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NONLINEAR EIGENVALUE PROBLEMS

Workshop Diff. Equations
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Contents

1 Introduction

Nonlinear eigenvalue problems of the type:

$$x'' = -\lambda f(x), \quad x(0) = 0,\ x(1) = 0, \tag{1}$$

(most recently P. Korman et al. [2], looking for multiple positive solutions).

We consider a two-parameter nonlinear problem:

$$x'' = -\lambda f(x^+) + \mu g(x^-), \quad x(0) = 0,\ x(1) = 0, \tag{2}$$

where f, g are positive valued Lipschitz functions such that $f(0) = g(0) = 0$.

The same equation in alternative form

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(-x), & \text{if} \quad x < 0. \end{cases} \tag{3}$$

If $f = g = x$ one has the Fučik equation:

$$x'' = -\lambda x^+ + \mu x^-, \quad x(0) = 0,\ x(1) = 0, \tag{4}$$

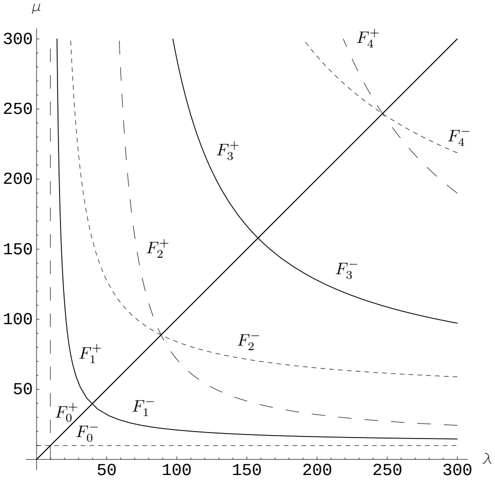

Fig. 1. The classical (λ, μ) Fučik spectrum.

$$\text{no zeros in } (0,1), x'(0) > 0: \ F_0^+ = \Big\{(\lambda,\mu): \ \lambda = \pi^2, \ \mu \geq 0\Big\},$$

$$\text{no zeros in } (0,1), x'(0) < 0: \ F_0^- = \Big\{(\lambda,\mu): \lambda \geq 0, \ \mu = \pi^2\Big\},$$

$$\text{one zero in } (0,1), x'(0) > 0: F_1^+ = \left\{(\lambda;\mu): \quad \frac{\pi}{\sqrt{\lambda}} + \frac{\pi}{\sqrt{\mu}} = 1\right\},$$

$$\text{one zero in } (0,1), x'(0) < 0: F_1^- = \left\{(\lambda;\mu): \quad \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} = 1\right\},$$

$$\text{two zeros in } (0,1), x'(0) > 0: F_2^+ = \left\{(\lambda;\mu): \quad 2\frac{\pi}{\sqrt{\lambda}} + \frac{1}{\sqrt{\mu}} = 1\right\},$$

$$\text{two zeros in } (0,1), x'(0) < 0: F_2^- = \left\{(\lambda;\mu): \quad \frac{\pi}{\sqrt{\lambda}} + 2\frac{1}{\sqrt{\mu}} = 1\right\}.$$

and so on.

The sample problem:

$$x'' = -\lambda(x^+)^{2\alpha+1} + \mu(x^-)^{2\beta+1}, \quad x(0) = 0,\ x(1) = 0, \qquad (5)$$

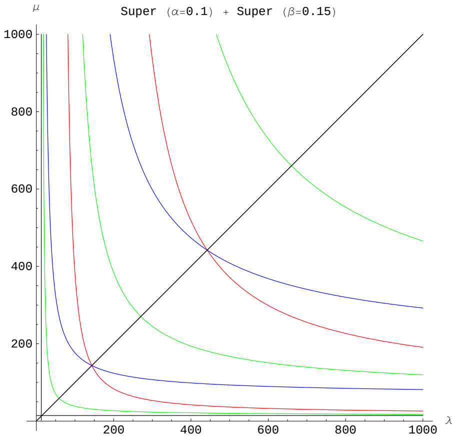

Fig. 2.

