POSITIVE SOLUTIONS OF MIXED BOUNDARY VALUE PROBLEMS WITH TIME AND SPACE SINGULARITIES

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Overview

1. Time and space singularities
2. Mixed problem with time and space singularities
3. Existence of positive solutions
4. Maximal positive solution

1. TIME AND SPACE SINGULARITIES

Let $q(t, x, y) \in Car((0, T) \times (0, \infty) \times \mathbb{R})$. We say that q *has a time singularity* at the point $t = 0$ $(t = T)$ if there exists $(x, y) \in (0, \infty) \times \mathbb{R}$ such that

$$\int_0^{\varepsilon} |q(t, x, y)| \, \mathrm{d}t = \infty \quad \left(\int_{T-\varepsilon}^{T} |q(t, x, y)| \, \mathrm{d}t = \infty \right)$$

for all sufficiently small $\varepsilon > 0$.

1. Time and space singularities

If $\lim_{x \to 0^+} |q(t,x,y)| = \infty$ for a.e. $t \in [0,T]$ and some $y \in \mathbb{R}$ then we say that q *has a space singularity at the point* $x = 0$.
We give a finer classification of space singularities. We say that q has *a weak* (*a strong*) *space singularity* at $x = 0$ if there exists $y \in \mathbb{R}$ such that

$$\int_0^\eta |q(t,x,y)| \,\mathrm{d}x < \infty \quad \left(\int_0^\eta |q(t,x,y)| \,\mathrm{d}x = \infty \right)$$

for a.e. $t \in [0,T]$ and all sufficiently small $\eta > 0$.

EXAMPLE. The function

$$q(t,x,y) = \frac{y}{t^\alpha (T-t)^\beta x^\gamma}, \quad \alpha,\beta,\gamma \in [1,\infty)$$

has a time singularity at $t = 0$, T and a strong space singularity at $x = 0$ (if $\gamma \in (0,1)$ then q has a weak space singularity at $x = 0$).

2. MIXED PROBLEM WITH TIME AND SPACE SINGULARITIES

The solvability of the mixed BVP

$$u'' = h(t, u, u'), \quad u(0) = 0, \ u'(T) = 0$$

in the set $C^0[0, T] \cap AC^1_{log}(0, T)$ was considered by **Agarwal and O'Regan** (2003), **Kiguradze** (2003), **Kiguradze and Shekhter** (1987), **O'Reagan** (1990), **Wang and Gao** (2000), **Zheng, Su and Zhang** (2005), ...
in the set $AC^1[0, T]$ by **Agarwal and O'Regan** (2003), **Agarwal and Staněk** (2003), **Kelevedjiev** (1999), **Rachůnková** (2006) ...

2. Mixed problem with time and space singularities

Berestycki, Lions and Peletier (1981), **Castro and Sudhasree** (1994), **Gidas, Ni and Nirenberg** (1981) considered

$$u'' - \frac{n-1}{1-t}u' + g(1-t, u) = 0, \quad u(0) = 0, \ u'(1) = 0$$

(radially symmetric solutions of nonlinear elliptic PDE)

Bertsch and Ughi (1990)

$$u'' = \frac{n-1}{1-t}u + \gamma\frac{|u'|^2}{u} - 1, \quad u(0) = 0, \ u'(1) = 0.$$

Zhou and Cai (2007 - preprint)

$$u'' = \lambda\frac{u}{1-t} + \gamma\frac{|u'|^2}{u} - f(t).$$

2. Mixed problem with time and space singularities

We consider the singular mixed boundary value problem

$$u'' = p(u')[f(t, u, u') - r(t)], \tag{1}$$

$$u(0) = 0, \quad u'(T) = 0, \tag{2}$$

where

(H_1) $p \in C^0(-a, a)$ is positive, $0 < a \le \infty$,

(H_2) $f \in Car((0, T) \times \mathcal{D})$, $\mathcal{D} = (0, \infty) \times \mathbb{R}$,

$$0 \le f(t, x, y) \le A\left[\frac{x^{\eta_0} + |y|^{\gamma_0}}{t^{\mu_0}} + \frac{|y|^{\gamma_1}}{(T-t)^{\mu_1}} + \frac{|y|^{\gamma}}{x^{\eta}}\right] + h(x, y)$$

for a.e. $t \in [0, T]$ and all $(x, y) \in \mathcal{D}$, where $h \in C^0([0, \infty) \times \mathbb{R})$ is nonnegative, $h(x, 0) = 0$ for $x \in [0, \infty)$ and A, η_0, μ_i, γ_i, γ and η are positive constants, $\mu_0 < 2\eta_0$, $\mu_i \le \gamma_i$ $(i = 0, 1)$, $\gamma \ge 2\eta$,

