On solvability of certain boundary value problems for second order linear functional differential equations

Petr Vodstrčil

Ostrava, Czech Republic

e-mail: `petr.vodstrcil@vsb.cz`

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Introduction

Consider the problem on the existence and uniqueness of a solution of equation

$$u''(t) = \ell(u)(t) + q(t), \tag{1}$$

where $\ell \in \mathcal{L}_{ab}$ (set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$) and $q \in L([a,b];\mathbb{R})$, satisfying one of the following three boundary conditions

$$u(a) = c_1, \quad u(b) = c_2, \tag{2}$$

$$u(a) = c_1, \quad u'(b) = c_2, \tag{3}$$

$$u(a) = c_1, \quad u(b) = u(t_0) + c_2, \tag{4}$$

where $c_1, c_2 \in \mathbb{R}$, and $t_0 \in]a,b[$.

By a solution of the equation (1) we understand a function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying equality (1) almost everywhere in $[a,b]$.

Along with the problems (1), (i), where $i \in \{2, 3, 4\}$, consider the corresponding homogeneous problems

$$u''(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2_0}$$

$$u(a) = 0, \quad u'(b) = 0, \tag{3_0}$$

$$u(a) = 0, \quad u(b) = u(t_0). \tag{4_0}$$

Special case of equation (1) is so-called equation with deviating argument

$$u''(t) = p(t)u(\tau(t)) + q(t),$$

where $\tau : [a, b] \to [a, b]$ is a measurable function.

The following result is well-known from the general theory of boundary value problems for functional differential equations.

$$u''(t) = \ell(u)(t) + q(t), \tag{1}$$

$$u(a) = c_1, \quad u(b) = c_2, \tag{2}$$

$$u(a) = c_1, \quad u'(b) = c_2, \tag{3}$$

$$u(a) = c_1, \quad u(b) = u(t_0) + c_2. \tag{4}$$

$$u''(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2_0}$$

$$u(a) = 0, \quad u'(b) = 0, \tag{3_0}$$

$$u(a) = 0, \quad u(b) = u(t_0). \tag{4_0}$$

Theorem 1. *Let $i \in \{2, 3, 4\}$. The problem (1), (i) is uniquely solvable iff the corresponding homogeneous problem (1_0), (i_0) has only the trivial solution.*

Differential Inequalities

Definition 1. *Let $i \in \{2, 3, 4\}$. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $V_i([a, b])$ if for every function $u \in \widetilde{C}'([a, b]; \mathbb{R})$ satisfying boundary condition (i_0) and*

$$u''(t) \geq \ell(u)(t) \quad \text{for } t \in [a, b],$$

the inequality

$$u(t) \leq 0 \quad \text{for } t \in [a, b]$$

holds.

Remark. Let $i \in \{2, 3, 4\}$. It follows from Definition 1 that if $\ell \in V_i([a, b])$, then the problem (1_0), (i_0) has only the trivial solution. Therefore, the problem (1), (i) is uniquely solvable (Theorem 1).

Moreover, if $q \in L([a, b]; \mathbb{R}_+)$, then the (unique) solution of the problem (1), (i_0) is nonpositive.

In the following, we will consider only the case, when $\ell \in \mathcal{P}_{ab}$ (set of linear bounded operators which transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$).

Remark.

- If
$$\ell(u)(t) = p(t)u(\tau(t)),$$
then $\ell \in \mathcal{P}_{ab}$ means that $p(t) \geq 0$ for $t \in [a,b]$.
- Analogously, $-\ell \in \mathcal{P}_{ab}$ means that $p(t) \leq 0$ for $t \in [a,b]$.
- Mention also that the operator
$$\ell(u)(t) = p(t)u(t), \quad p(t) \geq 0 \quad \text{for } t \in [a,b],$$
automatically belongs to the set $V_i([a,b])$ (ODE).

In the following, we will consider only the case, when $\ell \in \mathcal{P}_{ab}$ (set of linear bounded operators which transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$).

