On boundedness and Stability of Solutions of Nonlinear Delayed Differential Systems

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In the present paper we establish new effective conditions which guarantee, respectively, the boundedness, uniform stability and uniform asymptotic stability of solutions of nonlinear differential systems with delay.

Throughout the paper, the use will be made of the following notation:

$$R =]-\infty, +\infty[, \qquad R_+ = [0, +\infty[$$

δ_{ik} is Kronecker's symbol, i.e.,

$$\delta_{ik} = \begin{cases} 1 & \text{for } i=k \\ 0 & \text{for } i \neq k \end{cases};$$

$x = (x_i)_{i=1}^{n}$ and $X = (x_{ik})_{i,k=1}^{n}$ are the n-dimensional column vector and $n \times n$ -matrix with the elements x_i and $x_{ik} \in R\,(i,k = 1,\cdots,n)$ and the norms

$$\|x\| = \sum_{i=1}^{n} |x_i| , \qquad \|X\| = \sum_{i,k=1}^{n} |x_{ik}| ;$$

X^{-1} is the matrix inverse to X ;

$r(X)$ is the spectral radius of X;

E is the unit matrix ;

I is a compact or noncompact interval;

$C(I)$ is the space of bounded continuous functions $x : I \to R$ with the norm

$$\|x\|_{C(I)} = \sup\{|x(t)| : t \in I\} \;;$$

$\tilde{C}_{loc}(I)$ is the space of functions $x : I \to R$, absolutely continuous on every compact interval containing in I ;

$L(I)$ is the space of Lebesgue integrable functions $x : I \to R$;

$L_{loc}(I)$ is the space of functions $x : I \to R$, Lebesgue integrable on every compact interval containing in I .

Consider the differential system

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t))) \quad (i=1,\cdots,n). \tag{1}$$

Here $f_i : R_+ \times R^n \to R \quad (i=1,\cdots n)$ are functions satisfying the local Carathéodory conditions,

$$g_{0i} \in L_{loc}(R_+), \quad g_{0i}(t) \geq 0 \qquad \text{for} \qquad t \in R_+ \quad (i=1,\cdots,n)$$

and $\tau_i : R_+ \to R, \quad \tau_{ik} : R_+ \to R \quad (i,k=1,\cdots,n)$ are measurable on every finite interval functions such that

$$\tau_i(t) \leq t, \quad \tau_{i,k} \leq t \qquad \text{for} \qquad t \in R_+ \quad (i=1,\cdots,n) \quad .$$

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t)))\quad (i=1,\cdots,n) \tag{1}$$

Let

$$a\in R_+,\quad c_i\in C(]-\infty,a[),\quad c_{0i}\in R\quad (i=1,\cdots n)\ .$$

For the system (1) , we consider the Cauchy problem

$$x_i(t)=c_i(t)\qquad \text{for}\qquad t<a,\quad x_i(a)=c_{0i}\quad (i=1,\cdots n) \tag{2}$$

Suppose

$$\chi_a(t)=\begin{cases}1 & \text{for } t\geq a\\ 0 & \text{for } t<a\end{cases},\qquad \tau_{aik}(t)=\begin{cases}\tau_{ik}(t) & \text{for } t\geq a\\ a & \text{for } t<a\end{cases},\qquad (i,k=1,\cdots,n)$$

$$\tau_{ai}(t)=\begin{cases}\tau_i(t) & \text{for } t\geq a\\ a & \text{for } t<a\end{cases}\qquad (i=1,\cdots,n)$$

and introduce the following.

