Matrix theory and the solvability of nonlinear difference equations

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Continuous x Discrete Problems

$2n$-th order periodic problem

$$\begin{cases} (-1)^n \Delta^{2n} x(k-n) + q(k)x(k) = g(k, x(k)), & k = 1, 2, \dots, N, \\ \Delta^i x(1-n) = \Delta^i x(N+1-n), & i = 0, \dots, 2n-1, \end{cases} \tag{1}$$

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$$
\begin{bmatrix}
a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} & & & a_0 & a_1 & \cdots & a_{n-1} \\
a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} & & & a_0 & \cdots & a_{n-2} \\
\vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & & & \ddots & \vdots \\
a_1 & \cdots & a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} & & & a_0 \\
a_0 & a_1 & \cdots & a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} & & \\
 & \ddots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \ddots & \\
 & & a_0 & a_1 & \cdots & a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} \\
a_{2n} & & & a_0 & a_1 & \cdots & a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-1} \\
\vdots & \ddots & & & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & & & a_0 & a_1 & \cdots & a_{n-1} & a_n & a_{n+1} \\
a_{n+1} & \cdots & \cdots & a_{2n} & & & a_0 & a_1 & \cdots & a_{n-1} & a_n
\end{bmatrix}.
$$

where $a_i = (-1)^{n+i} \binom{2n}{i}$

Properties of matrices I

$$\overline{A}_2 := \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix}$$

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Eigenvalues of $\overline{A}_2$ are

$$\lambda_m = 4\sin^2\left(\frac{m\pi}{N}\right), \qquad \text{for} \quad m = 0, 1, \ldots, \frac{N}{2}, \text{if } N \text{ is even}$$

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Properties of matrices II

- $\overline{A}_{2n} = \overline{A}_2^n$

$$\sum_{m=0}^{k}(-1)^{m+1}\binom{2}{m}(-1)^{n+k+m}\binom{2n}{k-m} = (-1)^{n+1+k}\sum_{m=0}^{k}\binom{2}{m}\binom{2n}{k-m}$$
$$= (-1)^{n+1+k}\binom{2(n+1)}{k}.$$

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- eigenvalues of $\overline{A}_{2n}$ are the n-th powers of eigenvalues of $\overline{A}_2$.

$$(\lambda_m^n I - \overline{A}_{2n})v_m = \left(\lambda_m I - \overline{A}_2\right)\left(\lambda_m^{n-1} I + \lambda^{n-2}\overline{A}_2 + \ldots + \lambda\overline{A}_2^{n-2} + \overline{A}_2^{n-1}\right)v_m = o,$$

Theorem

Let us suppose that

$$q(k) \geq 0 \textit{ for all } k = 1, 2, \ldots, N, \quad q(\overline{k}) > 0 \textit{ for some } \overline{k} \in \{1, 2, \ldots, N\}.$$

Moreover, let us assume that for all $k = 1, 2, \ldots, N$, the functions g_k are continuous and that there exists $R > 0$ such that for each u with $0 < |u| \leq R$:

$$ug_k(u) \leq 0.$$

Then the problem (1) *has a solution.*

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Moreover, let us assume that for all $k = 1, 2, \ldots, N$, the functions g_k are continuous and that there exists $R > 0$ such that for each u with $0 < |u| \leq R$:

$$ug_k(u) < 0.$$

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Comparison of results

Green function + L&U solutions Atici, Cabada 2003 Atici, Guseinov 1999 Henderson 1989	uniqueness $g_k > 0$

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Matrix formulation + degree theory	no monotonicity $q \equiv 0$

Application of variational methods

$$F(x) := \frac{1}{2}\langle A_{2n}x, x\rangle - \widetilde{G}(x)$$

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Theorem

Let $q \geq 0$ be satisfied. Let us suppose that $g : \{1, 2, \dots, N\} \times \mathbb{R} \to \mathbb{R}$ is a function such that for each $k = 1, 2, \dots, N$:

(I) $g_k \in L^1_{loc}(\mathbb{R})$,

(L) *there exists $M > 0$ such that*

$$\lim_{u\to\infty} g_k(u) \leq -M, \quad \textit{and} \lim_{u\to-\infty} g_k(u) \geq M.$$

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Let us suppose that $g : \{1, 2, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$ *is a function such that for each* $k = 1, 2, \ldots, N$ *the assumptions (I), (L) hold and that*

(M) g_k *is nonincreasing,*

If $q \geq 0$ *and* $q(\overline{k}) > 0$ *for some* $\overline{k} \in \{1, 2, \ldots, N\}$*, then the solution of* (1) *is unique.*

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If $q \geq 0$ *and* $q(\overline{k}) > 0$ *for some* $\overline{k} \in \{1, 2, \ldots, N\}$, *then the solution of* (1) *is unique.*
Moreover, if

(M') g_k *is strictly decreasing,*

then the solution is unique also in the case when $q \equiv 0$.

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Matrix formulation + degree theory	no monotonicity $q \equiv 0$ no limit conditions

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Matrix formulation + degree theory	no monotonicity $q \equiv 0$ no limit conditions
Matrix formulation + var.methods	no continuity required uniqueness, even if $q \equiv 0$

- boundary conditions

$$\begin{cases} (-1)^n \Delta^n (\Delta^n x(k-n)) = f(k, x(k)), & k = 1, 2, \dots, N, \\ x(1-n+i) = C_i, \quad x(N+n-i) = D_i, & i = 0, \dots, n-1. \end{cases}$$

Extensions - Sturm - Liouville

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- different operators

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- variable steps

$$\begin{cases} -\left(p(\rho(t))x^{\Delta}(\rho(t))\right)^{\Delta} + q(t)x(t) = f(t, x(t)), \quad \text{on } [a, b]_{\mathbb{T}} \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = C, \quad \gamma x(\sigma(b)) + \delta x^{\Delta}(b) = D, \end{cases}$$

Final remarks

Fučík spectrum - Margulies & Margulies (1999) study matrix equations

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where A is 2×2 matrix

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Thank you for your attention