On the initial value problem for functional differential equations

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Cauchy problem

Question

Find a solution of an FDE possessing the prescribed properties

In the case of an ODE, the Cauchy problem is the most natural choice. For FDE studied insufficiently even in the case of ***a scalar first order linear equation***

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Issues

- The operator of translation along the trajectories is undefined
- The notion of a local solution does not make sense
- The tools of the ODE theory do not work (e. g., the classical existence and uniqueness theorems)
- The non-local character of an equation complicates the application of approximate methods

General theory of FDE

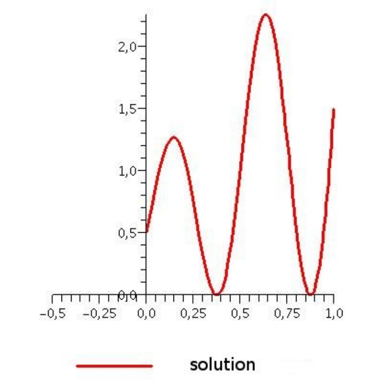
- *A. D. Myshkis*, General Theory of Delay Differential Equations// *Usp. Mat. Nauk*, 1949, No. 5, 99–141.
- *A. D. Myshkis*, Linear Differential Equations with Retarded Argument [in Russian], Moscow, 1951
- *S. B. Norkin*, Second Order Differential Equations with Retarded Argument [in Russian], Nauka, Moscow, 1965.
- *R. Bellman and K. Cooke*, Differential-Difference Equations, Academic Press, NY, 1963.
- *Yu. A. Mitropolsky, A. M. Samoilenko, and D. I. Martynyuk*, Systems of Evolution Equations with Periodic and Quasiperiodic Coefficients, Kluwer, Dordrecht, 1993.
- *V. B. Kolmanovsky and A. D. Myshkis*, Introduction to the Theory of Functional Differential Equations, Kluwer, Dordrecht, 1999.
- *J. K. Hale*, Theory of Functional Differential Equations, Springer-Verlag, NY, 1977.

More recent works

- *N. V. Azbelev, V. P. Maximov, and L. F. Rakhmatullina*, Introduction to the Theory of Functional Differential Equations (Moscow, 1991)
- *N. V. Azbelev, V. P. Maximov, and L. F. Rakhmatullina*, Methods of the Theory of Functional Differential Equations (Izhevsk, 2000)
- *R. Hakl, A. Lomtatidze, and J. Šremr,* Some Boundary Value Problems for First Order Scalar Functional Differential Equations (Brno, 2002)
- *I. Kiguradze and B. Půža*, Boundary Value Problems for Systems of Linear Functional Differential Equations (Brno, 2003)
- *Z. Sokhadze*, Cauchy Problem for Singular Functional Differential Equations (Kutaisi, 2005)

Cauchy problem setting

The "initial functions" and "continuous junction"



$$x'(t) = \frac{x(t)}{t+\frac{1}{2}} + 4\pi\left(t+\frac{1}{2}\right)\cos\left(4\pi t\right)$$

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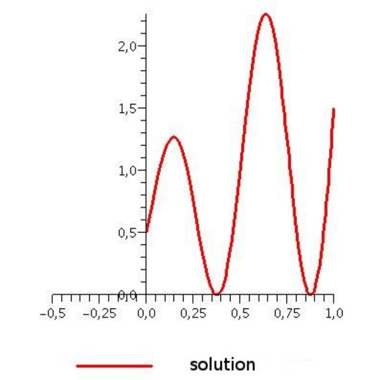

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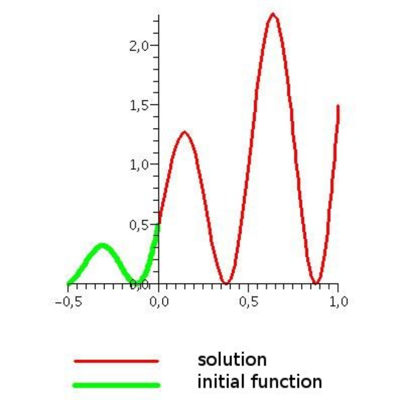

$$x'(t) = \frac{1-t}{t+\frac{1}{2}}x(t) + x\left(t-\tfrac{1}{2}\right) + 4\pi t\left(t+\tfrac{1}{2}\right)\cos\left(4\pi t\right)$$

Formulation

Formulation involving initial functions

Find a $u : [a,b] \longrightarrow \mathbb{R}$ such that

$$u'(t) = r(t)\, u(\eta(t)) + g(t), \quad t \in [a,b], \tag{1}$$

$$u(s) = \psi(s) \quad \text{for } s \notin [a,b], \tag{2}$$

where $r : [a,b] \to \mathbb{R}$, $\eta : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ and $\psi : \mathbb{R} \setminus [a,b] \to \mathbb{R}$ are given functions.