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t))) \quad (i=1,\cdots,n) \tag{1}$$

$$x_i(t)=c_i(t) \quad \text{for} \quad t<a, \quad x_i(a)=c_{0i} \quad (i=1,\cdots n) \tag{2}$$

Definition 1. Let $-\infty<a<b\le+\infty$ and $I=[a,b[$, or $-\infty<a<b<+\infty$ and $I=[a,b]$. The vector function $(x_i)_{i=1}^n : I \to R^n$ is said to be a **solution of the problem (1), (2)** , defined on I , if

$$x_i\in\tilde{C}_{loc}(I), \quad x_i(a)=c_i \quad (i=1,\cdots,n)$$

and almost everywhere on I the equality (1) is fulfilled, where

$$\begin{aligned} x_i(\tau_{ik}(t)) &= (1-\chi_a(\tau_{ik}(t)))c_i(\tau_{ik}(t))+ \\ &+\chi_a(\tau_{ik}(t))x_i(\tau_{aik}(t)) \quad (\mathrm{i,k}=1,\cdots \mathrm{n}), \end{aligned} \tag{3}$$

$$x_i(\tau_i(t))=(1-\chi_a(\tau_i(t)))c_i(\tau_i(t))+\chi_a(\tau_i(t))x_i(\tau_{ai}(t)) \quad (\mathrm{i}=1,\cdots \mathrm{n}), \tag{4}$$

and

$$c_i(t)=0 \quad \text{for} \quad t\ge a \quad (i=1,\cdots,n).$$

$$x_i'(t) + g_{0i}(t)\,x_i(\tau_i(t)) = f_i\big(t, x_1(\tau_{i1}(t)), \cdots, x_n(\tau_{in}(t))\big) \quad (i = 1, \cdots, n) \tag{1}$$

$$x_i(t) = c_i(t) \quad \text{for} \quad t < a, \quad x_i(a) = c_{0i} \quad (i = 1, \cdots n) \tag{2}$$

Definition 2. Let $-\infty < a < b < +\infty$ and $I = [a, b[$ ($I = [a, b]$). A solution $(x_i)_{i=1}^n$ of the problem (1), (2) is said to be **continuable** if there exist $\bar{b} \in [b, +\infty[$ $(b \in]b, +\infty[)$ and a solution $(\bar{x}_i)_{i=1}^n$ of that problem, defined on $[a, \bar{b}]$, such that

$$\bar{x}_i(t) = x_i(t) \quad \text{for} \quad t \in I \quad (i = 1, \cdots, n) \quad .$$

A solution $(x_i)_{i=1}^n$ is, otherwise, called **noncontinuable.**

If $f_i(t, 0, \cdots, 0) \equiv 0 \ (i = 1, \cdots, n)$, then the system (1) under the initial conditions $x_i(t) = 0$ for $t \leq 0$ has a trivial solution.

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t))) \quad (i=1,\cdots,n) \tag{1}$$

$$x_i(t)=c_i(t) \quad \text{for} \quad t<a, \quad x_i(a)=c_{0i} \quad (i=1,\cdots n) \tag{2}$$

Definition 3. A trivial solution of the system (1) is said to be **uniformly stable** if for any $\varepsilon>0$ there exist $\delta>0$ such that for arbitrary numbers and functions $a\in R_+$, $c_{0i}\in R$ and $c_i\in C(]-\infty,a[)$ $(i=1,\cdots,n)$, satisfying the condition

$$\sum_{i=1}^{n}\left(|c_{0i}|+\|c_i\|_{C(]-\infty,a[)}\right)<\delta, \tag{5}$$

every noncontinuable solution of the problem (1), (2) is defined on $[a,+\infty[$ and admits the estimate

$$\sum_{i=1}^{n}\|x_i\|_{C([a,+\infty[)}<\varepsilon. \tag{6}$$

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t)))\quad (i=1,\cdots,n) \tag{1}$$

$$x_i(t)=c_i(t) \quad \text{for} \quad t<a,\quad x_i(a)=c_{0i}\quad (i=1,\cdots n) \tag{2}$$

$$\sum_{i=1}^{n}\left(|c_{0i}|+\|c_i\|_{C(]-\infty,a[)}\right)<\delta, \tag{5}$$

$$\sum_{i=1}^{n}\|x_i\|_{C([a,+\infty[)}<\varepsilon. \tag{6}$$

Definition 4. A trivial solution of the system (1) is said to be **uniformly asymptotically stable** if for any $\varepsilon>0$ there exist $\delta>0$ such that for arbitrary numbers and functions $a\in R_+,\ c_{0i}\in R$ and $c_i\in C(]-\infty,a[)\quad (i=1,\cdots,n)$, satisfying the condition (5), every noncontinuable solution of the problem (1), (2) is defined on $[a,+\infty[$, admits the estimate (6) and is vanishing at infinity, i.e.

