A unified approach to singular problems arising in the membrane theory

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We will consider the singular problem

(1) $$(t^n u')' + t^n f(t,u) = 0,$$

(2) $$\lim_{t\to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A,$$

where $n \in \mathbb{N}$, $n \geq 2$, $a_0 \in (0,\infty)$, $a_1, A \in [0,\infty)$, and $f(t,x)$ is continuous on $(0,1] \times (0,\infty)$ and can have a time singularity at $t = 0$ and a space singularity at $x = 0$.

Consider the equation

$$(3) \qquad (t^3u')' + t^3\left(\frac{1}{8u^2} - \frac{\mu}{u} - \frac{\lambda^2}{2}t^{2\gamma-4}\right) = 0,$$

where $\mu \geq 0$, $\lambda > 0$, $\gamma > 1$.

Problem (3),(2) arises in the theory of shallow membrane caps and is a special case of problem (1),(2),where

$$f(t,x) = \frac{1}{8x^2} - \frac{\mu}{x} - \frac{\lambda^2}{2}t^{2\gamma-4}.$$

Dickey (Quart.Appl.Math. 1989),
Johnson (Quart.Appl.Math. 1997),
Kannan and O'Regan (J.Inequal.Appl. 2000),
Rachůnková, Koch, Pulverer, Weinmüller (JMAA 2007).

Consider the equation

$$u'' + \frac{3}{t}u' + \frac{q(t)}{u^2} = 0, \tag{4}$$

where q is continuous on $[0,1]$ and positive on $(0,1)$.
Problem (4),(2) describes a behaviour of symmetric circular membranes and can be easily transformed to a special case of problem (1),(2), where

$$f(t,x) = \frac{q(t)}{x^2}.$$

Agarwal and O'Regan (Dyn.Contin.Discrete Impuls.Syst. 2003).

An infinite interval problem of the form

$$(5) \qquad z'' + \frac{1}{s^3}\left(\frac{\lambda^2}{8s^{\gamma-2}} - \frac{1}{32z^2} + \frac{\mu}{4z}\right) = 0, \quad 1 < s < \infty,$$

$$(6) \qquad z \text{ is bounded for } s \to \infty, \quad b_0 z(1) - b_1 z'(1-) = A,$$

can be transformed to problem (3),(2) by the substitution $s = \frac{1}{t^2}$, $z(s) = u(t)$. Problem (5),(6) arises in the membrane theory and for $A > 0$ was solved by

Baxley and Robinson (J.Comp.Appl.Math. 1998),
Agarwal and O'Regan (Int.J.Non-Lin.Mech. 2004).

Assume that $f(t,x)$ is continuous on $(0,1] \times (0,\infty)$.

Definition

The function f has a time singularity at $t = 0$ if

$$\int_0^{\varepsilon} |f(t,x)|dt = \infty \quad \text{for some } x > 0 \ \text{ and for each } \varepsilon \in (0,1).$$

The function f has a space singularity at $x = 0$ if

$$\limsup_{x \to 0+} |f(t,x)| = \infty \quad \text{for} \quad t \in (0,1).$$

Example

From nonlinear PDEs to singular ODEs. Consider the Dirichlet problem

$$\Delta u + g(r, u) = 0 \quad \text{on } \Omega, \ u|_\Gamma = 0,$$

where Δ is the Laplace operator, Ω is the open unit disk in $\mathbb{R}^n$ centered at the origin, Γ is its boundary and r is the radial distance from the origin. When searching for positive radially symmetric solutions to this problem, we get the singular problem for an ordinary differential equation of the form

$$u'' + \frac{n-1}{t} u' + g(t, u) = 0, \quad u'(0) = 0, \ u(1) = 0.$$

- Berestycki, Lions and Peletier (Ind.Univ.Math.J. 1981),
- Gidas, Ni and Nirenberg (Adv.Math.Suppl.Studies 1981).

Example

In certain problems in fluid dynamics and boundary layer theory the generalized Emden-Fowler equation

$$u'' + \psi(t)u^{-\lambda} = 0$$

arises. Here $\lambda > 0$, $\psi \in C(0,1)$ and $\psi \notin L_1[0,1]$.

- Callegari and Friedman (J.Math.Anal.Appl. 1968),
- Callegari and Nachman (J.Math.Anal.Appl. 1978),
- Callegari and Nachman (SIAM J.Appl.Math. 1980).

We are interested in positive solutions of problem (1),(2).

