On the Cauchy problem for nonlinear functional-differential systems

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$$x'(t) = F(x)(t), \tag{1}$$
$$x(a) = c \tag{2}$$

$F : C([a, b]; \mathbb{R}^n) \to L([a, b]; \mathbb{R}^n)$ is a continuous operator such that

$$\sup\Big\{\big\|F(u)(\cdot)\big\| : u \in C([a, b]; \mathbb{R}^n),\ \|u\|_C \leq r\Big\} \in L([a, b]; \mathbb{R}_+) \quad \text{for } r > 0,$$

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Solution:

Absolutely continuous vector function $x : [a, b] \longrightarrow \mathbb{R}^n$ satisfying the system (1) almost everywhere on $[a, b]$ and verifying the initial condition (2).

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Remark. Since F is (in general) non-Volterra operator

- solutions of (1) have to be understood as global ones,
- notions like local solution and extendability of solutions have no sense.

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Problem:

To find conditions on F under which the problem (1), (2) is solvable (uniquely).

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Method of proofs of the main results

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- the nonlinear system (1) is compared with a suitable linear one
- to get solvability of the problem (1), (2) we apply the lemma on a priory estimate by I. Kiguradze, B. Půža (published in CMJ, 1997)
- to find a priory estimate of solutions of auxiliary problems we use the theorems on differential inequalities

$$x'(t) = \ell(x)(t) + q(t) \tag{3}$$

$\ell : C([a,b];\mathbb{R}^n) \to L([a,b];\mathbb{R}^n)$ is a linear bounded operator, $\quad q \in L([a,b];\mathbb{R}^n)$

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Components of the linear operator ℓ can be defined as follows:

- For any $i \in \{1, \ldots, n\}$ and $u \in C([a, b]; \mathbb{R}^n)$ we denote by $\ell_i(u)$ the i-th component of the vector funcion $\ell(u)$. Then

$$\ell_i : C([a, b]; \mathbb{R}^n) \to L([a, b]; \mathbb{R}) \quad \text{for } i = 1, \ldots, n.$$

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- For any $i,k \in \{1,\dots,n\}$ and $z \in C([a,b];\mathbb{R})$ we put

$$\ell_{ik}(z) = \ell_i\big(z_k\big), \qquad \text{where} \qquad z_k \equiv \begin{pmatrix} 0 \\ \vdots \\ z \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}k\text{-th}$$

Then

$$\ell_{ik} : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R}) \quad \text{for } i,k = 1,\dots,n,$$

are linear bounded operators.

$$x'(t) = \ell(x)(t) + q(t)$$

$$\updownarrow$$

$$\begin{aligned}
x_1'(t) &= \ell_{11}(x_1)(t) + \ell_{12}(x_2)(t) + \cdots + \ell_{1n}(x_n)(t) + q_1(t), \\
x_2'(t) &= \ell_{21}(x_1)(t) + \ell_{22}(x_2)(t) + \cdots + \ell_{2n}(x_n)(t) + q_2(t) \\
&\vdots \\
x_n'(t) &= \ell_{n1}(x_1)(t) + \ell_{n2}(x_2)(t) + \cdots + \ell_{nn}(x_n)(t) + q_n(t),
\end{aligned}$$

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Definition. We say that the theorem on differential inequalities holds for the system (3) if the implication

$$\left.\begin{array}{l} u \in AC([a,b];\mathbb{R}^n) \\ u'(t) \geq \ell(u)(t) \quad \text{for a. e. } t \in [a,b] \\ u(a) \geq 0 \end{array}\right\} \implies \quad u(t) \geq 0 \quad \text{for } t \in [a,b]$$

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2. the problem (3), $u(a) = c$ has a unique solution for every $q \in L([a,b];\mathbb{R}^n)$, $c \in \mathbb{R}^n$ and

$$c \geq 0, \qquad q(t) \geq 0 \quad \text{for a. e. } t \in [a,b] \implies u(t) \geq 0 \quad \text{for } t \in [a,b]$$

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3. the operator K_ℓ is inverse positive in the set B, where $B = \{x \in AC([a,b];\mathbb{R}^n) : x(a) \geq 0\}$ and $K_\ell : B \to L([a,b];\mathbb{R}^n)$ is defined by
$$K_\ell(v)(t) := v'(t) - \ell(v)(t) \quad \text{for a. e. } t \in [a,b]$$

$$x' = P(t)x + q(t) \tag{4}$$

$P = (p_{ik})_{i,k=1}^{n} : [a, b] \to \mathbb{R}^{n\times n}$ is an integrabe matrix function, $\quad q \in L([a, b]; \mathbb{R}^n)$

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Remark. If (3) is not "ordinary system" then (in general) some additional assumptions have to be imposed on the operator ℓ even in the scalar case.