The case $-\ell \in \mathcal{P}_{ab}$ is considered in the following papers:

[1] A. LOMTATIDZE, P. VODSTRČIL, On nonnegative solutions of second order linear functional differential equations. *Mem. Differential Equations Math. Phys.*, **32** (2004), 59–88.

[2] P. VODSTRČIL, On nonnegative solutions of a certain nonlocal boundary value problem for second order linear functional differential equations. *Georgian Math. J.*, **11** (2004), No. 3, 583–602.

Before we formulate the next results, we recall the definition of α–Volterra operator.

Definition 2. *We will say that $\ell \in \mathcal{L}_{ab}$ is an α–Volterra operator, where $\alpha \in [a, b]$, if for every $a_1 \in [a, \alpha]$, $b_1 \in [\alpha, b]$, $a_1 \neq b_1$ and $u \in C([a, b]; \mathbb{R})$ satisfying the condition*

$$u(t) = 0 \quad \textit{for } t \in [a_1, b_1],$$

we have

$$\ell(u)(t) = 0 \quad \textit{for } t \in [a_1, b_1].$$

Remark. Let

$$\ell(u)(t) = p(t)u(\tau(t)).$$

- If $\tau(t) \leq t$ for $t \in [a, b]$, then ℓ is an a–Volterra operator.
- Analogously, if $\tau(t) \geq t$ for $t \in [a, b]$, then ℓ is a b–Volterra operator.

Theorem 2. *Let $\ell \in \mathcal{P}_{ab}$ be a b–Volterra operator and let there exist $m, k \in \mathbb{N}, m > k$, such that*

$$\varphi_m(t) \leq \varphi_k(t) \quad \text{for } t \in [a, b],$$

where $\varphi_1 \in \widetilde{C}'([a, b]; \mathbb{R})$ satisfies

$$\varphi_1(t) > 0 \quad \text{for } t \in]a, b]$$

and

$$\varphi_{j+1}(t) \stackrel{def}{=} \int_a^t (t-s)\ell(\varphi_j)(s)\, ds \quad \text{for } t \in [a, b], \quad j \in \mathbb{N}.$$

Then $\ell \in V_i([a, b])$ $(i \in \{2, 3, 4\})$.

Theorem 3. *Let $\ell \in \mathcal{P}_{ab}$ be an a–Volterra operator and let there exist $m, k \in \mathbb{N}, m > k$, such that*

$$\psi_m(t) \le \psi_k(t) \quad \textit{for } t \in [a,b],$$

where $\psi_1 \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies

$$\psi_1(t) > 0 \quad \textit{for } t \in [a,b[$$

and

$$\psi_{j+1}(t) \stackrel{def}{=} \int\limits_t^b (s-t)\ell(\psi_j)(s)\,ds \quad \textit{for } t \in [a,b], \quad j \in \mathbb{N}.$$

Then $\ell \in V_i([a,b])$ $(i \in \{2,3,4\})$.

In particular, Theorems 2 and 3 imply the following assertion.

Corollary 1. *Let* $\ell \in \mathcal{P}_{ab}$ *be an* a*–Volterra* (*resp.* b*–Volterra*) *operator and*

$$\int_a^b (s-a)\ell(1)(s)\,ds \le 1 \qquad \left(\textit{resp.} \quad \int_a^b (b-s)\ell(1)(s)\,ds \le 1 \right).$$

Then $\ell \in V_i([a,b])$ $(i \in \{2,3,4\})$.

In the previous theorems (and their corollary), there is assumed that the operator ℓ is an a–Volterra, resp. b–Volterra operator. In the following two theorems, that assumption is omitted.

Theorem 4. *Let $\ell \in \mathcal{P}_{ab}$ and*

$$\int\limits_a^b (s-a)\ell(1)(s)\,ds \le 1, \qquad \int\limits_a^b (b-s)\ell(1)(s)\,ds \le 1.$$

Then the operator ℓ belongs to the set $V_i([a,b])$ $(i \in \{2,3,4\})$.

