empty set. For any pair of functions*

$$f_j = col\left(f_{j1}(x), \dots, f_{j,3n}(x)\right) : H \to \mathbb{R}^{3n}, j = 1, 2$$

we write

$$f_1 \triangleright_H f_2$$

if and only if there exist a function

$$k : H \to \{1, 2, \dots, 3n\}$$

such that

$$f_{1,k(x)}(x) > f_{2,k(x)}(x)$$

for all $x \in H$, which means that at every point $x \in H$ at least one of the components of the vector $f_1(x)$ is greater then the corresponding component of the vector $f_2(x)$.

Let us consider the set

$$\Omega = D_1 \times \Lambda_1 \times D_2, \tag{31}$$

where $D_1 \subset D_{\frac{T}{2}\delta_D(f)}$, $\Lambda_1 \subset \Lambda$, $D_2 \subset D$ — are certain bounded open sets.

Theorem 4. *Assume that conditions of Theorem 1 hold and, moreover, one can specify an $m \geq 1$ and a set $\Omega \subset \mathbb{R}^{3n}$ of the form (31) such that*

$$|\Phi_m| \triangleright_{\partial\Omega} \begin{pmatrix} \frac{10T}{27} K Q^m (I_n - Q)^{-1} \delta_D(f) \\ \frac{10}{9} \bar{B} Q^m (I_n - Q)^{-1} \delta_D(f) \\ \frac{5T}{9} Q^m (I_n - Q)^{-1} \delta_D(f) \end{pmatrix}, \tag{32}$$

holds, where $\partial\Omega$ is a bound of domain Ω.

If, in addition, the Brower degree of Φ_m over Ω with respect to zero satisfies the inequality

$$\deg\left(\Phi_m, \Omega, 0\right) \neq 0, \tag{33}$$

then there exist a triplet $(z^*, \lambda^*, \eta^*) \in \Omega$ such that the function

$$x^*(t) = x^*\left(t, z^*, \lambda^*, \eta^*\right) = \lim_{m \to \infty} x_m\left(t, z^*, \lambda^*, \eta^*\right) \tag{34}$$

is a solution of the nonlinear boundary–value problem (1) subjected to the integral boundary conditions (2) with the initial condition

$$x^*(0) = z^*. \tag{35}$$

Application

Consider the system

$$\begin{cases} \frac{dx_1}{dt} = 0.05x_2 + x_1x_2 - 0.005t^2 - 0.01t^3 + 0.1 = f_1(t, x_1, x_2), \\ \frac{dx_2}{dt} = 0.5x_1 - x_2^2 + 0.01t^4 + 0.15t = f_2(t, x_1, x_2), \end{cases} \tag{36}$$

where $t \in [0, \frac{1}{2}]$, with the linear two–point integral boundary conditions

$$Ax(0) + \int_0^{\frac{1}{2}} B(s)x(s)ds + Cx\left(\frac{1}{2}\right) = d, \tag{37}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; B(t) = \begin{pmatrix} 0 & t/2 \\ 1/2 & 1/4 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; d = \begin{pmatrix} 13/256 \\ 7/960 \end{pmatrix}.$$

It is easy to check that the exact solution of the problem (36), (37) is

$$\begin{cases} x_1^* = 0.1t, \\ x_2^* = 0.1t^2. \end{cases}$$

Suppose that the boundary–value problem (36), (37) is considered in the domain

$$D = \{(x_1, x_2) : |x_1| \leq 0.42, |x_2| \leq 0.4\}.$$

Let us introduce the following parameters:

$$
\begin{gathered}
z := x(0) = col\,(x_1(0), x_2(0)) = col\,(z_1, z_2)\,, \\
\lambda := \int_0^T B(s)x(s)ds = col\,(\lambda_1, \lambda_2) \\
\eta_2 := x_2\left(\tfrac{1}{2}\right)
\end{gathered} \tag{38}
$$

Using (38), the boundary restrictions (37) can be rewritten as linear ones that contain already non–singular matrix C_1

$$
Ax(0) + C_1 x\left(\frac{1}{2}\right) = d(\lambda, \eta), \tag{39}
$$

where $\eta = col(0, \eta_2)$, $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $d(\lambda, \eta) := d - \lambda + \eta$.