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$$\gamma(t) > 0 \quad \text{for } t \in [a,b], \tag{6}$$

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Theorem C. Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}^n$. Then

$$\ell_0 \in \mathcal{S}^n, \quad -\ell_1 \in \mathcal{S}^n \quad \Longrightarrow \quad \ell \in \mathcal{S}^n.$$

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$$x(a) = c \tag{2}$$

Theorem. Let there exist $\ell_0, \ell_1 \in \mathcal{P}^n$ and $q^* \in L([a,b]; \mathbb{R}^n_+)$ such that the inequality

$$\mathrm{Sgn}(v(t)) \Big[F(v)(t) + \ell_1(v)(t) \Big] \leq \ell_0(|v|)(t) + q^*(t) \quad \text{for a. e. } t \in [a,b] \tag{8}$$

holds on the set $C([a,b]; \mathbb{R}^n)$ and, moeover,

$$\ell_0 \in \mathcal{S}^n, \qquad -\ell_1 \in \mathcal{S}^n. \tag{9}$$

Then the problem (1), (2) has at least one solution.

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Remark. $-\ell_1 \in \mathcal{S}^n \implies$ the operator ℓ_1 is "diagonal", i.e., $\ell_{ik} \equiv 0$ for $i \neq k$.

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Theorem. Let there exist $\ell \in \mathcal{P}^1$ and $q^* \in L([a,b];\mathbb{R}_+)$ such that the inequality

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Then the problem (1), (2) has at least one solution.

$$x' = f(t,x), \qquad x(a) = c \tag{12}$$

Corollary. Let there exist $p \in L([a,b];\mathbb{R}_+)$ and $h \in L([a,b];\mathbb{R})$ such that

$$f(t,x) \cdot \operatorname{sgn}(x) \le p(t)\|x\| + h(t) \quad \text{for a. e. } t \in [a,b] \text{ and all } x \in \mathbb{R}^n. \tag{13}$$

Then the problem (12) has at least one solution.

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Wintner's type theorem. Let there exist $p \in L([a,b];\mathbb{R}_+)$ and $h \in L([a,b];\mathbb{R})$ such that

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Then every nonextendable solution of (12) is defined on the interval $[a,b]$.

$$x'(t) = F(x)(t), \tag{1}$$
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Theorem. Let there exist
$$\ell \in \mathcal{P}^1 \cap \mathcal{S}^1$$
such that, for any $u, v \in C([a,b];\mathbb{R}^n)$, the inequality
$$\Big[F(u)(t) - F(v)(t)\Big] \cdot \operatorname{sgn}\big(u(t) - v(t)\big) \le \ell(\|u - v\|)(t) \quad \text{for a. e. } t \in [a,b] \tag{14}$$
holds. Then the problem (1), (2) has a unique solution.

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Osgood's type corollary. Let there exist $p \in L([a,b];\mathbb{R}_+)$ such that

$$\Big[f(t,x) - f(t,y)\Big] \cdot \operatorname{sgn}(x - y) \leq p(t)\|x - y\| \quad \text{for a. e. } t \in [a,b] \text{ and all } x, y \in \mathbb{R}^n. \tag{16}$$

Then the problem (15) is uniquely solvable.

Remarks.

- The results presented can be generalized for the boundary condition

$$x(a) = \varphi(x)$$

where $\varphi : C([a,b];\mathbb{R}^n) \to \mathbb{R}$ is a continuous operator. The assumptions on φ has the form of one-sided restrictions, e. g.,

$$\operatorname{Sgn}\big(u(a)\big)\varphi(u) \le c \quad \text{for } u \in C([a,b];\mathbb{R}^n)$$

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- If F is a Volterra operator, i. e., for every $t_0 \in \,]a,b]$ and $x, y \in C([a,b];\mathbb{R}^n)$ satisfying

$$x(t) = y(t) \quad \text{for } t \in [a, t_0]$$

the relation

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- Results well-known for the ordinary nonlinear systems was generalized for the problem (1), (2) with a singular Volterra operator in works by I. Kiguradze and Z. Sokhadze.

$$x_1'(t) = 3x_2^3(t) - x_1\left(\frac{a+b}{2}\right) + \sin t,$$
$$x_2'(t) = x_2(b)\int_a^b x_1(s)\,\mathrm{d}s - e^{x_1(t)x_2(t)}x_2(t)$$