$$\lim_{t\to+\infty}x_i(t)=0\quad (i=1,\cdots,n). \tag{7}$$

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t)))\quad (i=1,\cdots,n) \tag{1}$$

$$x_i(t)=c_i(t) \quad \text{for} \quad t<a,\quad x_i(a)=c_{0i}\quad (i=1,\cdots n) \tag{2}$$

Theorem 1. Let there exist nonnegative constants ℓ_{ik} $(i,k=1,\cdots,n)$ and nonnegative functions $g_{ik}\in L_{loc}([a,+\infty[)$ and $f_{0i}\in L_{loc}([a,+\infty[)$ $(i,k=1,\cdots,n)$ such that the inequalities

$$|f_i(t,x_1,\cdots,x_n)|\le\sum_{k=1}^{n}g_{ik}(t)|x_k|+f_{0i}(t)\quad (i=1,\cdots,n), \tag{8}$$

$$g_{ik}(t)+g_{0i}(t)\int_{\tau_{ai}(t)}^{t}(g_{ik}(s)+\delta_{ik}g_{0k}(s))ds\le\ell_{ik}g_{0i}(t)\quad (i,k=1,\cdots,n) \tag{9}$$

are satisfied, respectively, on $[a,+\infty[\times R^n$ and $[a,+\infty[$. If, moreover,

$$\liminf_{t\to+\infty}\tau_i(t)>a\quad (i=1,\cdots,n), \tag{10}$$

$$\ell_{0i}=\sup\left\{\int_a^t\exp\left(-\int_s^t g_{0i}(x)dx\right)f_{0i}(s)ds+\int_{\tau_{ai}(t)}^{t}f_{0i}(s)ds:t\ge a\right\}<+\infty\quad (i=1,\cdots,n) \tag{11}$$

and the condition

$$r(\mathsf{L})<1 \quad , \quad \text{where} \quad \mathsf{L}=(\ell_{ik})_{i,k=1}^{n}, \tag{12}$$

is fulfilled, then every noncontinuable solution of the problem (1), (2) is defined on $[a,+\infty[$, is bounded and admits the estimate

$$\sum_{i=1}^{n}\|x_i\|_{C([a,+\infty[)}\le\rho\sum_{i=1}^{n}\left(\|c_i\|_{C(]-\infty,a[)}+\|c_{0i}\|+\ell_{0i}\right), \tag{13}$$

where ρ is a positive constant depending only on g_{0i}, g_{ik} and ℓ_{ik} $(i,k=1,\cdots,n)$.

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t))) \quad (i=1,\cdots,n) \tag{1}$$

$$x_i(t)=c_i(t) \quad \text{for} \quad t<a, \quad x_i(a)=c_{0i} \quad (i=1,\cdots n) \tag{2}$$

$$|f_i(t,x_1,\cdots,x_n)|\le\sum_{k=1}^{n}g_{ik}(t)|x_k|+f_{0i}(t) \quad (i=1,\cdots,n), \tag{8}$$

$$g_{ik}(t)+g_{0i}(t)\int_{\tau_{ai}(t)}^{t}(g_{ik}(s)+\delta_{ik}g_{0k}(s))ds\le\ell_{ik}g_{0i}(t) \quad (i,k=1,\cdots,n) \tag{9}$$

$$r(\mathsf{L})<1 \quad , \quad \text{where} \quad \mathsf{L}=(\ell_{ik})_{i,k=1}^{n}, \tag{12}$$