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Modern formulation ($\leftsquigarrow$ Perm Seminar on FDE; 1980s)

$$u'(t) = h(t)\,u(\omega(t)) + q(t), \quad t \in [a,b], \tag{3}$$

where $h(t) := \chi_\eta(t) r(t)$, $\omega(t) := \eta(t)\chi_\eta(t) + a\,(1 - \chi_\eta(t))$,

$$q(t) := \begin{cases} g(t) & \text{if } \eta(t) \in [a,b], \\ g(t) + r(t)\psi(\eta(t)) & \text{if } \eta(t) \notin [a,b], \end{cases}$$

and $\chi_\eta(t) = 1$ if $\eta(t) \in [a,b]$, $\chi_\eta(t) = 0$ if $\eta(t) \notin [a,b]$.

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The “tailless” form of (1), (2)

$$u'(t) = h(t)\, u(\omega(t)) + q(t), \quad t \in [a, b], \tag{3}$$

where $\omega : [a, b] \to [a, b]$ is measurable and $q \in L_1([a, b], \mathbb{R})$.

A scalar differential equation with an argument deviation

Find an absolutely continuous $u : [a, b] \to \mathbb{R}$ such that

$$u'(t) = h(t)\, u(\omega(t)) + q(t), \quad t \in [a, b], \tag{4}$$

where $\omega : [a, b] \to [a, b]$ is measurable and $q \in L_1([a, b], \mathbb{R})$.

- For the new equation (4) to make sense, it suffices to assume the integrability of r and g, the measurability of η in (1), and the continuity of ψ in (2).

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- The function ω in (3) transforms $[a, b]$ into itself, and thus the additional conditions of type (2) are redundant.

A scalar differential equation with an argument deviation

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- For the new equation (4) to make sense, it suffices to assume the integrability of r and g, the measurability of η in (1), and the continuity of ψ in (2).
- The function ω in (3) transforms $[a,b]$ into itself, and thus the additional conditions of type (2) are redundant.
- The Cauchy problem for Eq. (3) is posed at a ***single*** point $\tau \in [a,b]$:

$$u(\tau) = c.$$

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- Few sharp and efficient solvability conditions are known
- The majority of the results available concerns first order scalar equations

The Cauchy problem for a system of n linear first order FDEs with n^2 argument transformations

Find absolutely continuous $u_k : [a,b] \to \mathbb{R}$, $k = 1, 2, \ldots, n$, such that

$$u_k'(t) = \sum_{j=1}^{n} r_{kj}(t)\, u_j(\omega_{kj}(t)) + q_k(t), \quad t \in [a,b],\ k = 1, 2, \ldots, n, \tag{5}$$

$$u_k(\tau) = c_k, \qquad k = 1, 2, \ldots, n, \tag{6}$$

where $r_{kj} : [a,b] \to \mathbb{R}$, $k, j = 1, 2, \ldots, n$, and $q_k : [a,b] \to \mathbb{R}$, $k = 1, 2, \ldots, n$, are Lebesgue integrable functions, and ω_{kj}, $k, j = 1, 2, \ldots, n$, are arbitrary measurable functions mapping $[a,b]$ into itself.

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The terms q_k, $k = 1, 2, \dots, n$, contain everything that does not explicitly concern the operator of the equation (in particular, the initial functions).

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Important difference from the ODE theory

Without additional assumptions, the Cauchy problem (5), (6) ***may not have a unique solution*** even in the class of scalar linear equations with constant coefficients.

Example

The simplest scalar equation

$$u'(t) = \frac{u(b)}{b-a} + q(t), \quad t \in [a,b], \tag{7}$$

where $q : [a,b] \to \mathbb{R}$ is such that

$$\int_a^b q(s)\,ds \neq 0,$$

has no solutions u satisfying the condition

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"Passing to the limit" leads one to the uniquely solvable problem (8) for the equation $u' = q$ with $q \in L_{1;\,\mathrm{loc}}([a,+\infty))$? No "passing to the limit" is possible.