Definition

A function u is called a positive solution of problem (1),(2), if:

- $u \in C[0,1] \cap C^2(0,1)$,
- $u(t) > 0$ for $t \in (0,1)$,
- $(t^n u'(t))' + t^n f(t, u(t)) = 0$ for $t \in (0,1)$,
- $\lim_{t \to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A.$

Definition

A function σ is called a lower function of problem (1), (2), if:

- $\sigma \in C[0,1] \cap C^2(0,1)$,
- $(t^n(\sigma'(t)))' + t^n f(t,\sigma(t)) \geq 0 \quad \text{for} \quad t \in (0,1)$,
- $\lim_{t\to 0+} t^n\sigma'(t) \geq 0, \qquad a_0\sigma(1) + a_1\sigma'(1-) \leq A.$

If all the inequalities are reversed, then σ is called an upper funtion of problem (1), (2).

Note that $t^n\sigma'(t)$ can be unbounded at the endpoints $t = 0, t = 1$.

Some monographs using the method of lower and upper functions for regular problems:

- Coster and Habets, Springer 1996, Elsevier 2004,
- Ladde, Lakshmikantham and Vatsala, Pitman 1995,
- Vasiliev and Klokov, Zinatne 1978.

Some monographs which have extended this method on singular problems:

- Kiguradze and Shekhter, Viniti 1987,
- Rachůnková, Staněk and Tvrdý, Elsevier 2006.

In the next three theorems we assume that:

- σ_1 and σ_2 are lower and upper functions of problem (1),(2),
- $0 < \sigma_1(t) \leq \sigma_2(t)$ for $t \in (0,1)$,
- $\exists p < 2$ such that $\lim_{t \to 0+} t^p h(t) < \infty$,

where $h(t) = \sup\{|f(t,x)| : \sigma_1(t) \leq x \leq \sigma_2(t)\}$.

Note that:

- σ_1 and σ_2 can vanish at $t = 0$ and $t = 1$,
- f can have singularities at $t = 0$ and $x = 0$,
- therefore h can be unbounded, i.e.

$$\limsup_{t \to 0+} h(t) = \infty, \quad \limsup_{t \to 1-} h(t) = \infty.$$

(1) $$(t^n u')' + t^n f(t,u) = 0,$$

(2) $$\lim_{t\to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A,$$

Theorem

Let h be bounded on $[0,1]$. Then problem (1), (2) has a positive solution u which moreover belongs to $C^1[0,1]$ and satisfies $u'(0) = 0$ and

(7) $$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \textit{for } t \in [0,1].$$

Theorem 1 can be proved by the arguments which are used for regular problems.

Theorem

Let h be bounded at $t = 1$. Further assume that

$$\limsup_{t\to 0+} h(t) = \infty$$

and that there is $\delta_1 \in (0,1)$ such that

$$(8) \qquad (t^n \sigma_1'(t))' \geq 0, \quad (t^n \sigma_2'(t))' \leq 0 \quad \text{for } t \in (0, \delta_1).$$

Then problem (1), (2) has a positive solution u which moreover belongs to $C^1(0,1]$ and satisfies estimate (7).

Theorem

Let h be bounded at $t = 0$. Further assume that

$$\limsup_{t \to 1-} h(t) = \infty, \quad \sigma_1(1) = \sigma_2(1)$$

and that there are $\delta_2 \in (0,1)$, $K \in \mathbb{R}$ such that

$$(9) \qquad (t^n \sigma_1'(t))' \geq K, \quad (t^n \sigma_2'(t))' \leq K \quad \text{for } t \in (1 - \delta_2, 1).$$

Then $A = 0$ and problem (1), (2) has a positive solution u which moreover belongs to $C^1[0,1)$ and satisfies estimate (7) and $u'(0) = 0$.

We use the following approach to prove Theorems 2 and 3:

- the singular problem (1), (2) is approximated by a sequence of solvable regular problems,
- a sequence $\{u_n\}$ of solutions of the regular problems is generated,
- a convergence of a suitable subsequence $\{u_{k_n}\}$ is investigated and a limit $u = \lim_{n\to\infty} u_{k_n}$ is obtained,
- the type of the convergence determines the properties of u and implies that u is a solution of the original singular problem (1), (2).