2 One parameter problems

The nonlinear one-parameter eigenvalue problem:

$$x'' = -\lambda x^3, \quad x(0) = 0,\ x(1) = 0. \tag{6}$$

Looking for solutions without zeros in $(0, 1)$:
for any $\lambda > 0$ there exists a positive valued in $(0, 1)$ solution $x(t)$. The value $\max_{[0,1]} x(t) := \|x\|$ and λ relate as

$$\|x\| \cdot \lambda = 2\sqrt{2} \cdot \int_0^1 \frac{dx}{\sqrt{1 - x^4}}.$$

Therefore $F_0^+ = \Big\{(\lambda, \mu) : \ 0 < \lambda < +\infty,\ \mu \geq 0\Big\}$ for the problem (5).

In order to make the problem reasonable one should impose additional conditions. Let us require that

$$|x'(0)| = 1.$$

3 Two-parameter problems

3.1 Problem

$$x'' = -\lambda f(x^+) + \mu g(x^-), \quad x(0) = 0,\ x(1) = 0, \quad |x'(0)| = 1. \tag{7}$$

3.2 Assumptions

(A1) the first zero $t_1(\alpha)$ of a solution to the Cauchy problem

$$u'' = -f(u), \quad u(0) = 0,\ u'(0) = \alpha \tag{8}$$

exists for any $\alpha > 0$.

(A2) the first zero $\tau_1(\beta)$ of a solution to the Cauchy problem

$$v'' = g(-v), \quad v(0) = 0,\ v'(0) = -\beta \tag{9}$$

exists for any $\beta > 0$.

3.3 Formulas for the nonlinear spectrum

In presence of the conditions **(A1)** and **(A2)** the first branches of the spectrum are:

$$F_0^+ = \Big\{(\lambda,\mu):\ \lambda \text{ is a solution of } \frac{1}{\sqrt{\lambda}}\,t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1, \quad \mu \geq 0\Big\},$$

$$F_0^- = \Big\{(\lambda,\mu): \lambda \geq 0,\ \mu \text{ is a solution of } \frac{1}{\sqrt{\mu}}\,\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\},$$

$$F_1^+ = \Bigg\{(\lambda;\mu): \quad \frac{1}{\sqrt{\lambda}}\,t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{\sqrt{\mu}}\,\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Bigg\},$$

$$F_1^- = \Bigg\{(\lambda;\mu): \quad \frac{1}{\sqrt{\mu}}\,\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + \frac{1}{\sqrt{\lambda}}\,t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\Bigg\},$$

$$F_2^+ = \Bigg\{(\lambda;\mu): \quad 2\frac{1}{\sqrt{\lambda}}\,t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{\sqrt{\mu}}\,\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Bigg\},$$

$$F_{2i}^- = \Bigg\{(\lambda;\mu): \quad \frac{1}{\sqrt{\mu}}\,\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + 2\frac{1}{\sqrt{\lambda}}\,t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\Bigg\}.$$

3.4 Samples of time maps

Let equation be

$$x'' = -(r+1)x^r, \quad r > 0. \tag{10}$$

Then

$$t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 2A\,\lambda^{\frac{r-1}{2(r+1)}}, \quad A = \int_0^1 \frac{1}{\sqrt{1-\xi^{r+1}}}\,d\xi, \tag{11}$$

t_1 is decreasing in λ for $r \in (0,1)$,
t_1 is constant for $r = 1$,
t_1 is increasing in λ for $r > 1$.

The function

$$u(\lambda) = \frac{1}{\sqrt{\lambda}} t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 2A\,\lambda^{-\frac{1}{r+1}}$$

is decreasing for $r > 0$.

If $f(x)$ is a piece-wise linear function like in the picture then exact formulas are known for computation of t_1

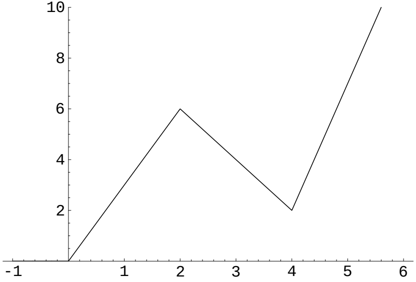

Fig. 3. Function $f(x)$.