It is easy to check that the matrix K from the Lipschitz condition (9) can be taken as

$$K = \begin{pmatrix} 0 & 0.05 \\ 0.5 & 0.8 \end{pmatrix},$$

and

$$r(K) < 0.84 < \frac{10}{3T},$$

when $T = \frac{1}{2}$.

Vector $\delta_D(f)$ can be chosen as

$$\delta_D(f) \leq \begin{pmatrix} 0.18925 \\ 0.3278125 \end{pmatrix}.$$

The set $D_{\frac{T}{2}\delta_D(f)}$ is defined by the inequalities:

$$|z_1| \leq 0.05078125 - \lambda_1,$$
$$|z_2| \leq 0.007291666667 - \lambda_2 + \eta_2 - z_2,$$

$\forall \lambda_1, \lambda_2 \in \Lambda,\ \eta_2 \in D.$

The set Λ is such that

$$\Lambda = \{(\lambda_1, \lambda_2) : |\lambda_1| \leq 0.105, |\lambda_2| \leq 0.31\}.$$

One can verify that, for the parametrized boundary–value problem (36), (39), all needed conditions are fulfilled. So, we can proceed with application of the numerical–analytic scheme described above and thus construct the sequence of approximate solutions.

The components of the iteration sequence (11) for the boundary–value problem (36) under the linear parametrized two–point boundary conditions (39) have the form

$$x_{m,1}(t,z,\lambda,\eta) := z_1 + \int_0^t f_1\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds - \\ -2t\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds + \\ +2t(0.05078125 - \lambda_1 - z_1), \tag{40}$$

$$x_{m,2}(t,z,\lambda,\eta) := z_2 + \int_0^t f_2\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds - \\ -2t\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right)) ds + \\ +2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2), \tag{41}$$

where $m = 1, 2, 3, \ldots,$

$$x_{0,1}(t,z,\eta,\lambda) = z_1 + 2t(0.05078125 - \lambda_1 - z_1), \tag{42}$$

$$x_{0,2}(t,z,\eta,\lambda) = z_2 + 2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2). \tag{43}$$

The system of approximate determining equations

$$\Delta_m(z,\lambda,\eta) = col(\Delta_{m,1}(z,\lambda,\eta), \Delta_{m,2}(z,\lambda,\eta))$$

of the form (24)–(26) depending on the number of iterations for the given example is

$$\Delta_{m,1}(z,\lambda,\eta) = -2\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds + \\ +2(0.05078125 - \lambda_1 - z_1) = 0, \quad (44)$$

$$\Delta_{m,2}(z,\lambda,\eta) = -2\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds + \\ +2(0.007291666667 - \lambda_2 + \eta_2 - 2z_2) = 0, \tag{45}$$

$$\int_0^{\frac{1}{2}} B(s)x_m(s,z,\lambda,\eta)ds = \lambda, \tag{46}$$

$$x_{m,2}\left(\frac{1}{2}, z, \lambda, \eta\right) = \eta_2, \tag{47}$$

$m = 1, 2, 3, \ldots.$

Using (40)–(43) as a result of the first iteration ($m = 1$), by Maple 13 we get:

$$\begin{gathered}
x_{11} = -0.0025t^4 + 0.1019859484t + 1.333333333t^3\lambda_1\lambda_2 - \\
-1.333333333t^3\lambda_1\eta_2 + 2.666666666t^3\lambda_1 z_2 + 1.333333333t^3 z_1\lambda_2 - \\
-1.333333333t^3 z_1\eta_2 + 2.666666666t^3 z_1 z_2 + t^2 z_1\eta_2 - t^2 z_1\lambda_2 - \\
-3t^2 z_1 z_2 - t^2\lambda_1 z_2 - 0.3333333334t\lambda_1\lambda_2 + \\
+0.3333333334t\lambda_1\eta_2 - 0.1666666667t\lambda_1 z_2 + 0.1666666666tz_1\lambda_2 - \\
-0.1666666666tz_1\eta_2 - 2.001215278tz_1 - 0.6770833333t^3\lambda_2 + \\
+0.06770833333t^3\eta_2 - 0.1354166667t^3 z_2 - 0.009722222219t^3\lambda_1 - \\
-0.009722222219t^3 z_1 - 0.05t^2\lambda_2 + 0.05t^2\eta_2 - 0.04921875t^2 z_2 + \\
+0.007291666665t^2 z_1 + 0.04192708333t\lambda_2 - 0.04192708333t\eta_2 - \\
-1.997569444t\lambda_1 + 0.0003645833334t^2 - 0.001172960069t^3 + z_1,
\end{gathered}$$

$$
\begin{aligned}
x_{12} = & -0.035719256 36t - 1.333333333t^3\lambda_2^2 - 1.333333333t^3\eta_2^2 - \\
& -5.333333333t^3z_2^2 - 0.5t^2\lambda_1 + 4t^2z_2^2 + 0.3333333334t\lambda_2^2 + \\
& +0.3333333334t\eta_2^2 + 0.25tz_1 + 0.0194444444t^3\lambda_2 - \\
& -0.0194444444t^3\eta_2 + 0.0388888888t^3z_2 - 0.01458333333t^2z_2 - 0.5t^2z_1 - \\
& -2.004861111t\lambda_2 + 2.004861111t\eta_2 + 0.25t\lambda_1 + 0.002t^5 + \\
& +2.666666666t^3\lambda_2\eta_2 - 5.333333333t^3\lambda_2z_2 + \\
& +5.333333333t^3\eta_2z_2 + 2t^2\lambda_2z_2 - 2t^2\eta_2z_2 - 0.6666666667t\lambda_2\eta_2 + \\
& +0.3333333334t\lambda_2z_2 - 0.3333333334t\eta_2z_2 + \\
& +0.100390625t^2 - 0.00007089120366t^3 + z_2,
\end{aligned}
$$

for all $t \in \left[0, \frac{1}{2}\right]$.

The computation shows that the approximate solutions of the determining system (44)–(47) for $m = 1$ are

$$\begin{aligned} z_1 := z_{11} &= -4.253290711 \cdot 10^{-7}, \\ z_2 := z_{12} &= 7.295492706 \cdot 10^{-7}, \\ \lambda_1 := \lambda_{11} &= 0.0007814848293, \\ \lambda_2 := \lambda_{12} &= 0.007290937121, \\ \eta_2 := \eta_{12} &= 0.0249993271. \end{aligned}$$

The first approximation to the first and second components of solution is

$$x_{11} = -0.0025t^4 + 0.09968792498t - 4.253290711 \cdot 10^{-7} + \\ + 0.001249955722t^2 - 8.714713042 \cdot 10^{-8} t^3,$$

$$x_{12} = 0.00008047566353t + 0.002t^5 + 7.295492706 \cdot 10^{-7} + \\ + 0.1000000588t^2 - 0.0008332398387t^3.$$

The error of the first approximation is

$$\max_{t\in[0,\frac{1}{2}]} |x_1^*(t) - x_{11}(t)| \le 2.1 \cdot 10^{-5},$$
$$\max_{t\in[0,\frac{1}{2}]} |x_2^*(t) - x_{12}(t)| \le 2.2 \cdot 10^{-6}.$$

The error of the second approximation is

$$\max_{t\in[0,\frac{1}{2}]} |x_1^*(t) - x_{21}(t)| \le -4.03 \cdot 10^{-8},$$
$$\max_{t\in[0,\frac{1}{2}]} |x_2^*(t) - x_{22}(t)| \le 1.2 \cdot 10^{-6}.$$

Continuing calculations one can get more approximate solutions of the original boundary–value problem with higher precision.

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THANK YOU FOR ATTENTION!