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On the BVPs for Φ-Laplacian type equation

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1 Introduction

Consider the Φ-Laplacian type equation

$$\boxed{\frac{d}{dt}\Phi(t,x') + f(t,x) = 0,} \tag{1}$$

$t \in I := [a,b]$,
$f \in C(I \times \mathbb{R}, \mathbb{R})$ is Lipschitz function with respect to x,
$\Phi \in C(I \times \mathbb{R}, \mathbb{R})$ is Lipschitz and monotone func. with respect to x',
together with the boundary conditions

$$\boxed{\begin{cases} x(a)\cos\alpha - \Phi(a, x'(a))\sin\alpha = 0, \\ x(b)\cos\beta - \Phi(b, x'(b))\sin\beta = 0, \end{cases}} \tag{2}$$

where $0 \le \alpha < \pi,\ 0 < \beta \le \pi$.

Denote $\quad y = \Phi(t,x'), \quad (1) \quad \Rightarrow \quad \boxed{\begin{cases} x' = \Phi^{-1}(t,y), \\ y' = -f(t,x). \end{cases}}$ (3)

and apply the quasilinearization process described in [3], [4], [7].

2 Quasilinearization process

ΦLE
$$\frac{d}{dt}\Phi(t,x') + f(t,x) = 0, \qquad \begin{array}{l} x(a)\cos\alpha - \Phi(a,x'(a))\sin\alpha = 0, \\ x(b)\cos\beta - \Phi(b,x'(b))\sin\beta = 0, \end{array}$$

$$\Downarrow$$

NLS
$$\begin{cases} x' = \Phi^{-1}(t,y), \\ y' = -f(t,x), \end{cases} \qquad \begin{array}{l} x(a)\cos\alpha - y(a)\sin\alpha = 0, \\ x(b)\cos\beta - y(b)\sin\beta = 0, \end{array}$$

$$\Updownarrow$$

QLS
$$\begin{cases} x' - ky = F_1(t,y), \\ y' + kx = F_2(t,x), \end{cases} \qquad \begin{array}{l} x(a)\cos\alpha - y(a)\sin\alpha = 0, \\ x(b)\cos\beta - y(b)\sin\beta = 0, \end{array}$$
$$\Omega = \{(t,x,y):\ a \le t \le b,\ |x| \le N_x,\ |y| \le N_y\}$$

$$\Downarrow$$

$$\begin{cases} x(t) = \int\limits_a^b (G_{11}(t,s)F_1(s,y(s)) + G_{12}(t,s)F_2(s,x(s)))ds, \\ y(t) = \int\limits_a^b (G_{21}(t,s)F_1(s,y(s)) + G_{22}(t,s)F_2(s,x(s)))ds, \end{cases}$$
$$|x(t)| \le (b-a)(\Gamma_{11}\cdot M_1 + \Gamma_{12}\cdot M_2) \le N_x,$$
$$|y(t)| \le (b-a)(\Gamma_{21}\cdot M_1 + \Gamma_{22}\cdot M_2) \le N_y.$$

“Key” inequalities

If $\boxed{|x(t)| \le N_x,\ |y(t)| \le N_y \quad \forall t \in [a,\, b]}$,
then we say that the nonlinear problem for the differential system (3) *allows for quasilinearization with respect to the extracted linear part*.

$$\boxed{\begin{cases} (b-a)(\Gamma_{11}\cdot M_1 + \Gamma_{12}\cdot M_2) \le N_x, \\ (b-a)(\Gamma_{21}\cdot M_1 + \Gamma_{22}\cdot M_2) \le N_y. \end{cases}} \tag{4}$$

If the inequalities (4) are fulfilled for different values of k, when the respective extracted linear parts are essentially different, then in this case we are able to obtain multiplicity results for boundary value problems under consideration.