(H_3) $r \in L_1[0, T]$,

$$r(t) \geq r_* > 0 \quad \text{for a.e. } t \in [0, T],$$

and

$$\min \left\{ \int_{-a}^{0} \frac{\mathrm{d}s}{p(s)}, \int_{0}^{a} \frac{\mathrm{d}s}{p(s)} \right\} > \int_{0}^{T} r(t)\,\mathrm{d}t.$$

(H_4) if $\mu_1 \geq 1$ in (H_2), then $\exists\, \nu \in (0, T)$ and $\exists\, \omega \in C^0[0, a)$, $\omega(0) = 0$, $\omega > 0$ on $(0, a)$ such that

$$f(t, x, y) \geq \frac{\omega(|y|)}{(T - t)^{\mu_1}}$$

for a.e. $t \in [\nu, T]$ and all $x \in (0, 1 + aT)$, $y \in (-a, a)$.

We say that $u \in AC^1[0, T]$ is *a positive solution* of problem (1), (2) if $u > 0$ on $(0, T]$, u satisfies (2) and
$u''(t) = p(u'(t))[f(t, u(t), u'(t)) - r(t)]$ holds for a.e. $t \in [0, T]$.

From assumption (H_2) it follows that $f(t, x, y)$ admits a time singularity at $t = 0$ and/or $t = T$ and a strong space singularity at $x = 0$. Hence problem (1), (2) admits mixed singularities.

REMARKS.

1) Under assumption (H_1), (1) can be written in the form

$$(\phi(u'))' = f(t, u, u') - r(t),$$

where $\phi(y) = \int_0^y \frac{\mathrm{d}s}{p(s)}$ for $y \in (-a, a)$.

2) If $\mu_1 \in (0, 1)$ in (H_2), then $f \in Car((0, T] \times \mathcal{D})$.

3. EXISTENCE OF POSITIVE SOLUTIONS

THEOREM 1. Let $(H_1)-(H_4)$ hold. Then there exists a positive solution $u \in AC^1[0, T]$ of problem (1), (2).

COROLLARY 1. Let $(H_1)-(H_4)$ hold. Then for all $\lambda > 0$, the problem

$$u'' = p(u')[\lambda f(t, u, u') - r(t)], \quad x(0) = 0, \quad x'(T) = 0$$

has a positive solution $u \in AC^1[0, T]$.

EXAMPLE. Let $\varepsilon = \pm 1$. Consider the differential equation

$$u'' = \frac{1}{1+\varepsilon(u')^2}\left(\frac{u^\alpha}{t^{\mu_0}} + \frac{|u'|^\beta}{(T-t)^{\mu_1}} + \frac{|u'|^\gamma}{u^\eta} + |u'|^\nu e^u - r(t)\right), \qquad (3)$$

where α, μ_0, μ_1, β, η, ν are positive numbers, $\mu_0 < 2\alpha$, $\mu_1 \le \beta$, $\gamma \ge 2\eta$, and $r \in L_1[0, T]$, $r(t) \ge r_* > 0$ for a.e. $t \in [0, T]$. In addition, if $\varepsilon = -1$ then $\int_0^T r(t)\,\mathrm{d}t < \frac{2}{3}$. By Theorem 1, problem (3), (2) has a positive solution $u \in AC^1[0, T]$.

Proof of Theorem 1. The proof is based on a combination of the method of lower and upper functions with a regularization and a sequential technique. In limit processes the Fatou lemma is used.

Step 1. *Regularization*

By (H_3), there exists $S \in (0, a)$ such that

$$\min\left\{\int_{-S}^{0} \frac{\mathrm{d}s}{p(s)}, \int_{0}^{S} \frac{\mathrm{d}s}{p(s)}\right\} > \int_{0}^{T} r(t)\,\mathrm{d}t.$$

Define $\hat{p}$, $\chi \in C^0(\mathbb{R})$ by the formulas

$$\hat{p}(y) = \begin{cases} p(S) & \text{for } y > S, \\ p(y) & \text{for } |y| \le S, \\ p(-S) & \text{for } y < -S, \end{cases}$$

$$\chi(y) = \begin{cases} 1 & \text{for } |y| \le S, \\ 2 - \frac{|y|}{S} & \text{for } S < |y| \le 2S, \\ 0 & \text{for } |y| > 2S. \end{cases}$$