Constant lower and upper functions

If $A > 0$ and if there exist $0 < r_1 \le \frac{A}{a_0}$ and $r_2 \ge \frac{A}{a_0}$ such that

$$(10) \qquad f(t, r_1) \ge 0, \quad f(t, r_2) \le 0 \quad \text{for} \quad t \in (0, 1),$$

then the constant function $\sigma_1(t) \equiv r_1$ is a lower function of problem (1), (2), and the constant function $\sigma_2(t) \equiv r_2$ is an upper function of problem (1), (2).

We demonstrate the application of Theorems 1–3 on the problem

$$(t^n u')' + t^n \left(\frac{a}{u^{2m}} - \frac{b}{u^m} - ct^{2\eta} \right) = 0, \tag{11}$$

$$\lim_{t \to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A, \tag{2}$$

where $a > 0$, $b, c \geq 0$, $\eta > -1$, $m, n \in \mathbb{N}$, $n \geq 2$, $a_0 > 0$, $a_1, A \geq 0$.

We can find lower and upper functions (nonconstant in general) for all values of the parameters.

$$c,\ 1 - t,\ 1 - t^2,\ t^{-\frac{\eta}{m}},\ (1 - t^2)^{\frac{1}{2m}}$$

For simplicity we show how to find lower and upper functions just for the problem

$$(11) \qquad (t^n u')' + t^n \left(\frac{a}{u^{2m}} - \frac{b}{u^m} - ct^{2\eta} \right) = 0,$$

$$(12) \qquad \lim_{t \to 0+} t^n u'(t) = 0, \quad u(1) = A,$$

where $b > 0$.

I. We assume: $A > 0$.

$$ax^2 - bx - c = 0, \quad x_1 = \frac{b + \sqrt{b^2 + 4ac}}{2a},$$

$$c_1 = \min\left\{A, \frac{1}{\sqrt[m]{x_1}}\right\}, c_2 = \max\left\{A, \sqrt[m]{\frac{a}{b}}\right\}, c_3 = \max\left\{A, \frac{1}{\sqrt[m]{x_1}}\right\}.$$

- $\eta \geq 0$: h is bounded

$$\sigma_1(t) = c_1, \quad \sigma_2(t) = c_2, \quad t \in [0,1],$$

- $\eta \in (-1, 0)$: h is unbounded at $t = 0$

$$\sigma_1(t) = c_1 t^{-\frac{\eta}{m}}, \quad \sigma_2(t) = c_3, \quad t \in [0,1].$$

II. We assume: $A = 0$.

$$ax^2 - bx - c = 0, \quad x_1 = \frac{b + \sqrt{b^2 + 4ac}}{2a},$$

$$\exists c_1 \in \left(0, \frac{1}{\sqrt[m]{x_1}}\right), \quad \exists c_2 > \sqrt[m]{\frac{a}{b}}$$

- $\eta \geq 0$: h is unbounded at $t = 1$

$$\sigma_1(t) = c_1(1 - t^2), \quad \sigma_2(t) = c_2(1 - t^2)^{\frac{1}{2m}}, \quad t \in [0, 1],$$

- $\eta \in (-1, 0)$: h is unbounded both at $t = 0$ and at $t = 1$

$$\sigma_1(t) = c_1 t^{-\frac{\eta}{m}}(1 - t), \quad \sigma_2(t) = c_2(1 - t^2)^{\frac{1}{2m}}, \quad t \in [0, 1].$$

Similar lower and upper functions can be found for other values of parameters b and a_1. Then, by Theorems 1–3, we get the following existence result for the problem

$$(t^n u')' + t^n \left(\frac{a}{u^{2m}} - \frac{b}{u^m} - ct^{2\eta} \right) = 0, \tag{11}$$

$$\lim_{t \to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A, \tag{2}$$

where $a > 0$, $b, c \geq 0$, $\eta > -1$, $m, n \in \mathbb{N}$, $n \geq 2$, $a_0 > 0$, $a_1, A \geq 0$.

Theorem

Problem (11), (2) has a positive solution u such that

- $\eta > -\frac{1}{2} \Longrightarrow u(0) > 0,\ u'(0+) = 0,$
- $\eta = -\frac{1}{2} \Longrightarrow u(0) > 0,\ u'(0+) = \frac{c}{n},$
- $\eta < -\frac{1}{2} \Longrightarrow u(0) \geq 0,\ u'(0+) = \infty,$

and

- $A > 0 \Longrightarrow u'(1-) \in \mathbb{R},$
- $A = 0, a_1 > 0 \Longrightarrow u'(1-) \in \mathbb{R},$
- $A = 0, a_1 = 0 \Longrightarrow u'(1-) = -\infty.$

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