3 Results for quasi-linear systems

Consider the quasi-linear system

$$\boxed{\begin{cases} x' - ky &= F_1(t,y), \\ y' + kx &= F_2(t,x), \end{cases}} \tag{5}$$

where functions F_1, F_2 are continuous, bounded and satisfy the Lipschitz conditions with respect to y and x respectively, together with the boundary conditions

$$\boxed{\begin{aligned} x(a)\cos\alpha - y(a)\sin\alpha = 0, \\ x(b)\cos\beta - y(b)\sin\beta = 0. \end{aligned}} \tag{6}$$

In order to classify the linear parts of (5) consider the respective homogeneous problem

$$\boxed{\begin{cases} x' - ky &= 0, \qquad x(a)\cos\alpha - y(a)\sin\alpha = 0, \\ y' + kx &= 0, \qquad x(b)\cos\beta - y(b)\sin\beta = 0. \end{cases}} \tag{7}$$

Introduce polar coordinates as

$$x(t) = r(t)\,\sin\phi(t), \quad y(t) = r(t)\,\cos\phi(t). \tag{8}$$

Then the angular function $\phi(t)$ for (7) satisfies $\boxed{\phi'(t) = k}$ and therefore the angular function $\phi(t)$ is monotonically increasing if $k > 0$.

The boundary conditions (6) in polar coordinates take the form

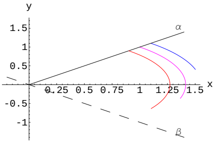
$$\boxed{\phi(a) = \alpha, \quad \phi(b) = \beta (\operatorname{mod} \pi).} \tag{9}$$

A linear part $\big(LX\big)(t)$ in system (7) is non-resonant with respect to the boundary conditions (6), if a coefficient $k > 0$ satisfies

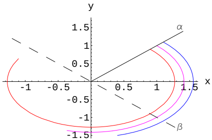
$$\boxed{\sin\big(\beta - \alpha - k(b-a)\big) \neq 0}.$$

All proper values of k form the intervals of non-resonance, in each of them the angular function $\phi(t)$ has distinctive properties.

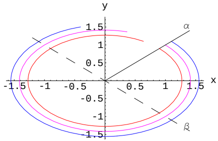
$$\left(0,\ \frac{\pi}{2}\right),\ \left(\frac{\pi}{2},\ \frac{3\pi}{2}\right),\ \left(\frac{3\pi}{2},\ \frac{5\pi}{2}\right),\ \ldots,\left(\frac{(2i-1)\pi}{2},\ \frac{(2i+1)\pi}{2}\right),\ldots$$



a) $k \in (0,\ \frac{\pi}{2})$



b) $k \in (\frac{\pi}{2},\ \frac{3\pi}{2})$



c) $k \in (\frac{3\pi}{2},\ \frac{5\pi}{2})$

Figure 1. Phase portraits of the solutions to the problem

$$\begin{cases} x' - k\,y &= 0, \qquad\qquad x(0) - y(0) &= 0, \\ y' + k\,x &= 0, \qquad\qquad x(1) + y(1) &= 0 \end{cases} \tag{10}$$

in the interval $t \in [0,\ 1]$ for different values of k.

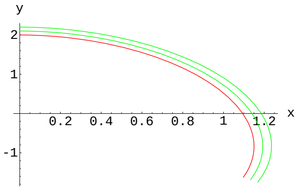
Definition 1. *A linear part* $\big(LX\big)(t)$ *in* (7) *is called i-nonresonant with respect to the boundary conditions* (6) *if the angular function* $\phi(t)$, *defined by the initial condition* $\phi(a) = \alpha$, *takes exactly* i *values of the form* $\beta(\operatorname{mod} \pi)$ *in the interval* (a, b) *and* $\phi(b) \neq \beta(\operatorname{mod} \pi)$.

If for different values of k the respective linear parts have different types of non-resonance then we say that these linear parts are *essentially different*.

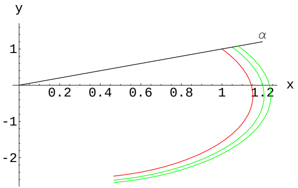
Let $\big(\xi(t), \eta(t)\big)$ be a solution of the quasi-linear problem (5), (6).