Theorem 1 (A. R., 2005)

Let there exist some constants $\{\gamma_k \mid k = 1, 2, \ldots, n\} \subset (0, +\infty)$ *and* $\alpha \in [1, +\infty)$ *for which at least one of the following conditions be satisfied:*

$$\max_{k=1,2,\ldots,n} \operatorname*{vrai\,max}_{t\in[a,b]\setminus\{\tau\}} \frac{1}{\gamma_k \left|t-\tau\right|^{\alpha-1}} \sum_{j=1}^{n} \gamma_j \left|r_{kj}(t)\right| \left|\omega_{kj}(t)-\tau\right|^{\alpha} < \alpha, \qquad (9)$$

$$\max_{k=1,2,\ldots,n} \sup_{t\in[a,b]\setminus\{\tau\}} \frac{\operatorname{sign}(t-\tau)}{\gamma_k \left|t-\tau\right|^{\alpha}} \sum_{j=1}^{n} \gamma_j \int_{\tau}^{t} \left|r_{kj}(s)\right| \left|\omega_{kj}(s)-\tau\right|^{\alpha} ds < 1. \qquad (10)$$

Then the initial value problem (5), (6) *is uniquely solvable for arbitrary constants* $\{c_k \mid k = 1, 2, \ldots, n\} \subset \mathbb{R}$ *and integrable functions* $q_k : [a, b] \to \mathbb{R}$, $k = 1, 2, \ldots, n$.

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Optimality of conditions of the theorem

None of the non-strict inequalities assumed in Theorem 1 ***cannot*** be replaced by the corresponding non-strict inequality because after such a replacement the corresponding assertions are lost. For example, the condition

$$\max_{k=1,2,\dots,n} \operatorname*{vrai\,max}_{t\in[a,b]\setminus\{\tau\}} \frac{1}{\gamma_k \left|t-\tau\right|^{\alpha-1}} \sum_{j=1}^{n} \gamma_j \left|r_{kj}(t)\right| \left|\omega_{kj}(t)-\tau\right|^{\alpha} \leq \alpha, \qquad (11)$$

which is a weakened version of condition (9), does not guarantee the unique solvability of problem (5), (6) for arbitrary forcing terms.

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The theorem quoted is not an exceptional case:

The conditions obtained in this way, as a rule, are optimal.

Theorem 2 (A. R., 2005)

Let there exist some constants $\{\gamma_k \mid k = 1, 2, \dots, n\} \subset (0, +\infty)$ *and* $\alpha \in [1, +\infty)$ *such that*

$$\operatorname*{vrai\,max}_{t\in[a,b]\setminus\{\tau\}} \sum_{l=1}^{n} \frac{|r_{kl}(t)|}{\gamma_k\,|t-\tau|^{\alpha-1}} \sum_{j=1}^{n} \gamma_j \left| \int_{\tau}^{\omega_{kl}(t)} |r_{lj}(s)||\omega_{lj}(s)-\tau|^{\alpha} ds \right| < \alpha.$$

for all $k = 1, 2, \dots, n$. *Then the initial value problem*

$$u_k'(t) = \sum_{j=1}^{n} r_{kj}(t)\, u_j(\omega_{kj}(t)) + q_k(t), \quad t \in [a, b], \tag{12}$$

$$u_k(\tau) = c_k, \qquad k = 1, 2, \dots, n, \tag{13}$$

is uniquely solvable for arbitrary $\{c_k \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ *and* $\{q_k, k = 1, 2, \dots, n\} \subset L_1([a, b], \mathbb{R})$.

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The strict inequality in the condition is essential:

Theorem

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$$u_k(\tau) = c_k, \qquad k = 1, 2, \ldots, n, \tag{15}$$

is uniquely solvable for arbitrary $\{c_k \mid k = 1, 2, \ldots, n\} \subset \mathbb{R}$ *and* $\{q_k, k = 1, 2, \ldots, n\} \subset L_1([a, b], \mathbb{R})$.

Monotone dependence of a solution on perturbations

Under additional conditions, one can claim not only the unique solvability of problem

$$u_k'(t) = \sum_{j=1}^{n} r_{kj}(t)\, u_j(\omega_{kj}(t)) + q_k(t), \quad t \in [a,b], \tag{16}$$

$$u_k(\tau) = c_k, \qquad k = 1, 2, \ldots, n, \tag{17}$$

but also a kind of the monotone dependence of its solution on q_k, $k = 1, 2, \ldots, n$, and c_k, $k = 1, 2, \ldots, n$.