Before we formulate the next theorem, introduce the notations.

$\sigma : L([a,b];\mathbb{R}) \longrightarrow C([a,b];\mathbb{R}_+)$ is an operator defined by

$$\sigma(p)(t) \stackrel{\text{def}}{=} \exp\left[\int_{\frac{a+b}{2}}^{t} p(s)\,ds\right] \qquad \text{for } t \in [a,b].$$

$\sigma_a, \sigma_b, \sigma_{ab} : L([a,b];\mathbb{R}) \longrightarrow C([a,b];\mathbb{R}_+)$ are operators defined by

$$\sigma_a(p)(t) \stackrel{\text{def}}{=} \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(s)\,ds, \quad \sigma_b(p)(t) \stackrel{\text{def}}{=} \frac{1}{\sigma(p)(t)} \int_t^b \sigma(p)(s)\,ds,$$

$$\sigma_{ab}(p)(t) \stackrel{\text{def}}{=} \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(s)\,ds \int_t^b \sigma(p)(s)\,ds \qquad \text{for } t \in [a,b].$$

Remark. If, for example, $p \equiv 0$, then

$$\begin{aligned}\sigma_a(p)(t) &= t - a,\\ \sigma_b(p)(t) &= b - t,\\ \sigma_{ab}(p)(t) &= (t-a)(b-t).\end{aligned}$$

Let $\ell \in \mathcal{L}_{ab}$. Put

$$\omega(t) \stackrel{\text{def}}{=} t - a \quad \text{for } t \in [a, b],$$

$$h(t) \stackrel{\text{def}}{=} \ell(\omega)(t) - \ell(1)(t)\omega(t) \quad \text{for } t \in [a, b],$$

$$r_a(t) \stackrel{\text{def}}{=} \exp\left[\frac{1}{\int_a^b \sigma(h)(s)\,ds} \int_a^t \sigma_{ab}(h)(s)\ell(1)(s)\,ds\right] \quad \text{for } t \in [a, b],$$

$$r_b(t) \stackrel{\text{def}}{=} \exp\left[\frac{1}{\int_a^b \sigma(h)(s)\,ds} \int_t^b \sigma_{ab}(h)(s)\ell(1)(s)\,ds\right] \quad \text{for } t \in [a, b].$$

Define operators $A \in \mathcal{L}_{ab}$ and $T \in \mathcal{L}_{ab}$ by

$$A(u)(t) \stackrel{\text{def}}{=} \int_a^t \int_a^s \ell(u)(\xi)\, d\xi\, ds,$$

$$T(u)(t) \stackrel{\text{def}}{=} \ell(A(u))(t) - \ell(\omega)(t) \int_a^t \ell(u)(\xi)\, d\xi -$$
$$- \ell(1)(t) \left[A(u)(t) - \omega(t) \int_a^t \ell(u)(\xi)\, d\xi \right].$$

Remark. If, for example,

$$\ell(u)(t) = p(t)u(\tau(t)),$$

then (it can be verified by direct calculation)

$$T(u)(t) = p(t) \int\limits_{t}^{\tau(t)} (\tau(t) - s)p(s)u(\tau(s))\, ds.$$

Theorem 5. *Let* $\ell \in \mathcal{P}_{ab}$ *and*

$$\int\limits_a^b \sigma_a(h)(s) r_a(s) T(1)(s)\, ds \leq 1, \qquad \int\limits_a^b \sigma_b(h)(s) r_b(s) T(1)(s)\, ds \leq 1.$$

Then the operator ℓ *belongs to the set* $V_i([a,b])$ $(i \in \{2,3,4\})$.

Unique Solvability

Let $i \in \{2,3,4\}$. It is trivial that the inclusion $\ell \in V_i([a,b])$ implies the unique solvability of the problem (1), (i). However, for unique solvability of our problem, it is not necessary to suppose that $\ell \in V_i([a,b])$. For example, the following theorem is valid.