Definition 2. *We say that* $\big(x(t;\delta),\, y(t;\delta)\big)$ *is a neighboring solution of a solution* $\big(\xi(t), \eta(t)\big)$ *to the quasi-linear problem* (5), (6), *if* $\big(x(t;\delta),\, y(t;\delta)\big)$ *solves the same quasi-linear system* (5), *satisfies the first boundary condition* $x(a;\delta)\cos\alpha - y(a;\delta)\sin\alpha = 0$ *and there exists* $\varepsilon > 0$ *such that* $\forall\, \delta \in (0, \varepsilon]$

$$y(a;\delta) = \eta(a) + \delta,\ \textit{if}\ \ \alpha = 0 \quad \textit{or} \quad x(a;\delta) = \xi(a) + \delta,\ \textit{if}\ \ \alpha > 0.$$



a) $\alpha = 0$



b) $0 < \alpha < \pi$

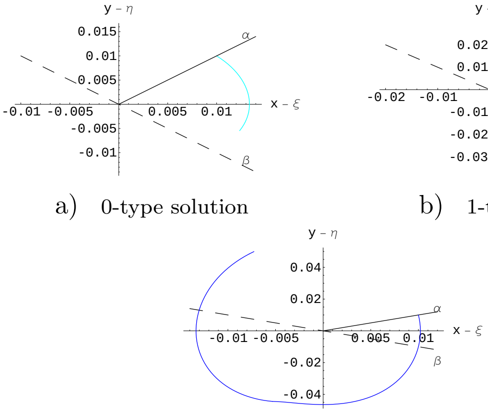
Figure 2. Neighboring solutions $\big(x(t;\,\delta), y(t;\,\delta)\big)$ of the solution $\big(\xi(t), \eta(t)\big)$.

In order to classify solutions of the quasi-linear problem under consideration introduce (local) polar coordinates for the difference between neighboring solution $\big(x(t;\delta), y(t;\delta)\big)$ and being investigated solution $\big(\xi(t), \eta(t)\big)$ as

$$\begin{aligned} x(t;\delta) - \xi(t) &= \rho(t)\, \sin\Theta(t;\delta), \\ y(t;\delta) - \eta(t) &= \rho(t)\, \cos\Theta(t;\delta), \end{aligned} \tag{11}$$

where $\Theta(a;\delta) = \alpha$.

Definition 3. *We say that $\big(\xi(t), \eta(t)\big)$ is an i-type solution of the quasi-linear problem (5), (6), if there exists some small number $\varepsilon > 0$ such that for $\delta \in (0, \varepsilon]$ the angular function $\Theta(t;\delta)$ of the difference between neighboring solution $\big(x(t;\,\delta), y(t;\,\delta)\big)$ and $\big(\xi(t), \eta(t)\big)$, defined by the initial condition $\Theta(a;\delta) = \alpha$, takes exactly i values of the form $\beta (\operatorname{mod} \pi)$ in the interval $(a,\, b)$ and $\Theta(b;\delta) \neq \beta (\operatorname{mod} \pi)$.*



c) 2-type solution

Figure 3. Phase portraits of differences between neighboring solution (x, y) and being investigated solution (ξ, η), $t \in [a,\, b]$.

Theorem 3.1.

If a linear part $\big(LX\big)(t)$ in the quasi-linear system (5) is i-nonresonant with respect to the boundary conditions (6), then the problem (5), (6) has an i-type solution.

This theorem was proved in [6], [7] for a quasi-linear system

$$\begin{cases} x' + a_{11}x + a_{12}y & = F_1(t,x,y), \\ y' + a_{21}x + a_{22}y & = F_2(t,x,y). \end{cases} \tag{12}$$

4 Green's matrix

The Green's matrix for the homogeneous problem (7) is constructed explicitly.

$$\mathbb{G}_k(t,\,s) = \begin{cases} \begin{pmatrix} \frac{\cos(\alpha-k(a-s))\sin(\beta-k(b-t))}{\sin(\beta-\alpha-k(b-a))} & \frac{-\sin(\alpha-k(a-s))\sin(\beta-k(b-t))}{\sin(\beta-\alpha-k(b-a))} \\ \frac{\cos(\alpha-k(a-s))\cos(\beta-k(b-t))}{\sin(\beta-\alpha-k(b-a))} & \frac{-\sin(\alpha-k(a-s))\cos(\beta-k(b-t))}{\sin(\beta-\alpha-k(b-a))} \end{pmatrix} \\ \qquad\qquad \text{if} \quad a \le s \le t \le b, \\ \begin{pmatrix} \frac{\sin(\alpha-k(a-t))\cos(\beta-k(b-s))}{\sin(\beta-\alpha-k(b-a))} & \frac{-\sin(\alpha-k(a-t))\sin(\beta-k(b-s))}{\sin(\beta-\alpha-k(b-a))} \\ \frac{\cos(\alpha-k(a-t))\cos(\beta-k(b-s))}{\sin(\beta-\alpha-k(b-a))} & \frac{-\cos(\alpha-k(a-t))\sin(\beta-k(b-s))}{\sin(\beta-\alpha-k(b-a))} \end{pmatrix} \\ \qquad\qquad \text{if} \quad a \le t < s \le b. \end{cases} \tag{13}$$

