Singular Nonlinear Problem for Ordinary Differential Equation of the Second Order

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Boundary value problem

We are looking for the assumptions ensuring the existence of at least one solution to the singular nonlinear boundary value problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \tag{1}$$

$$u(0) = 0, \quad u'(T) = \psi(u(T)). \tag{2}$$

where $f \in Car((0, T) \times D)$, $D = (0, \infty) \times \mathbb{R}$, can have time or space singularity, $\psi \in C[0, \infty)$.

Time and space singularities

Let $f \in Car((0,T) \times D)$, where $D = (0,\infty) \times \mathbb{R}$. We say that f has a time singularity at $t = 0$ and/or at $t = T$, if there exists $(x_1, y_1) \in D$ and/or $(x_2, y_2) \in D$ such that

$$\int_0^{\epsilon} |f(t, x_1, y_1)| \,\mathrm{d}t = \infty \quad \text{and/or} \quad \int_{T-\epsilon}^{T} |f(t, x_2, y_2)| \,\mathrm{d}t = \infty$$

for each sufficiently small $\epsilon > 0$. The point $t = 0$ and/or $t = T$ will be called a singular point of f.
We say that f has a space singularity at $x = 0$ if

$$\limsup_{x \to 0+} |f(t, x, y)| = \infty \quad \text{for a. e. } t \in [0, T] \text{ and for some } y \in \mathbb{R}.$$

Solution of the problem (1),(2)

By solution of the problem

(1) $$u''(t) + f(t, u(t), u'(t)) = 0,$$

(2) $$u(0) = 0, \quad u'(T) = \psi(u(T))$$

we understand the function

$$u \in AC^1[0, T]$$

satisfying the differential equation (1) and boundary conditions (2).

Main assumptions

To prove the existence of at least one solution of the problem (1), (2) we assume that

(i) $f \in Car((0,T) \times D)$, where $T > 0$, $D = (0,\infty) \times \mathbb{R}$, with possible time singularities at $t = 0$ and/or $t = T$ and a space singularity at $x = 0$.

(ii) there exist $\epsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu,\infty)$ and $\epsilon_0 \in (0,\infty)$ such that

$$f(t,ct,c) = 0 \qquad \text{for a. e. } t \in [0,T],$$

$$0 = \psi(0), \quad \psi(cT) \le c$$

$$0 \le f(t,x,y)$$

for a. e. $t \in [0,T]$, each $x \in (0,ct]$, $y \in [\min_{t\in[0,cT]} \psi(t), c]$,

$$\epsilon \le f(t,x,y)$$

for a. e. $t \in [T-\nu, T]$, each $x \in (0,ct]$, $y \in (-\epsilon_0, \nu]$.

The existence theorem

Theorem Let assumptions (i) and (ii) hold. Then there exists a solution u of the problem (1), (2) such that

$$0 < u(t) \le ct$$

for each $t \in (0, T]$ (the constant c is from the assumption (ii)).

Proof. *Step* 1. For $k \ge 3/T$ we define

$$\alpha_k(t,x) = \begin{cases} c/k & \text{for} \quad x < c/k, \\ x & \text{for} \quad c/k \le x \le ct, \\ ct & \text{for} \quad x > ct, \end{cases}$$

for each $t \in [1/k, T - 1/k]$, $x \in \mathbb{R}$,

$$\beta(y) = \begin{cases} \min_{t\in[0,cT]} \psi(t) & \text{for} \quad y < \min_{t\in[0,cT]} \psi(t), \\ y & \text{for} \quad \min_{t\in[0,cT]} \psi(t) \le y \le c, \\ c & \text{for} \quad y > c, \end{cases}$$

and

$$\gamma(y) = \begin{cases} \epsilon & \text{for} \quad y < \nu, \\ \epsilon \frac{c-y}{c-\nu} & \text{for} \quad \nu \le y \le c, \\ 0 & \text{for} \quad y > c, \end{cases}$$

for each $y \in \mathbb{R}$ and

$$f_k(t,x,y) = \begin{cases} 0 & \text{for} \quad t \in [0, 1/k), \\ f(t, \alpha_k(t,x), \beta(y)) & \text{for} \quad t \in [1/k, T - 1/k], \\ \gamma(y) & \text{for} \quad t \in (T - 1/k, T], \end{cases}$$

for each $x,\ y \in \mathbb{R}$.

Let us define regular problem

$$(R_k) \quad u'' + f_k(t, u, u') = 0, \qquad u(0) = 0, \quad u'(T) = \psi(u(T)).$$

Step 2. From the assumptions it follows that

$$\sigma_1(t) = 0 \qquad \text{is a lower function,}$$
$$\sigma_2(t) = ct \quad \text{is an upper function}$$

of the regular problems (R_k) for each k.
Then using the existence theorem for regular problem we get

$$0 \le u_k(t) \le ct$$

for each solution u_k of the problem (R_k). Moreover we can find $\omega > 0$ such that

$$u_k(t) \ge \omega t \qquad \text{for each } t \in [0, T] \text{ and almost each } k \in \mathbb{N}.$$

The existence theorem

Step 3. We can find a convergent subsequence of the sequence $\{u_k\}$ and prove that its limit is a solution of the problem (1), (2).

Example Let α, $\beta \in (0,\infty)$. Then, by the existence theorem the problem

$$u'' + (u^{-\alpha} + u^{\beta} + t^2 + 1)(1 - (u')^3) = 0, \quad u(0) = 0, \quad u'(1) = -(u(1))^2$$

has a solution $u \in AC^1[0,1]$ such that

$$0 < u(t) \le t \qquad \text{for each } t \in (0,1].$$

References

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