Definition. $\mathcal{P}^n$ is the set of linear operators ℓ for which

$$\ell(v)(t) \geq 0 \quad \text{for a. e. } t \in [a,b]$$

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Example. Let $\ell : C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$ is given by the formula

$$\ell(v)(t) \overset{\text{def}}{=} \begin{pmatrix} h_{11}(t)v_1\big(\tau_{11}(t)\big) + h_{12}(t)v_2\big(\tau_{12}(t)\big) \\ h_{21}(t)v_1\big(\tau_{21}(t)\big) + h_{22}(t)v_2\big(\tau_{22}(t)\big) \end{pmatrix} \quad \text{for a. e. } t \in [a,b],$$

where $h_{ik} \in L([a,b];\mathbb{R})$ and $\tau_{ik} : [a,b] \to [a,b]$ are measurable functions $(i,k = 1,2)$. Then, for any $i,k \in \{1,2\}$ and $z \in C([a,b];\mathbb{R})$, we have

$$\ell_{ik}(z)(t) = h_{ik}(t)z\big(\tau_{ik}(t)\big) \quad \text{for a. e. } t \in [a,b].$$

Moreover,

$$\ell \in \mathcal{P}^2 \qquad \Longleftrightarrow \qquad h_{ik}(t) \geq 0 \quad \text{for a. e. } t \in [a,b].$$

If $x = (x_k)_{k=1}^n \in \mathbb{R}^n$ then we put

$$\mathrm{Sgn}(x) = \begin{pmatrix} \operatorname{sgn} x_1 & 0 & \dots & 0 \\ 0 & \operatorname{sgn} x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \operatorname{sgn} x_n \end{pmatrix}.$$

$$x_i'(t) = \sum_{k=1}^{2} p_{ik}(t) x_k\big(\tau_{ik}(t)\big) + f_i\Big(t, x_1(t), x_2(t), x_1(\mu_{i1}(t)), x_2(\mu_{i2}(t))\Big), \;\; i = 1, 2, \quad (17)$$

$$x_i(a) = c_i, \quad i = 1, 2, \quad (18)$$

Corollary. Let there exist $q_1, q_2 \in L([a,b]; \mathbb{R}_+)$ such that, for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$, the inequalities

$$f_i\big(t, x_1, x_2, y_1, y_2\big) \operatorname{sgn} x_i \le q_i(t) \quad \text{for a. e. } t \in [a,b], \;\; i = 1, 2$$

hold. If, moreover,

$$\tau_{ii}(t) \le t \quad \text{for a. e. } t \in [a,b], \;\; i = 1, 2,$$

$$\int_{\tau_{11}(t)}^{t} [p_{11}(s)]_- ds \le \frac{1}{e}\,, \quad \int_{\tau_{22}(t)}^{t} [p_{22}(s)]_- ds \le \frac{1}{e} \quad \text{for a. e. } t \in [a,b],$$

and

$$\int_{t}^{\tau_{12}(t)} p(s) ds \le \frac{1}{e}\,, \quad \int_{t}^{\tau_{21}(t)} p(s) ds \le \frac{1}{e} \quad \text{for a. e. } t \in [a,b],$$

where

$$p(t) \stackrel{\text{def}}{=} \max\big\{[p_{11}(t)]_+ + |p_{12}(t)|, [p_{22}(t)]_+ + |p_{21}(t)|\big\} \quad \text{for a. e. } t \in [a,b].$$

Then the problem (17), (18) has at least one solution.

If $x = (x_k)_{k=1}^n \in \mathbb{R}^n$ then we put

$$\operatorname{sgn}(x) = \begin{pmatrix} \operatorname{sgn} x_1 \\ \operatorname{sgn} x_2 \\ \vdots \\ \operatorname{sgn} x_n \end{pmatrix}.$$

$$
\begin{aligned}
x_1'(t) &= p_1(t)x_2\big(\tau(t)\big) + g_1(t)e^{\,x_1(t)x_2(t)}|x_2(t)|x_1(t) + q_1(t),\\
x_2'(t) &= p_2(t)x_1\big(\tau(t)\big) - g_2(t)e^{\,x_1(t)x_2(t)}|x_1(t)|x_2(t) + q_2(t),
\end{aligned} \tag{19}
$$

$$
x_1(a) = c_1, \quad x_2(a) = c_2 \tag{20}
$$

Corollary. The problem (19), (20) has at least one solution provided that

$$
g_1(t) \le g_2(t) \quad \text{for a. e. } t \in [a,b]
$$

and

$$
\int_t^{\tau(t)} \widetilde{p}(s)ds \le \frac{1}{e}\ ,
$$

where

$$
\widetilde{p}(t) \stackrel{\text{def}}{=} \max\big\{|p_1(t)|, |p_2(t)|\big\} \quad \text{for a. e. } t \in [a,b].
$$