$$\left|G_k^{ij}(t,\,s)\right| \le \frac{1}{|\sin(\beta-\alpha-k(b-a))|} =: \Gamma_k, \qquad (i,j=1,2). \tag{14}$$

5 Application

Consider the boundary value problem (1), (2),

$$\text{where} \quad \boxed{\Phi(t,x') = r(t)|x'|^{\frac{1}{p}} \operatorname{sgn} x', \quad f(t,x) = q(t)|x|^p \operatorname{sgn} x}, \tag{15}$$

$t \in I := [a,b]$, $p > 1$, $r, q \in C(I; (0,+\infty))$. Denote $\Phi(t,x') = y$, then obtain a two-dimensional differential system

$$\begin{cases} x' = (r(t))^{-p}|y|^p \operatorname{sgn} y, \\ y' = -q(t)|x|^p \operatorname{sgn} x, \end{cases} \tag{16}$$

together with the boundary conditions (6). The obtained system (16) is equivalent to a system

$$\begin{cases} x' - k\,y = (r(t))^{-p}|y|^p \operatorname{sgn} y - k\,y, \\ y' + k\,x = k\,x - q(t)|x|^p \operatorname{sgn} x, \end{cases} \tag{17}$$

where the coefficient $k > 0$ satisfies $\sin(\beta - \alpha - k(b-a)) \neq 0$.

Then we wish to make bounded the right sides in the system (17) $U_k(t,y) := (r(t))^{-p}|y|^p \operatorname{sgn} y - k\,y, \quad V_k(t,x) := k\,x - q(t)|x|^p \operatorname{sgn} x.$

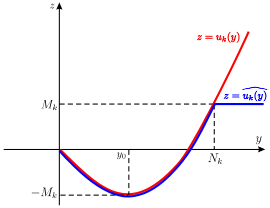


Figure 4. Number N_k existence.

Computation gives that

$$m_y(t^*) = |U_k(t^*, y_0)| = \left(\frac{k}{p}\right)^{\frac{p}{p-1}} (p-1)\big(r(t^*)\big)^{\frac{p}{p-1}}, \tag{18}$$

$$n_y(t^*) = k^{\frac{1}{p-1}} \big(r(t^*)\big)^{\frac{p}{p-1}} \gamma, \tag{19}$$

where a constant γ is a root of the equation $\gamma^p = \gamma + (p-1)p^{\frac{p}{1-p}}$.

$$m_x(t^*) = |V_k(t^*, x_0)| = \left(\frac{k}{p}\right)^{\frac{p}{p-1}} (p-1)\big(q(t^*)\big)^{\frac{1}{1-p}}, \tag{20}$$

$$n_x(t^*) = k^{\frac{1}{p-1}} \big(q(t^*)\big)^{\frac{1}{1-p}} \gamma. \tag{21}$$

Instead of the functions $U_k(t,y)$, $V_k(t,x)$ consider

$$\begin{aligned}\widehat{U_k(t,y)} &:= U_k(t,\delta(-N_y,y,N_y)),\\ \widehat{V_k(t,x)} &:= V_k(t,\delta(-N_x,x,N_x)),\end{aligned}$$

where $N_y = \min\{n_y(t) : t \in [a,b]\}$ and $N_x = \min\{n_x(t) : t \in [a,b]\}$, besides

$$\begin{aligned}\sup|\widehat{U_k(t,y}| &= M_y = \max\{m_y(t) : t \in [a,b]\},\\ \sup|\widehat{V_k(t,x)}| &= M_x = \max\{m_x(t) : t \in [a,b]\}.\end{aligned}$$

