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DMF

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NEW FUČÍK SPECTRA

Workshop on Differential Equations
September 16- 20, 2007, Hejnice, Czech Republic

Contents

1. The second order Fučík problem

$$-x'' = \mu x^+ - \lambda x^-, \tag{1.1}$$

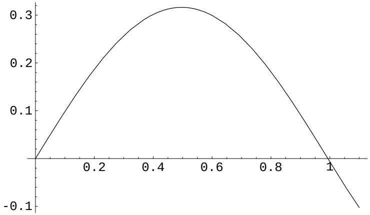
$$x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\},$$

$$x(0) = x(1) = 0. \tag{1.2}$$

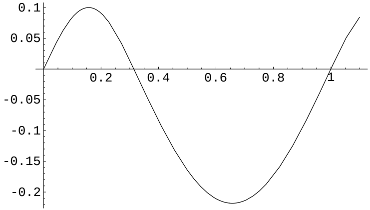
The Fučík spectrum is a set of points (μ, λ) such that the problem has nontrivial solutions.

The Fučík spectrum consists of a set of curves F_i^+ and F_i^-, $i = 0, 1, 2, \ldots$.

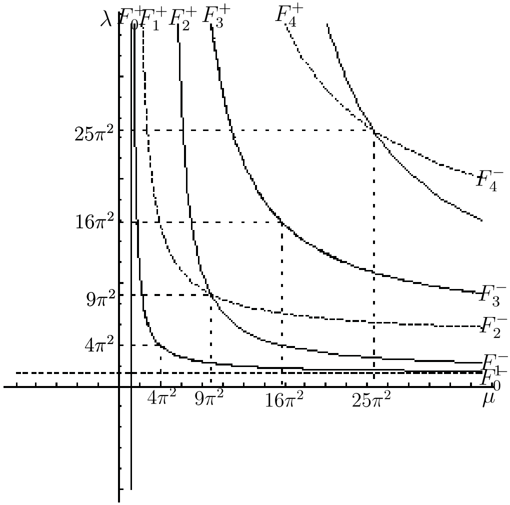
The lower index shows how many zeroes has the respective solution in the interval $(0; 1)$, but the upper index shows the sign of the derivative of a solution at $t = 0$.



1.1. fig.



1.2. fig.



1.3. fig. The Fučík spectrum for the problem (1.1), (1.2)

Theorem: *The Fučík spectrum for the problem* (1.1), (1.2) *consists of the branches given by*

$$F_0^+ = \left\{ \left(\pi^2, \lambda \right) \right\},$$

$$F_0^- = \left\{ \left(\mu, \pi^2 \right) \right\},$$

$$F_{2i-1}^+ = \left\{ (\mu, \lambda) \Big| \ \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} = 1 \right\},$$

$$F_{2i}^+ = \left\{ (\mu, \lambda) \Big| \ \frac{(i+1)\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} = 1 \right\},$$

$$F_{2i-1}^- = \left\{ (\mu, \lambda) \Big| \ \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} = 1 \right\},$$

$$F_{2i}^+ = \left\{ (\mu, \lambda) \Big| \ \frac{i\pi}{\sqrt{\mu}} + \frac{(i+1)\pi}{\sqrt{\lambda}} = 1 \right\},$$

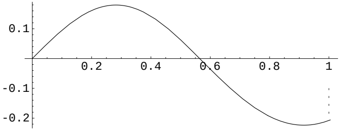
where $i = 1, 2, \ldots$.

2. The second order BVP with nonlocal condition

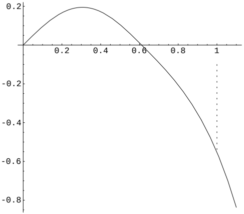
$$-x'' = \mu x^+ - \lambda x^-, \tag{2.1}$$

