Duffing equation with potential Landesman-Lazer condition

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Introduction

Let us consider the nonlinear problem

$$u''(x) + c\,u' + (m^2 - \frac{c^2}{4})\,u + g(u) = f(x)\,, \quad x \in \langle 0, \pi \rangle\,, \qquad (1)$$
$$u(0) = u(\pi) = 0\,,$$

where $c \in \mathbb{R}\,, m \in \mathbb{N}\,,$ a nonlinearity $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function and $f \in L^1(0, \pi)$.

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First we solve the homogenous equation

$$\begin{aligned} u''(x) + \lambda\, u(x) = 0\,, \quad & x \in (0, \pi)\,, \\ u(0) = u(\pi) = 0\,. \end{aligned} \tag{2}$$

We obtain a nontrivial solution to (2) if $\lambda = m^2$, $m \in \mathbb{N}$. The number $\lambda = m^2$ is called an eigenvalue of (2) and corresponding solution $\sin mx$ is called eigenfunction of (2).

We will investigate the problem (1) on the space H spanned by functions $\sin x, \sin 2x, \sin 3x, \dots$.

Now we investigate the linear problem

$$\begin{aligned} u''(x) + \lambda\, u(x) = f(x)\,, \quad x \in (0, \pi)\,, \\ u(0) = u(\pi) = 0\,. \end{aligned} \tag{3}$$

We obtain the solution to (3) for all $f \in L^1(0, \pi)$ if $\lambda \neq m^2$, $m \in \mathbb{N}$.

If $\lambda = m^2$ then

$$\int_0^\pi f(x) \sin mx \, dx = 0\,. \tag{4}$$

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We define $\lim_{s\to\infty} g(s) = g_+$ and $\lim_{s\to-\infty} g(s) = g_-$. Then the problem

$$\begin{aligned} u''(x) + m^2\, u + g(u) = f(x)\,, \quad & x \in \langle 0, \pi\rangle\,, \\ u(0) = u(\pi) = 0\,, & \end{aligned} \tag{5}$$

has at least one solution provided that

$$\begin{aligned} &\int_0^\pi \left[g_-(\sin mx)^+ - g_+(\sin mx)^-\right] dx \\ &\quad < \int_0^\pi f(x) \sin mx\, dx < \int_0^\pi \left[g_+(\sin mx)^+ - g_-(\sin mx)^-\right] dx\,, \end{aligned} \tag{6}$$

$(\sin mx)^+ = \max\{\sin mx, 0\}\,,\ (\sin mx)^- = \max\{-\sin mx, 0\}.$

It is worth mentioning that if we suppose

$$g_- < g(s) < g_+ \qquad \text{for any} \quad s \in \mathbb{R}\,,$$

the conditon (6) is also necessary for the solvability of (5).

We suppose

$$\lim_{|s|\to\infty} \frac{g(s)}{s} = 0\,.$$

We can generalized the condition (6) if we define

$$\liminf_{s\to\infty} g(s) = g_+ \quad \text{and} \quad \limsup_{s\to-\infty} g(s) = g_-\,.$$

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We study (5) by using of varitional method. More precisely, we look for critical points of the functional J, which is defined by

$$J(u) = \frac{1}{2}\int_0^\pi \left[(u')^2 - m^2u^2\right] dx - \int_0^\pi \left[G(u) - fu\right] dx\,,$$

where

$$G(s) = \int_0^s g(t)\,dt\,.$$

Every critical point $u \in H$ of the functional J satisfies

$$\langle J'(u), v\rangle = \int_0^\pi \left[u'v' - m^2uv\right] dx - \int_0^\pi \left[g(u)v - f\,v\right] dx = 0$$

for all $v \in H$. Then u is also a weak solution of (5) and vice versa.

We define

$$G_+ = \liminf_{s \to +\infty} \frac{G(s)}{s}, \quad G_- = \limsup_{s \to -\infty} \frac{G(s)}{s}.$$

We assume that the following potential Landesman-Lazer type condition holds:

$$\int_0^\pi \left[G_-(\sin mx)^+ - G_+(\sin mx)^- \right] dx \tag{7}$$
$$< \int_0^\pi f(x) \sin mx \, dx < \int_0^\pi \left[G_+(\sin mx)^+ - G_-(\sin mx)^- \right] dx .$$

Then the problem (5) has at least one solution.

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We multiply (1) $u''(x) + c\,u' + (m^2 - \frac{c^2}{4})\,u + g(u) = f(x)$ by the function $e^{\frac{c}{2}x}$. We put $w(x) = e^{\frac{c}{2}x}u(x)$ and obtain an equivalent Dirichlet problem

$$\begin{gathered} w''(x) + m^2\,w(x) + e^{\frac{c}{2}x}\,g(\frac{w}{e^{\frac{c}{2}x}}) = e^{\frac{c}{2}x}\,f(x)\,, \\ w(0) = w(\pi) = 0\,. \end{gathered} \tag{8}$$

Now we investigate the functional $J : H \to \mathbb{R}$, which is defined by

$$J(w) = \frac{1}{2}\int_0^\pi \Big[(w')^2 - m^2\,w^2\Big]\,dx - \int_0^\pi \Big[e^{cx}\,G(\frac{w}{e^{\frac{c}{2}x}}) - e^{\frac{c}{2}x}\,f\,w\Big]\,dx\,.$$

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Generalized potential Landesman-Lazer condition

We will suppose that g is bounded and for the potential G the following condition holds :

$$\begin{aligned}
&\int_0^\pi \left[e^{\frac{c}{2}x}(G_-(x)(\sin mx)^+ - G_+(x)(\sin mx)^-)\right] dx \\
&\quad < \int_0^\pi [e^{\frac{c}{2}x} f(x) \sin mx]\, dx \\
&\qquad < \int_0^\pi \left[e^{\frac{c}{2}x}(G_+(x)(\sin mx)^+ - G_-(x)(\sin mx)^-)\right] dx\,.
\end{aligned} \tag{9}$$

Then the problem (1) has at least one solution in H.

Literature

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