The nonlinear system (17) and the quasi-linear one

$$\begin{cases} x' - k\,y = \widehat{U_k(t,y)}, \\ y' + k\,x = \widehat{V_k(t,x)} \end{cases} \tag{22}$$

are equivalent in a domain

$$\Omega_k = \{(t,\, x,\, y) :\; a \le t \le b,\, |x(t)| \le N_x,\, |y(t)| \le N_y\}. \tag{23}$$

The modified quasi-linear problem (22), (6) is solvable if $k > 0$ satisfies $\sin(\beta - \alpha - k(b-a)) \neq 0$. The respective solution $\big(x_k(t),\, y_k(t)\big)$ can be written in the integral form and can be estimated. Since the elements of Green's matrix have same estimate therefore "key" inequalities for BVP under consideration take the form

$$\boxed{\begin{gathered}(b-a)\Gamma_k\Big(M_y + M_x\Big) \;<\; N_x,\\ (b-a)\Gamma_k\Big(M_y + M_x\Big) \;<\; N_y\end{gathered}} \tag{24}$$

or, equivalently

$$\boxed{\frac{(b-a)}{|\sin(\beta-\alpha-k(b-a))|}\Big(M_y + M_x\Big) < \min\{N_x,\, N_y\}.} \tag{25}$$

If "key" inequality (25) holds then the nonlinear problem (17), (6) (or, equivalently, the original problem (1), (2)) allows for quasilinearization and therefore has a solution of definite type.

Suppose that in (15) $0 < r_1 \le r(t) \le r_2$ and $0 < q_1 \le q(t) \le q_2$ $\forall t \in [a,b]$.

Taking into consideration the expressions for $M_y,\ M_x,\ N_y,\ N_x,$ γ (19) we obtain the following inequality

$$\frac{k(b-a)}{|\sin(\beta-\alpha-k(b-a))|}\cdot p^{\frac{p}{1-p}}\cdot(p-1)\cdot\left(r_2^{\frac{p}{p-1}}+q_1^{\frac{1}{1-p}}\right)<A\cdot\gamma, \quad (26)$$

where $A=\min\{r_1^{\frac{p}{p-1}},q_2^{\frac{1}{1-p}}\}$. Thus a fulfilment of the inequality (26) is a sufficient condition for existence of a solution of definite type to the problem (1), (2). Depending on the functions $r(t)$ and $q(t)$ and parameter p there are possible 4 different cases. Denote:

$$\begin{array}{llll}
*\ \mu=\dfrac{1}{r_2^p\cdot q_2}, & \text{if}\quad r_1^{-p}<q_2 & \text{and} & r_2^{-p}<q_1; \\
*\ \mu=r_1^p\cdot q_1, & \text{if}\quad r_1^{-p}>q_2 & \text{and} & r_2^{-p}>q_1; \\
*\ \mu=\left(\dfrac{r_1}{r_2}\right)^p, & \text{if}\quad r_1^{-p}>q_2 & \text{and} & r_2^{-p}<q_1; \\
*\ \mu=\dfrac{q_1}{q_2}, & \text{if}\quad r_1^{-p}<q_2 & \text{and} & r_2^{-p}>q_1.
\end{array} \quad (27)$$

Then the inequality (26) is fulfilled if the following inequality holds

$$\boxed{\frac{2k(b-a)}{|\sin(\beta-\alpha-k(b-a))|}\cdot p^{\frac{p}{1-p}}\cdot(p-1)\cdot\mu^{\frac{1}{1-p}}<\gamma.}$$

Dirichlet boundary conditions ($\alpha = 0$, $\beta = \pi$):

$$x(a) = 0, \quad x(b) = 0. \tag{28}$$

Neumann boundary conditions ($\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$):

$$x'(a) = 0, \quad x'(b) = 0. \tag{29}$$

Mixed boundary conditions ($\alpha = 0$, $\beta = \frac{\pi}{2}$):

$$x(a) = 0, \quad x'(b) = 0. \tag{30}$$

The linear part $\Big(LX\Big)(t)$ under consideration is i-nonresonant

- with respect to the boundary conditions (28) and (29), if

$$\boxed{k \in \left(\frac{i\pi}{(b-a)}, \frac{(i+1)\pi}{(b-a)}\right), \quad i \in \mathbb{N} \cup \{0\}};$$