3. Existence of positive solutions

Let ε_* be a positive constant such that

$$\varepsilon_* < \min\left\{1, \frac{T}{4}, \sqrt[2\eta_0-\mu_0]{\frac{r_*}{4A}}\right\}$$

For each $\varepsilon \in (0, \varepsilon_*)$ define $\sigma_\varepsilon : \mathbb{R} \to [\varepsilon^2, \infty)$ by

$$\sigma_\varepsilon(x) = \begin{cases} 1 + ST & \text{for } x > 1 + ST, \\ x & \text{for } \varepsilon^2 < x \le 1 + ST, \\ \varepsilon^2 & \text{for } x \le \varepsilon^2. \end{cases}$$

Let $\{t_n\} \subset (T - \varepsilon_*, T)$ be an increasing sequence such that $\lim_{n\to\infty} t_n = T$ and inequality in (H_2) holds for $t = t_n$, $(x, y) \in \mathcal{D}$ and $n \in \mathbb{N}$. Put $\varepsilon_n := T - t_n$. Finally, let

$$f_n(t, x, y) = \begin{cases} f(t, x, y) & \text{for } t \in (0, t_n),\ (x, y) \in \mathcal{D}, \\ f(t_n, x, y) & \text{for } t \in [t_n, T],\ (x, y) \in \mathcal{D}. \end{cases}$$

3. Existence of positive solutions

Consider the regular mixed problem

$$u'' = \chi(u')\hat{p}(u')[f_n(t, \sigma_{\varepsilon_n}(u), u') - r(t)], \tag{4}$$

$$u(\varepsilon_n) = \varepsilon_n^2, \quad u'(T) = 0, \tag{5}$$

on the interval $[\varepsilon_n, T]$.

Step 2. *A priori bounds*

We show that if $u_n \in AC^1[\varepsilon_n, T]$ is a solution of (4), (5), then

$$\varepsilon_n^2 \le u_n(t) < 1 + ST \quad \text{for } t \in [\varepsilon_n, T], \tag{6}$$

$$|u_n'(t)| < S \quad \text{for } t \in [\varepsilon_n, T]. \tag{7}$$

Step 3. *Lower function of* (4), (5)

We prove that there exists $n_* \in \mathbb{N}$ and $C_* \in (0,1)$ such that the function

$$\alpha_n(t) = \begin{cases} C_*[(t-\varepsilon_n)(2T - t - 3\varepsilon_n) + \varepsilon_n]^2 & \text{for } t \in [\varepsilon_n, t_n], \\ C_*[(2t_n - T)^2 + T - t_n]^2 & \text{for } t \in (t_n, T] \end{cases}$$

is a lower function of (4), (5) for $n \ge n_*$.

Step 4. *Existence of a solution of* (4), (5)
We prove that for all $n \geq n_*$ there exists a solution $u_n \in AC^1[\varepsilon_n, T]$ of problem (4), (5) satisfying (6), (7) and the inequality

$$u_n(t) \geq \alpha_n(t) \quad \text{for } t \in [\varepsilon_n, T]. \tag{8}$$

Here we apply the method of lower and appear functions with the lower function $\alpha_n(t)$ and the upper function $\beta(t) := (2T^2 + 1)^2 + 2St$.

Step 5. *Existence of a positive solution of* (1), (2)
By Step 4, for each $n \geq n_*$ there exists a solution $u_n \in AC^1[\varepsilon_n, T]$ of (4), (5) satisfying inequalities (6)-(8). By the Arzelà-Ascoli theorem, the diagonalization principle and the Fatou lemma, we can choose a function $\hat{u} \in C^0(0, T] \cap C^1(0, T)$ and a subsequence $\{k_n\}$ of $\{n\}$ such that

$$\begin{aligned} \lim_{n\to\infty} u_{k_n}(t) &= \hat{u}(t) \text{ locally uniformly on } (0, T], \\ \lim_{n\to\infty} u'_{k_n}(t) &= \hat{u}'(t) \text{ locally uniformly on } (0, T). \end{aligned}$$

We can show that there exists $u \in AC^1[0, T]$ such that $u = \hat{u}$ on $(0, T]$ and that $u(0) = 0$ is a solution of (1) on $[0, T]$. Here we distinguish if $\mu_1 \in (0, 1)$ or $\mu_1 \geq 1$. Finally, by (H_4), we show that $u'(T) = 0$.