Let $t_1(\alpha)$ be the first positive zero of a solution of the IVP

$$x'' = -f(x), \quad x(0) = 0, \quad x'(0) = \alpha > 0. \tag{12}$$

Denote $F(x) = \int_0^x f(s)\,ds$. Direct calculations show that

1. if $0 \le \alpha \le \sqrt{2F(a_1)}$, then $t_1(\alpha) = \pi\sqrt{\frac{a_1}{b_1}}$;

2. if $\sqrt{2F(a_1)} \le \alpha \le \sqrt{2F(a_2)}$, then

$$t_1(\alpha) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} + \\ + \sqrt{\frac{a_2 - a_1}{b_1 - b_2}} \ln \frac{D_2(\alpha)}{\left(-2b_1 + 2\sqrt{\frac{b_1 - b_2}{a_2 - a_1}}\sqrt{\alpha^2 - a_1 b_1}\right)^2},$$

3. if $\alpha \ge \sqrt{2F_2(a_2)}$, then

$$t_1(\alpha) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} + \sqrt{\frac{a_3 - a_2}{b_3 - b_2}} \left[\pi - 2 \arcsin \frac{2b_2}{\sqrt{D_3(\alpha)}}\right] + \\ + 2\sqrt{\frac{a_2 - a_1}{b_1 - b_2}} \ln \left| \frac{-b_2 + \sqrt{\frac{b_1 - b_2}{a_2 - a_1}}\sqrt{\alpha^2 - a_1 b_1 - (a_2 - a_1)(b_1 + b_2)}}{-b_1 + \sqrt{\frac{b_1 - b_2}{a_2 - a_1}}\sqrt{\alpha^2 - a_1 b_1}} \right|,$$

where

$$D_2(\alpha) = 4\,\frac{b_1 - b_2}{a_1 - a_2}\alpha^2 + 4b_1\,\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}, \quad D_3(\alpha) = 4\,\frac{b_2 - b_3}{a_2 - a_3}\alpha^2 +$$

$$+\,4\,\frac{-a_2 b_1 b_2 + a_1 b_2^2 + a_3 b_2^2 + a_2 b_1 b_3 - a_1 b_2 b_3 + a_2 b_2 b_3}{a_2 - a_3}.$$

The first zero function is asymptotically linear:

$$\lim_{\alpha \to +\infty} t_1(\alpha) = \sqrt{\frac{a_3 - a_2}{b_3 - b_2}}\,\pi.$$

4 Some properties of spectra

Let

$$u(\lambda) := \frac{1}{\sqrt{\lambda}} t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) \quad v(\mu) := \frac{1}{\sqrt{\mu}} \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big). \tag{13}$$

Spectrum is a union of the roots of equations

$$\begin{array}{llllll} u(\lambda) & + & v(\mu) & = 1, & & F_1^{\pm} \\ 2u(\lambda) & + & v(\mu) & = 1, & & F_2^{+} \\ u(\lambda) & + & 2v(\mu) & = 1, & & F_2^{-} \\ 2u(\lambda) & + & 2v(\mu) & = 1, & & F_3^{\pm} \\ 3u(\lambda) & + & 2v(\mu) & = 1, & & F_4^{+} \\ 2u(\lambda) & + & 3v(\mu) & = 1, & & F_4^{-} \\ \dots & & & & & \end{array} \tag{14}$$

The coefficients at $u(\lambda)$ and $v(\mu)$ give the numbers of “positive” and “negative” humps of the respective eigenfunctions.

4.1 Monotone $u(\lambda), v(\mu)$

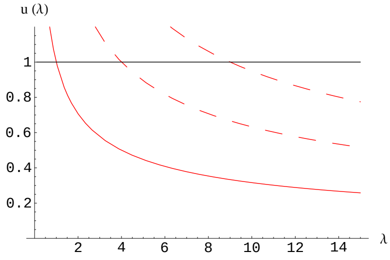

Fig. 4. The graphs of $u(\lambda), 2u(\lambda), 3u(\lambda)$.