- with respect to the mixed boundary conditions (30), if

$$\boxed{k \in \left(\frac{(2i-1)\pi}{2(b-a)}, \frac{(2i+1)\pi}{2(b-a)}\right), \quad i \in \mathbb{N}}.$$

Theorem 5.1. If there exists some number $k_i \in \left(\frac{i\pi}{(b-a)}, \frac{(i+1)\pi}{(b-a)}\right)$, $i \in \mathbb{N} \cup \{0\}$, which satisfies the inequality

$$\boxed{\frac{2k_i(b-a)}{|\sin k_i(b-a)|} \cdot p^{\frac{p}{1-p}} \cdot (p-1) \cdot \mu^{\frac{1}{1-p}} < \gamma}, \tag{31}$$

where γ is a root of the equation $\gamma^p = \gamma + (p-1) \cdot p^{\frac{p}{1-p}}$ and μ is number of the form (27), then there exists an i-type solution of the nonlinear problem (1), (15), (28) (or (1),(15), (29)).

Theorem 5.2. If there exists some number $k_i \in \left(\frac{(2i-1)\pi}{2(b-a)}, \frac{(2i+1)\pi}{2(b-a)}\right)$ $i \in \mathbb{N}$, which satisfies the inequality

$$\boxed{\frac{2k_i(b-a)}{|\cos k_i(b-a)|} \cdot p^{\frac{p}{1-p}} \cdot (p-1) \cdot \mu^{\frac{1}{1-p}} < \gamma}, \tag{32}$$

where γ is a root of the equation $\gamma^p = \gamma + (p-1) \cdot p^{\frac{p}{1-p}}$ and μ is number of the form (27), then there exists an i-type solution of the nonlinear problem (1), (15),(30).

Denote:

- τ_i is a root of the equation $\tau = \tan\tau$, which belongs to the interval $\left(i\pi, \dfrac{(2i+1)\pi}{2}\right)$, $i \in \mathbb{N} \cup \{0\}$;
- ν_i is a root of the equation $\nu = -\cot\nu$, which belongs to the interval $\left(\dfrac{(2i-1)\pi}{2}, \pi i\right)$, $i \in \mathbb{N}$.

The results of calculations are provided in the following tables. For certain values of p and μ these tables show which numbers k_i of the form $k_i = \dfrac{\tau_i}{(b-a)}$ or $k_i = \dfrac{\nu_i}{(b-a)}$ satisfy the inequality (31) or (32) respectively.

Table 1. Results of calculations for the BVP (1), (15),(28) (or for the BVP (1), (15), (29)).

p	γ	μ	k_i
$\frac{4}{3}$	1.2703	$\mu \geq 0.9144$	$k_0;\ k_1$
$\frac{5}{4}$	1.2813	$\mu \geq 0.8760$	$k_0;\ k_1$
		$\mu \geq 0.9991$	$k_0;\ k_1;\ k_2$
$\frac{6}{5}$	1.2884	$\mu \geq 0.8630$	$k_0;\ k_1$
		$\mu \geq 0.9588$	$k_0;\ k_1;\ k_2$
$\frac{7}{6}$	1.2933	$\mu \geq 0.8596$	$k_0;\ k_1$
		$\mu \geq 0.9384$	$k_0;\ k_1;\ k_2$
		$\mu \geq 0.9931$	$k_0;\ k_1;\ k_2;\ k_3$

Table 2. Results of calculations for the BVP (1), (15), (30).

p	γ	μ	k_i
$\frac{3}{2}$	1.2509	$\mu \geq 0.8390$	$k_0;\ k_1$
$\frac{4}{3}$	1.2703	$\mu \geq 0.7903$	$k_0;\ k_1$
$\frac{5}{4}$	1.2813	$\mu \geq 0.7852$	$k_0;\ k_1$
		$\mu \geq 0.9437$	$k_0;\ k_1;\ k_2$
$\frac{6}{5}$	1.2884	$\mu \geq 0.7907$	$k_0;\ k_1$
		$\mu \geq 0.9161$	$k_0;\ k_1;\ k_2$
		$\mu \geq 0.9949$	$k_0;\ k_1;\ k_2;\ k_3$
$\frac{7}{6}$	1.2933	$\mu \geq 0.7991$	$k_0;\ k_1$
		$\mu \geq 0.9034$	$k_0;\ k_1;\ k_2$
		$\mu \geq 0.9702$	$k_0;\ k_1;\ k_2;\ k_3$