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In other words:

1. Existence of Green's operator for (16), (17)

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In other words:

1. Existence of Green's operator for (16), (17)
2. Positivity of Green's operator

Theorem 3 (A. R., 2005)

Let $\operatorname{vrai\,min}_{t\in[a,b]} r_{kj}(t)\operatorname{sign}(t-\tau) \geq 0$ *for all* k, j. *Moreover, let there exist* $\alpha \in [1, +\infty)$ *and* $\{\gamma_k \mid k = 1, 2, \ldots, n\} \subset (0, +\infty)$ *such that, for every* $k = 1, 2, \ldots, n$, *at least one of the following conditions is satisfied:*

$$\operatorname*{vrai\,max}_{t\in[a,b]\setminus\{\tau\}} \frac{\operatorname{sign}(t-\tau)}{\gamma_k |t-\tau|^{\alpha-1}} \sum_{j=1}^{n} \gamma_j r_{kj}(t) \left|\omega_{kj}(t) - \tau\right|^{\alpha} < \alpha, \tag{18}$$

$$\sup_{t\in[a,b]\setminus\{\tau\}} \frac{1}{\gamma_k |t-\tau|^{\alpha}} \sum_{j=1}^{n} \gamma_j \int_{\tau}^{t} r_{kj}(s) \left|\omega_{kj}(s) - \tau\right|^{\alpha} ds < 1. \tag{19}$$

Then the Cauchy problem (5), (6) *has a unique solution for all* $\{(c_k, q_k) \mid k = 1, 2, \ldots, n\} \subset \mathbb{R} \times L_1([a, b], \mathbb{R})$. *If, moreover,*

$$\min_{t\in[a,b]} \int_{\tau}^{t} q_k(s)ds \geq -c_k \tag{20}$$

for all k, *then the solution* $u = (u_k)_{k=1}^{n}$ *of problem* (5), (6) *is non-negative.*

Theorem 3 (A. R., 2005)

Let $\operatorname{vrai\,min}_{t\in[a,b]} r_{kj}(t)\operatorname{sign}(t-\tau) \geq 0$ *for all* k, j. *Moreover, let there exist* $\alpha \in [1, +\infty)$ *and* $\{\gamma_k \mid k = 1, 2, \dots, n\} \subset (0, +\infty)$ *such that, for every* $k = 1, 2, \dots, n$, *at least one of the following conditions is satisfied:*

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$$u'(t) = \frac{\alpha\, |t-\tau|^{\alpha-1}}{|\theta-\tau|^{\alpha}}\, \mathrm{sign}\,(t-\tau)\, u(\theta), \qquad t \in [a,b], \tag{21}$$

where $\theta \in [a,b] \setminus \{\tau\}$ and $\alpha \in [1,+\infty)$.

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- The weakened versions of the conditions are satisfied
- The Cauchy problem

  $$u(\tau) = 0$$

  for Eq. (21) has the family of solutions

  $$u(t) = \lambda\,|t-\tau|^{\alpha}, \qquad t \in [a,b],$$

  where $\lambda \in \mathbb{R}$ is arbitrary.

An integral functional equation

The methods developed also allow one to study other problems

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Another application

Unique solvability conditions for the equation

$$x(t) = \int_0^1 h(t,s)\,x(\omega(s))\,ds + q(t), \quad t \in [0,1], \tag{22}$$

where $q \in C([0,1],\mathbb{R})$, $h(t,\cdot) \in L_1([0,1],\mathbb{R})$ for all $t \in [0,1]$, $h(\cdot,s) \in C([0,1],\mathbb{R})$ for a. e. $s \in [0,1]$, and $\omega : [0,1] \to [0,1]$ is measurable.

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where $q \in C([0,1],\mathbb{R})$, ω is measurable, $h(t,\cdot) \in L_1([0,1],\mathbb{R})$ for all $t \in [0,1]$, and $h(\cdot,s) \in C([0,1],\mathbb{R})$ for a. e. $s \in [0,1]$.

Theorem 4 (A. R., 2005)

Let there exist some constants $\tau \in [0,1]$ and $\gamma \geq 0$ such that

$$\operatorname*{vrai\,max}_{t\in[0,1]\setminus\omega^{-1}(\tau)} \frac{1}{|\omega(t)-\tau|^\gamma} \int_0^1 |\omega(s)-\tau|^\gamma\, |h(\omega(t),s)|\, ds < 1, \tag{24}$$

$$\operatorname*{vrai\,max}_{t\in[0,1]\setminus\omega^{-1}(\tau)} \frac{1}{|\omega(t)-\tau|^\gamma} \int_0^1 |h(\omega(t),s)|\, ds < +\infty. \tag{25}$$

In the case where $\text{més}\,\omega^{-1}(\tau) > 0$, *assume also that*

$$h(\tau,s) = 0 \quad \textit{for a. e. } s \in [0,1]. \tag{26}$$

Then Eq. (23) *has a unique solution for all* $q \in C([0,1],\mathbb{R})$.

