Anti-maximum principle for p-Laplacians and its application to weakly singular periodic problems

Milan Tvrdý

Institute of Mathematics
Acad. Sci. of the Czech Rep.

[based on the joint work with **Alberto Cabada** and **Alexander Lomtatidze**]

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$$u'' + \left(\frac{\pi}{T}\right)^2 u = 0, \quad u(0) = u(T),\ u'(0) = u'(T)$$

has Green's function

$$G(t,s) = \frac{T}{2\pi} \sin\left(\frac{\pi}{T}|t-s|\right) \quad \text{for} \ \ t,s \in [0,T].$$

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Let: $\mu \in L_1[0,T]$, $\overline{\mu} > 0$ and $0 \leq \mu \leq (\frac{\pi}{T})^2$ a.e. on $[0,T]$.

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Then

$$\left.\begin{array}{l} v \in AC^1[0,T], \\ v''(t) + \mu(t)\,v(t) \geq 0 \text{ a.e. on } [0,T], \\ v(0) = v(T),\ v'(0) \geq v'(T) \end{array}\right\} \implies v \geq 0 \quad \text{on}\ \ [0,T].$$

Let

$$H_0^1 = \{u \in AC : \ u' \in L_2 \ \wedge \ u(0) = u(T) = 0\}$$

and

$$K(p) = \inf\Big\{ \frac{\|u'\|_2^2}{\|u\|_p^2} : \ u \in H_0^1 \setminus \{0\} \Big\} \quad \text{for } \ 1 \le p \le \infty,$$

i.e. $K(p)$ is the best Sobolev constant for the inequality $C\,\|u\|_p^2 \le \|u'\|_2^2$.

It is known that

$$K(p) = \begin{cases} \dfrac{2\,\pi}{p\,T^{1+\frac{2}{p}}} \left(\dfrac{2}{2+p}\right)^{1-\frac{2}{p}} \left(\dfrac{\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{2}+\frac{1}{p})}\right)^2 & \text{if } 1 \le p < \infty, \\ \frac{4}{T} & \text{if } p = \infty. \end{cases}$$

In particular, $K(2) = \left(\frac{\pi}{T}\right)^2$ = the **first eigenvalue** of the related **Dirichlet problem**.

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Moreover, if $\|\mu\|_q < K(2\,q^*)$, then the corresponding Green's function is strictly positive.

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We denote $\phi_p(y) = |y|^{p-2}\, y$ for $y \in \mathbb{R}$.

$$\left(\phi_p(u')\right)' + \lambda\, \phi_p(u) = 0,\ u(t_0) = 0,\ u'(t_0) = d \quad \text{iff} \quad u(t) = d\, \lambda^{-1/p} \sin_p(\lambda^{1/p}(t - t_0))$$

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λ is an **eigenvalue** for $\big(\phi_p(u')\big)' + \lambda\,\phi_p(u),\ u(a) = u(b) = 0$ iff

$$\lambda \in \left\{ \left(\frac{n\,\pi_p}{b-a}\right)^p : n \in \mathbb{N} \cup \{0\} \right\},$$

where

$$\pi_p = 2\,(p-1)^{1/p} \int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s = 2\,(p-1)^{1/p}\,\frac{\pi/p}{\sin(\pi/p)}\,.$$

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In particular, if

$$(\phi_p(u'))' + \left(\tfrac{\pi_p}{T}\right)^p\,\phi_p(u) = 0, \quad u(a) = u(b) = 0$$

has a nontrivial solution, then $b - a \geq T$.

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- $\max\{v(t) : t \in [0,T]\} > 0$
- $\min\{v(t) : t \in [0,T]\} < 0 \implies \exists\, a, b \in \mathbb{R},\ a < b:\ v > 0$ on (a,b),

  $v(a) = v(b) = 0$ and $b - a < T$

  $\implies$ v is **LOWER FUNCTION** for (D)

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  - **LET** $a' < a < b < b', \; b' - a' = T$ and

    $w(t) = d \, \sin_p\left(\frac{\pi_p\,(t-a')}{T}\right)$ with $d > 0$ so large that $w > v$ on $[a,b]$.

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    THEN $(\phi_p(w(t)'))' + \left(\frac{\pi_p}{T}\right)^p \phi_p(w(t)) = 0$ a.e. on $[a', b']$,

    $w(a') > 0,\ w(b') > 0,$ $w > v$ on $[a,b]$

    $\Rightarrow w$ is an **UPPER FUNCTION** for (D)

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$\Rightarrow \exists$ SOLUTION u to (D) such that $0 < v \leq u \leq w$ on (a,b)

CONTRADICTION to **PROPOSITION** !!!

$\Rightarrow \quad \min\{v(t) : t \in [0,T]\} \geq 0$

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THEOREM 1 [Cabada, Lomtatidze & Tvrdý (2006/7))

Assume:

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Then

$$\left.\begin{array}{l} \phi_p(v') \in AC[0,T], \\ (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \ge 0 \text{ a.e. on } [0,T], \\ v(0) = v(T),\ v'(0) \ge v'(T) \end{array}\right\} \implies v \ge 0 \quad \text{on} \quad [0,T].$$

Main result

(P) $$\big(\phi_p(u')\big)' = f(t,u), \quad u(0)=u(T),\ u'(0)=u'(T)$$

THEOREM 2 (Cabada, Lomtatidze & Tvrdý (2006))

Assume: $f \in Car([0,T]\times(0,\infty))$ and there are $r>0, A\ge r, B>A, \beta\in L_1$ such that

- $\overline{\beta}\le 0$ and $f(t,x)\le\beta(t)$ for a.e. $t\in[0,T], x\in[A,B]$;
- $f(t,x)\ge -\left(\frac{\pi_p}{T}\right)^p \phi_p(x-r)$ for a.e. $t\in[0,T]$ and all $x\in[r,B]$,
- $B-A\ge\frac{T}{2}\,\|m\|_1^{q-1},\ \frac{1}{p}+\frac{1}{q}=1,$
  $m(t)=\max\big\{\sup\{f(t,x)\colon x\in[r,A]\},\beta(t),0\big\}$ for a.e. $t\in[0,T]$.