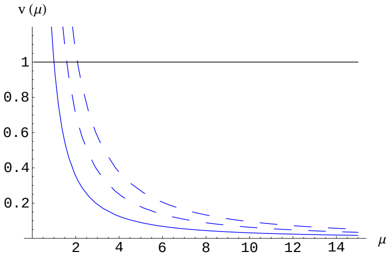

Fig. 5. The graphs of $v(\mu), 2v(\mu), 3v(\mu)$.

Let λ_1, λ_2, λ_3 be points of intersection of $u(\lambda), 2u(\lambda), 3u(\lambda)$ ("red" curves) and the horizontal line $u = 1$. Respectively μ_1, μ_2, μ_3 for "blue" curves.

The branches $F_1^{\pm}$ coincide and look like hyperbola with the vertical asymptote at $\lambda = \lambda_1$ and horizontal asymptote at $\mu = \mu_1$.

The branch F_2^{+} has the vertical asymptote at $\lambda = \lambda_2$ and horizontal asymptote at $\mu = \mu_1$.

The branch F_2^{-} has the vertical asymptote at $\lambda = \lambda_1$ and horizontal asymptote at $\mu = \mu_2$. The branches F_2^{+} and F_2^{-} need not to cross at the bisectrix unless $g \equiv f(-x)$.

The branches $F_3^{\pm}$ coincide and have the vertical asymptote at $\lambda = \lambda_2$ and horizontal asymptote at $\mu = \mu_2$.

The branch F_4^{+} has the vertical asymptote at $\lambda = \lambda_3$ and horizontal asymptote at $\mu = \mu_2$.

The branch F_4^{-} has the vertical asymptote at $\lambda = \lambda_2$ and horizontal asymptote at $\mu = \mu_3$. The branches F_4^{+} and F_4^{-} need not to cross at the bisectrix.

4.2 Non-monotone $u(\lambda), v(\mu)$

It is possible that the functions $u(\lambda) = \frac{1}{\sqrt{\lambda}} t_1\left(\frac{1}{\sqrt{\lambda}}\right)$ and $v(\mu) := \frac{1}{\sqrt{\mu}} \tau_1\left(\frac{1}{\sqrt{\mu}}\right)$ are non-monotone.

Then spectra may differ essentially from those in the monotone case.

Remark. Suppose that $iu(\lambda)$ and $iv(\mu)$ are monotonically decreasing starting from some $\lambda_\bigstar$ and $\mu_\bigstar$, $iu(\lambda_\bigstar) = 1$, $iv(\mu_\bigstar) = 1$, where i is some positive integer. Then branches $F^{\pm}_{2i-1}$ and higher $(F^{\pm}_n,\ n > 2i-1)$ behave like those in the monotone case.

4.3 Nonmonotonicity over u,v=1

Consider equation $x'' = -\lambda f(x) + \mu f(-x)$, where $f(x)$ is a piece-wise linear function depicted in Fig. 3., parameters of the piece-wise linear function $f(x)$ are

$$a_1 = 0.1, \qquad a_2 = 0.3, \qquad a_3 = 0.31,$$
$$b_1 = 9, \qquad b_2 = 0.5, \qquad b_3 = 150.$$

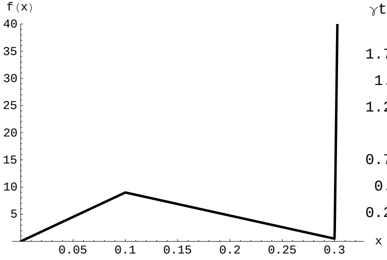

Fig. 6. The graph of $y = f(x)$.

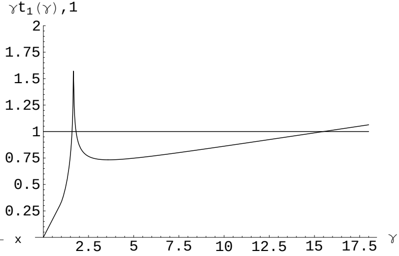

Fig. 7. The graphs of $y = \gamma t_1(\gamma)$ and $y = 1$.