Theorem 2. Let there exist nonnegative constants ℓ_{ik} $(i,k=1,\cdots,n)$ and γ and nonnegative functions $g_0\in L_{loc}([a,+\infty[)$, $g_{ik}\in L_{loc}([a,+\infty[)$ and $f_i\in L_{loc}([a,+\infty[)$, such that the inequalities (8) are satisfied on $[a,+\infty[\times R^n$ and the inequalities

$$t-\tau_i(t)\le\gamma, \quad t-\tau_{ik}(t)\le\gamma \quad (i,k=1,\cdots,n), \quad g_{0i}(t)\ge g_0(t) \quad (i=1,\cdots,n) \tag{14}$$

along with (9) are satisfied on $[a,+\infty[$. Let, moreover,

$$\sup\left\{\int_0^t\exp\left(-\int_s^t g_{0i}(x)dx\right)\tilde f_{0i}(s)ds+\int_{\tau_{ai}(t)}^{t}\tilde f_{0i}(s)ds : t\ge a\right\}<+\infty \quad (i=1,\cdots,n) \tag{15}$$

Where

$$\tilde f_i(t)=\exp\left(\int_a^t g_0(s)ds\right)f_{0i}(t) \quad (i=1,\cdots,n), \tag{16}$$

and let the conditions (12) and $\int_a^{+\infty}g_0(x)dx=+\infty$ be fulfilled. Then every noncontinuable solution of the problem (1), (2) is defined on $[a,+\infty[$ and is vanishing at infinity.

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t))) \quad (i=1,\cdots,n) \tag{1}$$

$$g_{ik}(t)+g_{0i}(t)\int_{\tau_{ai}(t)}^{t}(g_{ik}(s)+\delta_{ik}g_{0k}(s))ds \le \ell_{ik}g_{0i}(t) \quad (i,k=1,\cdots,n) \tag{9}$$

$$r(\mathsf{L})<1 \quad , \quad \text{where} \quad \mathsf{L}=(\ell_{ik})_{i,k=1}^{n}, \tag{12}$$

Theorem 3. Let there exist constants $\delta_0>0, \quad \ell_{ik}\ge 0 \quad (i,k=1,\cdots,n)$ and nonnegative functions $g_{ik}\in L_{loc}(R_+) \quad (i,k=1,\cdots,n)$ such that, respectively, on the set $\{(t,x_1,\cdots,x_n):t\in R_+,|x_k|\le\delta_0 \quad (k=1,\cdots,n)\}$ and on the interval R_+ the inequalities

$$|f_i(t,x_1,\cdots,x_n)|\le\sum_{k=1}^{n}g_{ik}(t)|x_k| \quad (i=1,\cdots,n) \tag{17}$$

and (9) are satisfied. If, moreover,

$$\liminf_{t\to+\infty}\tau_i(t)>0 \quad (i=1,\cdots,n)$$

and the condition (12) is fulfilled, then the trivial solution of the system (1) is uniformly stable.

$$x_i'(t)+g_{0i}(t)x_i(\tau_i(t))=f_i\left(t,x_1(\tau_{i1}(t)),\cdots,x_n(\tau_{in}(t))\right)\quad (i=1,\cdots,n) \tag{1}$$

$$g_{ik}(t)+g_{0i}(t)\int_{\tau_{ai}(t)}^{t}\left(g_{ik}(s)+\delta_{ik}g_{0k}(s)\right)ds\le \ell_{ik}g_{0i}(t)\quad (i,k=1,\cdots,n) \tag{9}$$

$$r(\mathsf{L})<1 \quad , \quad \text{where} \quad \mathsf{L}=(\ell_{ik})_{i,k=1}^{n}, \tag{12}$$

$$t-\tau_i(t)\le\gamma,\quad t-\tau_{ik}(t)\le\gamma\quad (i,k=1,\cdots,n),\quad g_{0i}(t)\ge g_0(t)\quad (i=1,\cdots,n) \tag{14}$$

$$\left|f_i(t,x_1,\cdots,x_n)\right|\le\sum_{k=1}^{n}g_{ik}(t)|x_k|\quad (i=1,\cdots,n) \tag{17}$$

Theorem 4. Let there exist constants $\delta_0>0,\ \ \ell_{ik}\ge 0\ \ (i,k=1,\cdots,n)$ and nonnegative functions $g_{ik}\in L_{loc}(R_+)\ \ (i,k=1,\cdots,n)$ such that on the set $\{(t,x_1,\cdots,x_n):t\in R_+,|x_k|\le\delta_0\ \ (k=1,\cdots,n)\}$ the inequalities (17) are fulfilled, while on the interval R_+ the inequalities (9) and (14) hold. If, moreover, the conditions (12) and $\int_a^{+\infty}g_0(x)\,dx=+\infty$ are fulfilled, where

$$g_0(t)=\min\{g_{0i}(t):\ \ i=1,\cdots,n\},$$

then the trivial solution of the system (1) is uniformly asymptotically stable.