Then (P) has a solution u such that $r\le u\le B$ on $[0,T]$.

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- $\overline{\beta} \leq 0$ and $f(t,x) \leq \beta(t)$ for a.e. $t \in [0,T]$, $x \in [A,B]$;
- $f(t,x) \geq -\left(\frac{\pi_p}{T}\right)^p \phi_p(x-r)$ for a.e. $t \in [0,T]$ and all $x \in [r,B]$,
- $B - A \geq \frac{T}{2}\, \|m\|_1^{q-1}$, $\frac{1}{p} + \frac{1}{q} = 1$,
  $m(t) = \max\left\{\sup\{f(t,x) \colon x \in [r,A]\}, \beta(t), 0\right\}$ for a.e. $t \in [0,T]$.

Then (P) has a solution u such that $r \leq u \leq B$ on $[0,T]$.

COROLLARY

Assume: $f(t,x) = g(x) + e(t)$, $g \in C(0,\infty)$, $e \in L_1[0,T]$,

- $\overline{e} + \limsup\limits_{x\to\infty} g(x) < 0$,
- $\exists\, r > 0:\ e(t) + g(x) + \left(\frac{\pi_p}{T}\right)^p (x-r)^{p-1} \geq 0$ for a.e. $t \in [0,T]$ and all $x \geq r$.

Then (P) has a solution u such that $u(t) \geq r$ on $[0,T]$.

Example $(p = 2)$

$$\text{(E)}\quad u'' + k\,u = \frac{a}{u^{\lambda}} + e(t),\ u(0) = u(T),\ u'(0) = u'(T) \quad \big[a > 0,\ \lambda > 0,\ k \geq 0,\ e \in L_1\big]$$

(E) **has a solution if**:

- $k = 0,\ \lambda \geq 1,\ \overline{e} < 0$ [**Lazer & Solimini**],
- $k \neq \left(n\,\frac{\pi}{T}\right)^2$ for all $n \in \mathbb{N},\ \lambda \geq 1,\ e \in C$ [**del Pino, Manásevich & Montero**]
- $0 < k < \left(\frac{\pi}{T}\right)^2,\ \lambda \geq 1,\ e \in L_\infty$ [**Omari & Ye**],
- $k = 0,\quad \overline{e} < 0,\quad \inf\operatorname{ess}_{t\in[0,T]} e(t) > -\left(\frac{1}{T^2\,\lambda\,a}\right)^{\frac{\lambda}{\lambda+1}} (\lambda + 1)\,a,$

  $0 < k < \left(\frac{\pi}{T}\right)^2,\quad \inf\operatorname{ess}_{t\in[0,T]} e(t) > -\left(\frac{\pi^2 - T^2 k}{T^2\,\lambda\,a}\right)^{\frac{\lambda}{\lambda+1}} (\lambda + 1)\,a$

  [supplementary results by **Torres**],

  $k = \left(\frac{\pi}{T}\right)^2,\quad \inf\operatorname{ess}_{t\in[0,T]} e(t) > 0$ [**Rachunková, Tvrdý & Vrkoč**],

  [improvements by **Bonheure & De Coster**].

Example $(1 < p < \infty)$

(E) $\quad (\phi_p(u'))' + k\,\phi_p(u) = \frac{a}{u^\lambda} + e(t), \quad u(0) = u(T),\ u'(0) = u'(T),$

$\left[a > 0,\ \lambda > 0,\ k \ge 0,\ e \in L_1\right]$

(E) **has a solution if**:

- $0 < k < (\frac{\pi_p}{T})^p,\ \lambda \ge 1$ **[Jebelean & Mawhin, Liu Bing, Rachunková & Tvrdý]**,
- $1 < p < \infty,\ k = 0,\ \overline{e} < 0,\ \ \inf\operatorname{ess}_{t \in [0,T]} e(t) + a\,\frac{\lambda+p-1}{p-1}\left(\frac{(p-1)(\frac{\pi_p}{T})^p}{\lambda\,a}\right)^{\frac{\lambda}{\lambda+p-1}} > 0,$
- $1 < p < \infty,\ 0 < k < (\frac{\pi_p}{T})^p,\ \inf\operatorname{ess}_{t \in [0,T]} e(t) + a\,\frac{\lambda+p-1}{p-1}\left(\frac{(p-1)\left((\frac{\pi_p}{T})^p - k\right)}{\lambda\,a}\right)^{\frac{\lambda}{\lambda+p-1}} > 0,$
- $1 < p \le 2,\quad k = (\frac{\pi_p}{T})^p,\qquad \inf\operatorname{ess}_{t \in [0,T]} e(t) > 0.$

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- PART I. HIGHER ORDER SINGULAR PROBLEMS
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