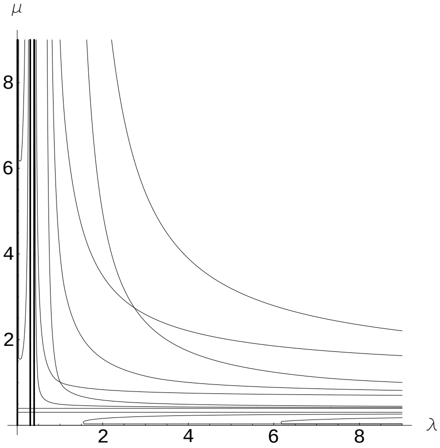

Fig. 8. The branch F_0^+ in the (λ, μ)-plane.

The branch F_0^+ consists of three vertical lines which corresponds to three solutions of the equation $\frac{1}{\sqrt{\lambda}} t_1(\frac{1}{\sqrt{\lambda}}) = 1$.

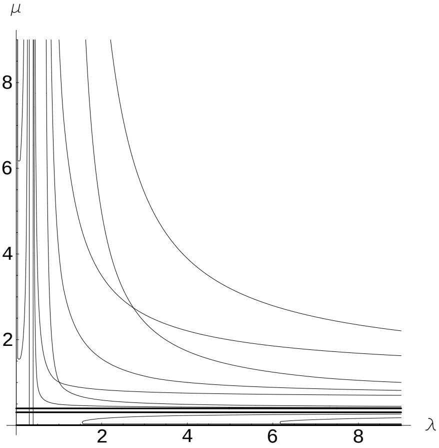

Fig. 9. The branch F_0^- in the (λ, μ)-plane.

The branch F_0^- consists of horizontal lines which correspond to solutions of the equation $\frac{1}{\sqrt{\mu}}\tau_1(\frac{1}{\sqrt{\mu}}) = 1$.

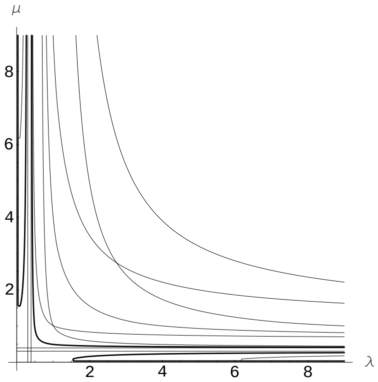

Fig. 10. The branches $F_1^+ = F_1^-$ in the (λ, μ)-plane.Properties of the branches $F_1^\pm$ depend on solutions of the equation $u(\lambda) + v(\mu) = 1$. A set of solutions of this equation consists of exactly three components due to non-monotonicity of the functions a)

4.4 Nonmonotonicity beneath u,v=1

Consider equation $x'' = -\lambda f(x) + \mu f(-x)$, where $f(x)$ is a piece-wise linear function depicted in Fig. 3., parameters of the piece-wise linear function $f(x)$ are

$$a_1 = 0.1, \qquad a_2 = 0.2, \qquad a_3 = 0.22,$$
$$b_1 = 0.2, \qquad b_2 = 0.1, \qquad b_3 = 120.$$

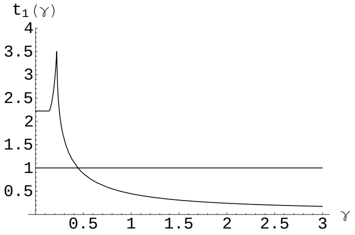

Fig. 11. Funkcija $t_1(\gamma)$.

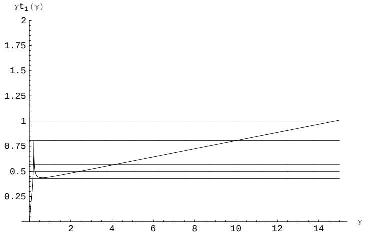

Fig. 12. Funkcija $u(\gamma) = \gamma t_1(\gamma)$.