Theorem 6. *Let $i \in \{2,3,4\}$ and $\ell \in \mathcal{P}_{ab}$. Let, moreover, at least one of the following conditions is fulfilled:*

$$\frac{1}{2}\ell \in V_i([a,b])$$

or

$$\frac{1}{3}\ell \in V_i([a,b]), \qquad -\frac{1}{3}\ell \in V_i([a,b]).$$

Then the problem (1), (i) *is uniquely solvable.*

Equations With Deviating Argument

Now, we present several theorems for the case, when the operator $\ell \in \mathcal{L}_{ab}$ is defined by equality

$$\ell(u)(t) \stackrel{\text{def}}{=} p(t)u(\tau(t)),$$

where $p \in L([a,b];\mathbb{R}_+)$ and $\tau : [a,b] \longrightarrow [a,b]$ is a measurable function.

Then, the equation (1) takes the form

$$u''(t) = p(t)u(\tau(t)) + q(t). \tag{5}$$

Together with equation (5) we again consider one of the following (above–mentioned) boundary conditions:

$$u(a) = c_1, \quad u(b) = c_2, \tag{2}$$

$$u(a) = c_1, \quad u'(b) = c_2, \tag{3}$$

$$u(a) = c_1, \quad u(b) = u(t_0) + c_2. \tag{4}$$

Theorem 7. *Let* $i \in \{2, 3, 4\}$ *and*

$$\tau(t) \geq t \quad \textit{for } t \in [a, b].$$

Moreover, let either

$$\int_a^b (b - s)p(s)\, ds \leq 2$$

or the following two conditions are satisfied:

$$\int_a^b (b - s)p(s)\, ds > 2$$

and

$$\int_t^{\tau(t)} \left(\int_a^s p(\xi)\, d\xi \right) ds \leq \frac{2}{e} \qquad \textit{for } t \in [a, b].$$

Then the problem (5), (i) *is uniquely solvable.*

Theorem 8. *Let* $i \in \{2, 3, 4\}$ *and*

$$\tau(t) \le t \quad \textit{for } t \in [a, b].$$

Moreover, let either

$$\int_a^b (s-a)p(s)\,ds \le 2$$

or the following two conditions are satisfied:

$$\int_a^b (s-a)p(s)\,ds > 2$$

and

$$\int_{\tau(t)}^{t} \left(\int_s^b p(\xi)\,d\xi \right) ds \le \frac{2}{e} \qquad \textit{for } t \in [a, b].$$

Then the problem (5), (*i*) *is uniquely solvable.*

Put

$$\lambda = \max\left\{ \int_a^b (s-a)p(s)\,ds, \int_a^b (b-s)p(s)\,ds \right\}.$$

Theorem 9. *Let $i \in \{2,3,4\}$ and either*

$$\lambda < 3$$

or $\lambda \geq 3$ and (for almost all $t \in [a,b]$) the inequality

$$\left(e^{e^{\lambda}-1} - 1\right) \int_t^{\tau(t)} (\tau(t)-s)p(s)\,ds \leq 2\left(20^4 + 20^3 + 20^2 + 20 + 1\right)$$

is fulfilled.

Then the problem (5), (i) is uniquely solvable.

$$u''(t) = p(t)u(\tau(t)) + q(t), \tag{5}$$

$$u(a) = c_1, \quad u(b) = c_2, \tag{2}$$
$$u(a) = c_1, \quad u'(b) = c_2, \tag{3}$$
$$u(a) = c_1, \quad u(b) = u(t_0) + c_2. \tag{4}$$

Remark.

- In Theorem 7 and Theorem 8 we suppose that $\tau(t) \geq t$, resp. $\tau(t) \leq t$.
- There is no additional assumption on the function τ in Theorem 9.
- If $\tau(t) = t$ (ODE with nonnegative coefficient p), then all assumptions of previous three theorems are fulfilled. Therefore, the problem (5), (i) ($i \in \{2, 3, 4\}$) is uniquely solvable.
- If the difference between $\tau(t)$ and t is „small enough“, then the problem (5), (i) is uniquely solvable.