Power argument transformation

$$x(t) = \int_0^1 h(t,s)\, x(s^\alpha)\, ds + q(t), \quad t \in [0,1], \tag{27}$$

where $\alpha \in (0,+\infty)$, $q \in C([0,1],\mathbb{R})$.

Corollary 5

Let the following conditions be satisfied for some $\tau \in [0,1]$ and $\gamma \in [0,+\infty)$:

$$\sup_{t\in[0,1]\setminus\{\tau\}} \frac{1}{|t-\tau|^\gamma} \int_0^1 |s^\alpha - \tau|^\gamma\, |h(t,\, s)|\, ds < 1, \tag{28}$$

$$\sup_{t\in[0,1]\setminus\{\tau\}} \frac{1}{|t-\tau|^\gamma} \int_0^1 |h(t,\, s)|\, ds < +\infty. \tag{29}$$

Then Eq. (27) *has a unique solution for an arbitrary* $q \in C([0,1],\mathbb{R})$.

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$$x(t) = \int_0^1 h(t,s)\, x(s^\alpha)\, ds + q(t), \quad t \in [0,1], \tag{27}$$

where $\alpha \in (0,+\infty)$, $q \in C([0,1],\mathbb{R})$.

Corollary 6

Equation (27) is uniquely solvable for arbitrary continuous q provided that there exists some $\tau \in [0,1]$ for which

$$\operatorname*{vrai\,max}_{s\in[0,1]} \sup_{t\in[0,1]\setminus\{\tau\}} \left|\frac{h(t,s)}{t-\tau}\right| < \frac{\alpha+1}{2\alpha\tau^{1+\frac{1}{\alpha}} - (\alpha+1)\tau + 1}. \tag{30}$$

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The conditions indicated ***cannot be weakened***. A counterexample:

$$x(t) = \frac{(\alpha+1)\,|t-\tau|}{2\alpha\tau^{1+\frac{1}{\alpha}} - \tau(\alpha+1) + 1} \int_0^1 x(s^\alpha)\,ds, \qquad t \in [0,1],$$

where $\alpha \in (0,+\infty)$ and $\tau \in [0,1]$ are certain constants.

A theorem on differential inequalities

$$u_k'(t) = (l_k u)(t) + q_k(t), \qquad t \in [a,b],\ k = 1, 2, \ldots, n;$$
$$u_k(\tau) = c_k, \qquad k = 1, 2, \ldots, n,$$

where $n \in \mathbb{N}$, $l_k : C([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R})$, $k = 1, 2, \ldots, n$, are bounded linear operators, $-\infty < a \leq \tau \leq b < +\infty$, c_k, $k = 1, 2, \ldots, n$ are constants, and $q_k : [a,b] \to \mathbb{R}$, $k = 1, 2, \ldots, n$ are Lebesgue integrable.

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A ***solution*** is an absolutely continuous function $u = (u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ with property (6) which satisfies equations (5) almost everywhere on $[a,b]$.

A theorem on differential inequalities

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Definition 7

We say that $p : C([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R})$ is τ-***positive*** if

$$\operatorname*{vrai\,min}_{t\in[a,b]} (pu)(t)\ \operatorname{sign}(t-\tau) \geq 0$$

for any $u = (u_k)_{k=1}^n$ with non-negative components.

$$u_k'(t) = (l_k u)(t) + q_k(t), \qquad t \in [a,b],\ k = 1,2,\ldots,n; \tag{31}$$

$$u_k(\tau) = c_k, \qquad k = 1,2,\ldots,n \tag{32}$$

Theorem 8 (A. R., 2003)

Let $l_k : C([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R})$, $k = 1,2,\ldots,n$ be τ-positive. Let there exists an absolutely continuous $y = (y_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$

$$y_k(\tau) = 0, \quad k = 1,2,\ldots,n, \tag{33}$$

$$y_k(t) > 0, \quad t \in [a,b] \setminus \{\tau\}, \quad k = 1,2,\ldots,n, \tag{34}$$

and a $\varrho \in (1,+\infty)$ such that

$$\min_{k=1,2,\ldots,n} \operatorname*{vrai\,min}_{t\in[a,b]} \left(y_k'(t) - \varrho\,(l_k y)(t)\right) \operatorname{sign}(t-\tau) \geq 0. \tag{35}$$

Then (31), (32) is uniquely solvable for any forcing terms. If, in addition,

$$\min_{k=1,2,\ldots,n} \min_{t\in[a,b]} \int_\tau^t q_k(s)ds \geq -c_k,$$

then its solution is non-negative.

Thank you