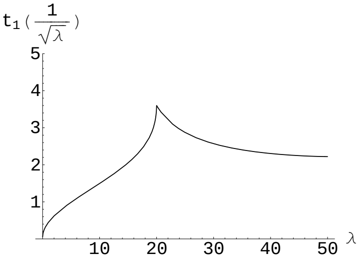

Fig. 13. Funkcija $t_1\left(\frac{1}{\sqrt{\lambda}}\right)$.

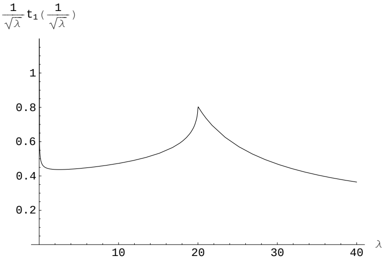

Fig. 14. Funkcija $\frac{1}{\sqrt{\lambda}}t_1\left(\frac{1}{\sqrt{\lambda}}\right)$.

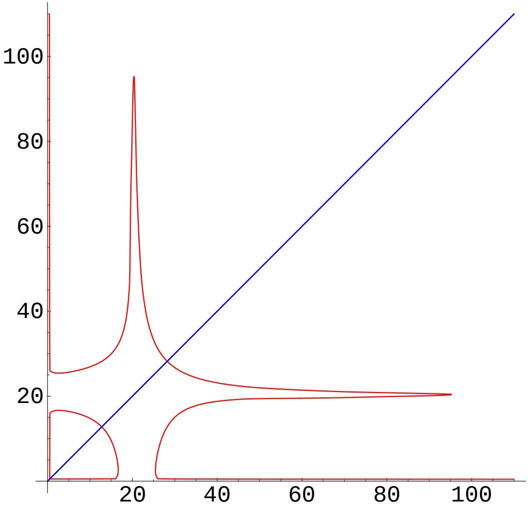

Fig. 15. The branch $F_1^+ = F_1^-$ Case $max + min > 1$.

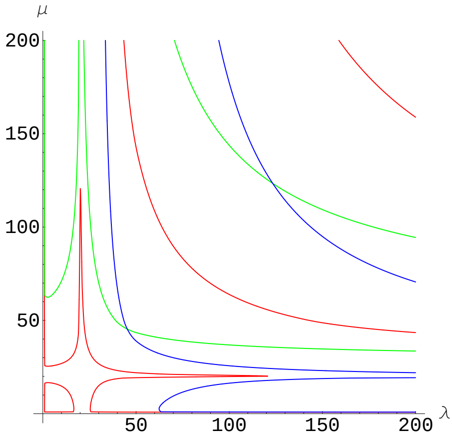

Fig. 16.

- Red $F_1^+ = F_1^-$, $F_3^+ = F_3^-$, $F_5^+ = F_5^-$, the branch $F_1^+ = F_1^-$ consists of 2 components.
- Blue F_2^+, F_4^+, the branch F_2^+ consists of 2 components.

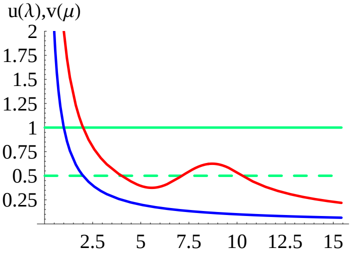

Fig. 17. $u(\lambda)$-red, $v(\mu)$-blue.

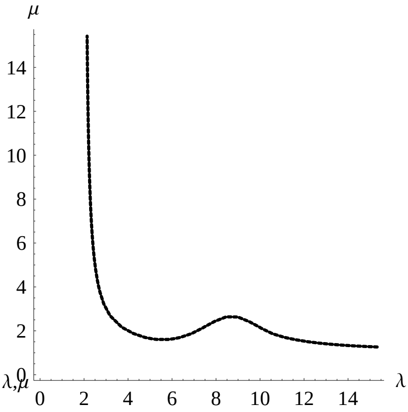

Fig. 18. The branch F_1^+.