$$x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\},$$

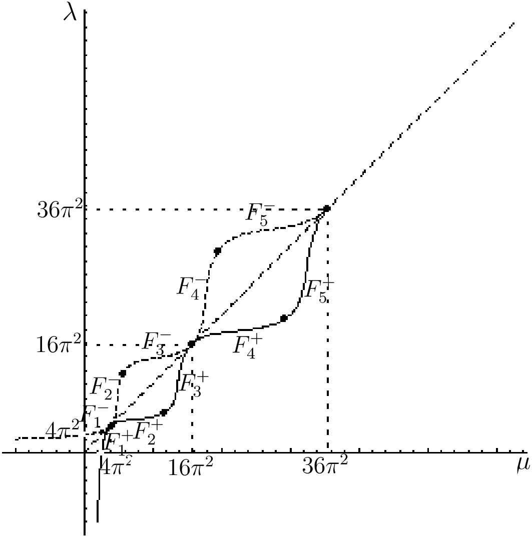
$$x(0) = 0, \quad \int_0^1 x(s)ds = 0. \tag{2.2}$$



2.1. fig.



2.2. fig.



2.3. fig. The Fučík spectrum for the problem (2.1), (2.2).

Theorem: *The Fučík spectrum for the problem* (2.1), (2.2) *consists of the branches* F_i^+ *un* F_i^- *given by*

$$F_1^+ = F_{1+}^+ \cap F_{1-}^+$$

$$F_{1+}^+ = \left\{ (\mu,\lambda) \Big| \; \frac{2\sqrt{\lambda}}{\sqrt{\mu}} - \frac{\sqrt{\mu}}{\sqrt{\lambda}} - \frac{\sqrt{\mu}\cos(\sqrt{\lambda} - \frac{\sqrt{\lambda}\pi}{\sqrt{\mu}} + \pi)}{\sqrt{\lambda}} = 0, \right.$$

$$\left. \mu > 0,\ \lambda > 0,\ \frac{\pi}{\sqrt{\mu}} \leq 1,\ \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} > 1 \right\};$$

$$F_{1-}^+ = \left\{ (\mu,\lambda) \Big| \; -2\lambda + \mu - \mu\cosh\left(\sqrt{-\lambda} - \sqrt{-\frac{\lambda}{\mu}}\pi\right) = 0, \right.$$

$$\left. \mu > 0,\ \lambda < 0 \right\};$$

$$F_1^- = F_{1+}^- \cap F_{1-}^-$$

$$F_{1+}^- = \Bigg\{(\mu,\lambda)\Big|\ \frac{2\sqrt{\mu}}{\sqrt{\lambda}} - \frac{\sqrt{\lambda}}{\sqrt{\mu}} - \frac{\sqrt{\lambda}\cos(\sqrt{\mu} - \frac{\sqrt{\mu}\pi}{\sqrt{\lambda}} + \pi)}{\sqrt{\mu}} = 0,$$

$$\mu > 0,\ \lambda > 0,\ \frac{\pi}{\sqrt{\lambda}} \leq 1,\ \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} > 1\Bigg\};$$

$$F_{1-}^- = \Big\{(\mu,\lambda)\Big|\ -2\mu + \lambda - \lambda\cosh\left(\sqrt{-\mu} - \sqrt{-\frac{\mu}{\lambda}}\pi\right) = 0,$$

$$\mu < 0, \lambda > 0\};$$

$$F_{2i}^{+} = \left\{ (\mu,\lambda) \Big| \; \frac{(2i+1)\sqrt{\lambda}}{\sqrt{\mu}} - \frac{2i\sqrt{\mu}}{\sqrt{\lambda}} - \frac{\sqrt{\lambda}\cos(\sqrt{\mu} - \frac{\sqrt{\mu}\pi i}{\sqrt{\lambda}} + \pi i)}{\sqrt{\mu}} = 0, \right.$$

$$\left. \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} \le 1, \; \frac{(i+1)\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} > 1 \right\};$$

$$F_{2i+1}^{+} = \left\{ (\mu,\lambda) \Big| \; \frac{2i\sqrt{\lambda}}{\sqrt{\mu}} - \frac{(2i-1)\sqrt{\mu}}{\sqrt{\lambda}} - \frac{\sqrt{\mu}\cos(\sqrt{\lambda} - \frac{\sqrt{\lambda}\pi i}{\sqrt{\mu}} + \pi i)}{\sqrt{\lambda}} = 0, \right.$$

$$\left. \frac{i\pi}{\sqrt{\mu}} + \frac{(i-1)\pi}{\sqrt{\lambda}} \le 1, \; \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} > 1 \right\};$$