As an example, let us consider the linear differential system

$$x_i'(t)=\sum_{k=1}^{n} p_{ik}(t)\,x_i(\tau_{ik}(t)) \quad (i=1,\cdots,n) \tag{18}$$

where $p_{ik}\in L_{loc}(R_+)\quad (i,k=1,\cdots,n)$, and $\tau_{ik}:R_+\to R\quad (i,k=1,\cdots,n)$ are measurable on every finite segment functions satisfying the inequalities

$$\tau_{ik}(t)\le t \quad (i,k=1,\cdots,n).$$

The system (18) is said to be **uniformly stable** (**uniformly asymptotically stable**) if its trivial solution is uniformly stable (uniformly asymptotically stable).

Suppose

$$\tau_{0i}(t)=\begin{cases}\tau_{ii}(t) & \text{for } \tau_{ii}(t)\ge 0\\ 0 & \text{for } \tau_{ii}(t)<0\end{cases} \qquad (i=1,\cdots,n).$$

$$r(\mathsf{L}) < 1 \,, \qquad \text{where} \qquad \mathsf{L} = (\ell_{ik})_{i,k=1}^{n} \,, \tag{12}$$

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t)\, x_i(\tau_{ik}(t)) \quad (i = 1, \cdots, n) \tag{18}$$

From Theorem 3 we have

Corollary 1. Let almost everywhere on R_+ the inequalities

$$p_{ii}(t) \le 0, \qquad \int_{\tau_{0i}(t)}^{t} |p_{ii}(s)|\, ds \le \ell_{ii} \quad (i = 1, \cdots, n), \tag{19}$$

$$|p_{ik}(t)| + |p_{ii}(t)| \int_{\tau_{0i}(t)}^{t} |p_{ik}(s)|\, ds \le \ell_{ik} \,|p_{ii}(t)| \quad (i, k = 1, \cdots, n; \quad i \ne k) \tag{20}$$

be satisfied, where ℓ_{ik} $(i, k = 1, \cdots, n)$ are nonnegative constants, satisfying the condition (12). If, moreover,

$$\liminf_{t \to +\infty} \tau_{ii}(t) > 0 \quad (i = 1, \cdots, n),$$

then the system (18) is uniformly stable.

$$r(\mathrm{L}) < 1, \quad \text{where} \quad \mathrm{L} = (\ell_{ik})_{i,k=1}^{n}, \tag{12}$$

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t) x_i(\tau_{ik}(t)) \quad (i = 1, \cdots, n) \tag{18}$$

Theorem 4 results in

Corollary 2. Let almost everywhere on R_+ the inequalities

$$p_{ii}(t) \le 0, \qquad \int_{\tau_{0i}(t)}^{t} |p_{ii}(s)| ds \le \ell_{ii} \quad (i = 1, \cdots, n),$$

$$|p_{ik}(t)| + |p_{ii}(t)| \int_{\tau_{0i}(t)}^{t} |p_{ik}(s)| ds \le \ell_{ik} |p_{ii}(t)| \quad (i, k = 1, \cdots, n; \quad i \ne k)$$

be satisfied, where $\ell_{ik} \quad (i, k = 1, \cdots, n)$ are nonnegative constants, satisfying the condition (12). If, moreover,

$$\text{vrai max}\{t - \tau_{ik}(t) : t \in R_+\} < +\infty \quad (i, k = 1, \cdots, n), \qquad \int_{0}^{+\infty} p(t) = +\infty,$$

where

$$p(t) = \min\{|p_{ii}(t)| : \quad i = 1, \cdots, n\},$$

then the system (18) is uniformly asymptotically stable.