$max + min = 1$

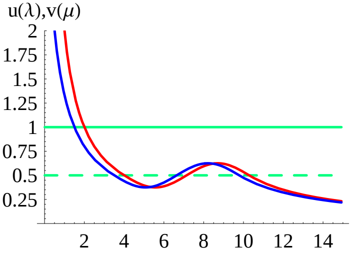

Fig. 19.$u(\lambda)$-red, $v(\mu)$-blue.

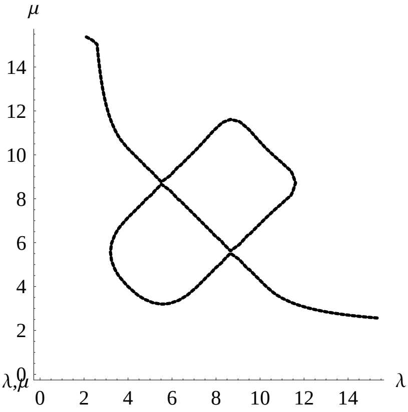

Fig. 20. The branch F_1^+.

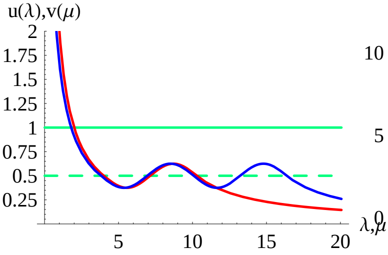

Fig. 21. $u(\lambda)$-red, $v(\mu)$-blue.

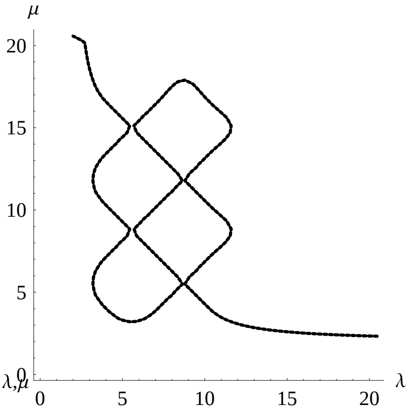

Fig. 22. The branch F_1^+.

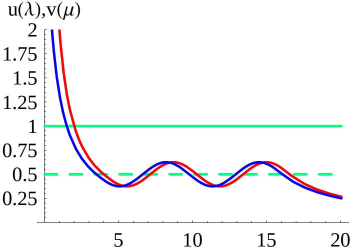

Fig. 23. $u(\lambda)$-red, $v(\mu)$-blue.

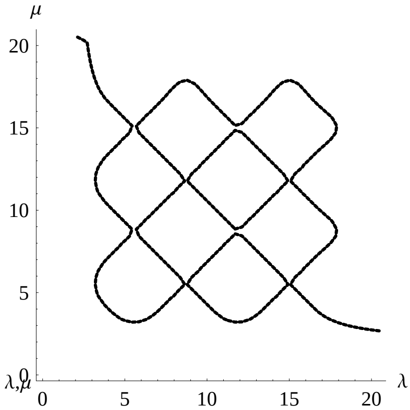

Fig. 24. The branch F_1^+.

References

[1] A. Gritsans and F. Sadyrbaev. On nonlinear eigenvalue problems. *Proceedings LU MII "Mathematics. Differential Equations", vol. 6 (2006), 76 - 86.*

[2] P. Korman. Global solution branches and exact multiplicity of solutions for two-point boundary value problems. *In: Handbook of Diff. Equations, ODE, Vol. III. Elsevier - North Holland, Amsterdam, 2006, 548 - 606.* 3

[3] F. Sadyrbaev and A. Gritsans. Nonlinear Spectra for Parameter Dependent Ordinary Differential Equations. *Nonlinear Analysis: Modelling and Control, 2007, 12, No.2, 253 - 267.*

[4] R. Schaaf. Global solution branches of two-point boundary value problems. *Lect. Notes Math. 1458. Springer-Verlag, Berlin - Heidelberg - New York, 1990.*