$$F_{2i}^{-} = \left\{ (\mu,\lambda) \Big| \; \frac{(2i+1)\sqrt{\mu}}{\sqrt{\lambda}} - \frac{2i\sqrt{\lambda}}{\sqrt{\mu}} - \frac{\sqrt{\mu}\cos(\sqrt{\lambda} - \frac{\sqrt{\lambda}\pi i}{\sqrt{\mu}} + \pi i)}{\sqrt{\lambda}} = 0, \right.$$
$$\left. \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} \leq 1, \; \frac{i\pi}{\sqrt{\mu}} + \frac{(i+1)\pi}{\sqrt{\lambda}} > 1. \right\},$$

$$F_{2i+1}^{-} = \left\{ (\mu,\lambda) \Big| \; \frac{2i\sqrt{\mu}}{\sqrt{\lambda}} - \frac{(2i-1)\sqrt{\lambda}}{\sqrt{\mu}} - \frac{\sqrt{\lambda}\cos(\sqrt{\mu} - \frac{\sqrt{\mu}\pi i}{\sqrt{\lambda}} + \pi i)}{\sqrt{\mu}} = 0, \right.$$
$$\left. \frac{(i-1)\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} \leq 1, \; \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} > 1. \right\}$$

3. Connection between the spectra

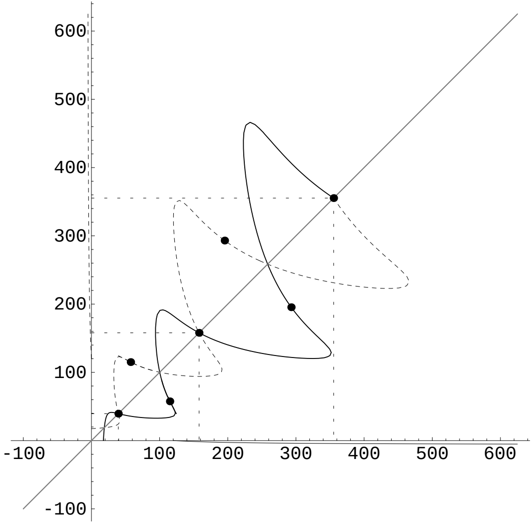
$$-x'' = \mu x^+ - \lambda x^-, \tag{3.1}$$

$$x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\},$$

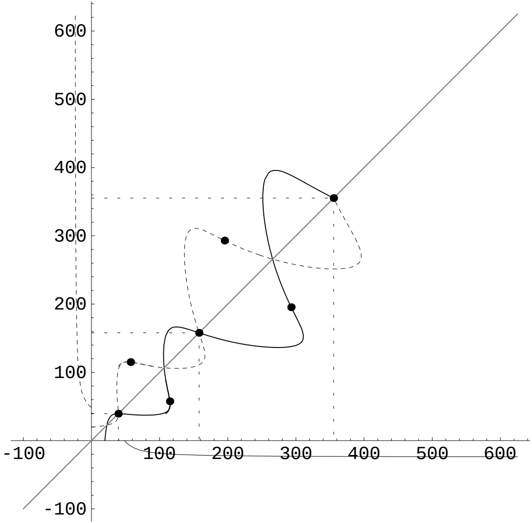
$$x(0) = 0, \tag{3.2}$$

$$(1-\alpha)x(1) + \alpha \int_0^1 x(s)ds = 0, \quad \alpha \in [0,1]. \tag{3.3}$$

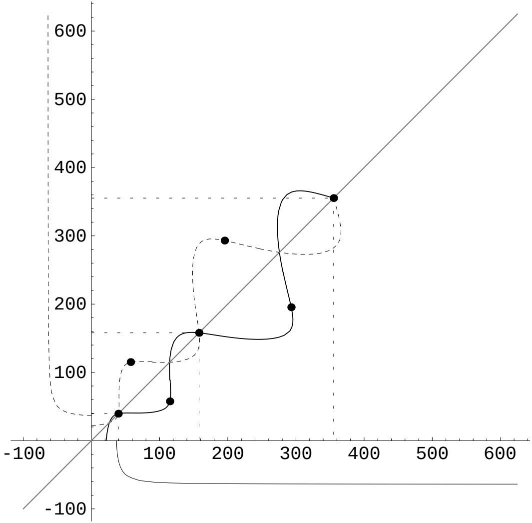
This problem connects two problems: the problem with Dirichlet conditions (in the case when $\alpha = 0$) and the problem with nonlocal one (in the case when $\alpha = 1$).