4. EXISTENCE OF THE POSITIVE MAXIMAL SOLUTION

Let conditions $(H_1) - (H_4)$ be satisfied. Then Theorem 1 guarantees the existence of a solution $u \in AC^1[0, T]$ of problem $(1), (2)$ fulfilling $u(t) > 0$ for $t \in (0, T]$. Define

$$\mathcal{A} = \{u \in AC^1[0, T] : u \text{ is a positive solution of problem } (1), (2)\}.$$

Then $\mathcal{A}$ is a nonempty set. We say that $\overline{u} \in \mathcal{A}$ is *the maximal positive solution of problem* $(1), (2)$ if $\overline{u} \geq u$ on $[0, T]$ for each $u \in \mathcal{A}$.

4. Existence of the positive maximal solution

It is obvious that if problem (1), (2) has the maximal positive solution, then is unique. In order to prove the existence of the maximal positive solution of problem (1), (2) we use generalized lower and upper functions of the differential equation

$$u'' = \chi(u')\hat{p}(u')[f(t, u, u') - r(t)] \tag{9}$$

on intervals of the type $[t_1, t_2] \subset [0, T]$, where the functions χ and $\hat{p}$ are given in Section 3.

4. Existence of the positive maximal solution

Let $[t_1, t_2] \subset [0, T]$. By Kiguradze and Shekher (1987), we say that a positive function $\gamma \in AC[t_1, t_2]$ is a *generalized lower* (*upper*) *function* of the differential equation (9) on the interval $[t_1, t_2]$ if

(i) γ' can be written in the form $\gamma'(t) = \xi(t) + \xi_0(t)$ where $\xi \in AC[t_1, t_2]$ and $\xi_0 : [t_1, t_2] \to \mathbb{R}$ is nondecreasing (nonincreasing) and its derivative vanishes a.e. on $[t_1, t_2]$,

(ii) the inequality

$$\begin{aligned} \gamma''(t) &\geq \chi(\gamma'(t))\hat{\gamma}'(t))[f(t, \gamma(t), \gamma'(t)) - r(t)] \\ (\gamma''(t) &\leq \chi(\gamma'(t))\hat{\gamma}'(t))[f(t, \gamma(t), \gamma'(t)) - r(t)]) \end{aligned}$$

holds for a.e $t \in [t_1, t_2]$.

Let us choose $u_* \in \mathcal{A}$ and put

$$\mathcal{A}_* = \{u \in \mathcal{A} : u(t) \geq u_*(t) \text{ for } t \in [0, T]\}.$$

THEOREM 2. Assume that conditions $(H_1) - (H_4)$ hold. Then there exists the maximal positive solution of problem $(1), (2)$.
Proof. *Step* 1. We show that each $u \in \mathcal{A}$ satisfies the inequality

$$|u'(t)| < S \quad \text{for } t \in [0, T].$$

and if $\{u_n\} \subset \mathcal{A}_*$, then there exist its subsequence $\{u_{k_n}\}$ and $u \in \mathcal{A}_*$ such that $\lim_{n\to\infty} u_{k_n}(t) = u(t)$ uniformly on $[0, T]$.

Step 2. Put

$$M = \sup \left\{ \int_0^T u(t)\,\mathrm{d}t : u \in \mathcal{A}_* \right\}.$$

Then $M < ST^2$. Let $\{u_n\} \subset \mathcal{A}_*$ be such that

$$\lim_{n\to\infty} \int_0^T u_n(t)\,\mathrm{d}t = M.$$

In view of Step 1 there exist $\overline{u} \in \mathcal{A}_*$ and a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that $\lim_{n\to\infty} u_{k_n}(t) = \overline{u}(t)$ uniformly on $[0, T]$. Hence

$$M = \int_0^T \overline{u}(t)\,\mathrm{d}t.$$

4. Existence of the positive maximal solution

Step 3. We will proof that $\overline{u}$ is the maximal positive solution of problem (1), (2). To prove this, suppose the contrary. Then there exists $z \in \mathcal{A}_*$ such that either

(i) $z > \overline{u}$ on (t_1, t_2), $0 < t_1 < t_2 < T$, and $z(t_j) = \overline{u}(t_j)$ for $j = 1, 2$ or
(ii) $z > \overline{u}$ on $(0, t_3)$, $0 < t_3 < T$, and $z(t_3) = \overline{u}(t_3)$ or
(iii) $z > \overline{u}$ on (t_4, T), $0 < t_4 < T$, and $z(t_4) = \overline{u}(t_4)$.

We now consider separately cases (i)-(iii). Applying the method of generalized lower and upper functions together and a sequential technique we show that there exist $w \in \mathcal{A}_*$ such that $w(t) \geq \max\{\overline{u}(t), z(t)\}$ for $t \in [0, T]$. Hence $\int_0^T w(t)\,\mathrm{d}t > M$, which is impossible.