6 Example

Consider the problem

$$\frac{d}{dt}(0.005(95+\sin\frac{\pi}{2}t)|x'|^{\frac{5}{6}}\operatorname{sgn}x')+0.05(49-\frac{4}{\pi}\arctan(2-t))|x|^{\frac{6}{5}}\operatorname{sgn}x=0,$$

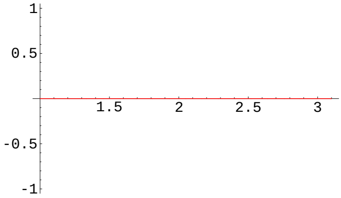
$$x(1)=0, \qquad x(3)=0, \tag{33}$$

that is a special case of the problem (1), (15), (28) with $p=\frac{6}{5}$, $r(t)=0.005(95+\sin\frac{\pi}{2}t)$ and $q(t)=0.05\big(49-\frac{4}{\pi}\arctan(2-t)\big)$.

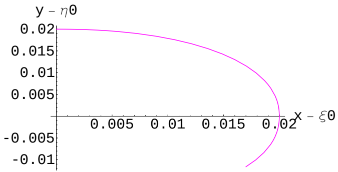
$\forall t\in[1,\,3]\quad 0.47\le r(t)\le 0.48\quad$ and $\quad 2.4\le q(t)\le 2.5$, therefore that is the 4-th possible case $r_1^{-p}<q_2$ and $r_2^{-p}>q_1$, thus $\mu=\frac{q_1}{q_2}$, $\mu=0.96$.

In accordance with calculations (Table 1) there exist at least three different solutions of the problem (33), of 0-type, 1-type and 2-type respectively. We have computed them.

$\forall \delta \in (0,\ 0.12]$



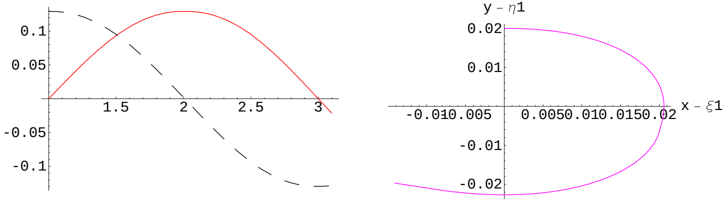
a)



b)

Figure 5. 0-type solution of the BVP (33):
a) the trivial solution $\xi_0(t) \equiv 0$;
b) phase portrait of the difference between neighboring solution $\big(x(t;\delta),\, y(t;\delta)\big)$ and $\big(\xi_0(t),\, \eta_0(t)\big)$, $t \in [1,\ 3]$, if $\delta = 0.02$.

$\forall \delta \in (0,\, 4]$



a) b)

Figure 6. 1-type solution of the BVP (33):
a) $\xi_1(1) = 0,\ \xi_1'(1) = 0.207219\,;$
b) phase portrait of the difference between neighboring solution $\big(x(t;\delta),\, y(t;\delta)\big)$ and $\big(\xi_1(t),\, \eta_1(t)\big)$, $\ t \in [1,\, 3]$, if $\delta = 0.02$.

$\forall \delta \in (0,\ 29]$

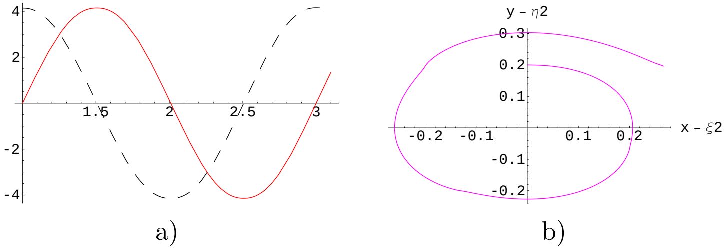


Figure 7. 2-type solution of the BVP (33):
a) $\xi_2(1) = 0,\ \xi_1'(2) = 13.2705\,;$
b) phase portrait of the difference between neighboring solution $\big(x(t;\delta),\ y(t;\delta)\big)$ and $\big(\xi_2(t),\ \eta_2(t)\big)$, $t \in [1,\ 3]$, if $\delta = 0.2$.

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