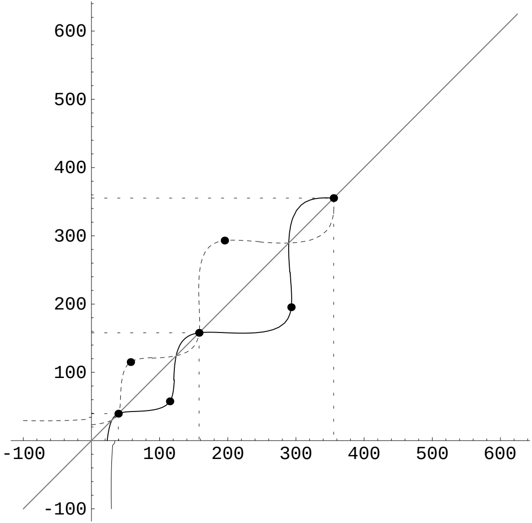
3.1. fig. The Fučík spectrum for the problem (3.1) - (3.3) in the case of $\alpha = \frac{3}{4}$.



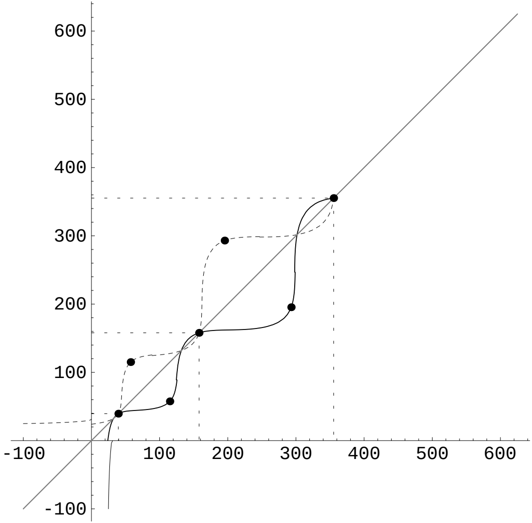
3.2. fig. The Fučík spectrum for the problem (3.1) - (3.3) in the case of $\alpha = \frac{5}{6}$.



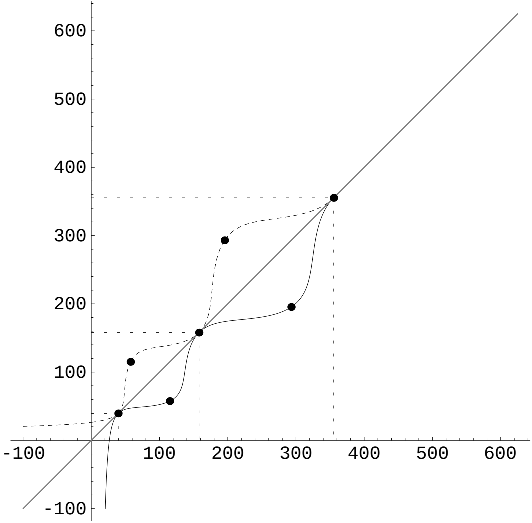
3.3. fig. The Fučík spectrum for the problem (3.1) - (3.3) in the case of $\alpha = \frac{8}{9}$.



3.4. fig. The Fučík spectrum for the problem (3.1) - (3.3) in the case of $\alpha = \frac{13}{14}$.



3.5. fig. The Fučík spectrum for the problem (3.1) - (3.3) in the case of $\alpha = \frac{19}{20}$.



3.6. fig. The Fučík spectrum for the problem (3.1) - (3.3) in the case of $\alpha = 1$.

The interesting feature of this transformation is that all positive ($x'(0) = 1$) branches which are located in both regions over the bisectrix and below it for $\alpha = 0$, for $\alpha = 1$ (which correspond the fully integral condition) are located in the region below the bisectrix.

The negative ($x'(0) = -1$) branches for $\alpha = 1$ are located over the bisectrix.

4. References

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