Neumann boundary value problem with ϕ-Laplacian

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Outline

1. **Introduction to problem**
   - Motivation
   - Problem that we study
   - Lower and upper functions, solution

2. **Existence results**
   - Conditions of the sign type
   - Onesided growth conditions

Motivation

div$(|\nabla u|^{p-2}\nabla u) = f(t,u,u')$

- non-Newtonian fluid theory, diffusion of flows in a porous medium, image processing
- div$(|\nabla u|^{p-2}\nabla u)$ – n-dimensional p-Laplacian

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- $\text{div}(|\nabla u|^{p-2}\nabla u)$ – n-dimensional p-Laplacian

p-Laplacian

$$(|v'|^{p-2}v')' = f(t,v,v')$$

- radially symmetric solutions of equations with multi-dimensional p-Laplacian, turbulent flow of a gas in a porous medium
- $u \to (|u'|^{p-2}u')'$, $p > 1$ – one-dimensional p-Laplacian

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ϕ-Laplacian

$$u \to (\phi(u'))'$$

- $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$

Assumptions

- $T > 0$, $[0, T] \subset \mathbb{R}$, $A, B \in \mathbb{R}$
- $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$
- $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$

Definition

The function f satisfies the Carathéodory conditions on the set $[0, T] \times \mathbb{R}^2$ if

- $f(\cdot, x, y) : [0, T] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathbb{R}^2$,
- $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous for a.e. $t \in [0, T]$,
- for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K \in L_1[0, T]$ such that $|f(t, x, y)| \le m_K(t)$ for a.e. $t \in [0, T]$ and all $(x, y) \in K$.

Formulation of the problem

Boundary value problem

$$(\phi(u'))' = f(t, u, u'), \tag{Eq}$$

$$u'(0) = A, \quad u'(T) = B. \tag{NC}$$

Lower and upper functions

Definition

Functions $\sigma_1, \sigma_2 : [0, T] \to R$ are respectively *lower and upper functions* of problem (Eq), (NC) if $\phi(\sigma_i') \in AC[0, T]$ for $i \in \{1, 2\}$ and

$$(\phi(\sigma_1'(t)))' \geq f(t, \sigma_1(t), \sigma_1'(t)), \quad \text{for a.e.} t \in [0, T], \quad \sigma_1'(0) \geq A, \sigma_1'(T) \leq B,$$

$$(\phi(\sigma_2'(t)))' \leq f(t, \sigma_2(t), \sigma_2'(t)), \quad \text{for a.e.} t \in [0, T], \quad \sigma_2'(0) \leq A, \sigma_2'(T) \geq B.$$

Solution of Neumann problem

Definition (solution of (Eq), (NC))

A function $u : [0, T] \to \mathbb{R}$ with $\phi(u') \in AC[0, T]$ is called *a solution* of problem (Eq), (NC) if u satisfies

$$(\phi(u'(t)))' = f(t, u(t), u'(t))$$

for a.e. $t \in [0, T]$ and fulfils (NC).

Bounded right-hand sides of equations

Theorem

Let σ_1, σ_2 be respectively lower and upper functions of (Eq), (NC) and let $\sigma_1 \leq \sigma_2$ on $[0, T]$. Let there is $f_0 \in L_1[0, T]$ such that $|f(t, x, y)| \leq f_0(t)$ for a.e. $t \in [0, T]$ and for all $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}$. Then problem (Eq), (NC) has a solution $u \in C^1[0, T]$ with $\phi(u') \in AC[0, T]$ such that

$$\sigma_1 \leq u \leq \sigma_2 \text{ on } [0, T].$$

Proof:

- application of the Schauder fixed point to a modified problem

Conditions of the sign type

Theorem

Let σ_1, σ_2 be respectively lower and upper functions of (Eq), (NC) and let $\sigma_1 \leq \sigma_2$ on $[0,T]$. Let there exist functions $\varphi_1, \varphi_2 \in C[0,T]$ such that $\phi(\varphi_1), \phi(\varphi_2) \in AC[0,T]$ and

$$\varphi_1(0) \leq A, \varphi_1(T) \leq B, \quad \varphi_2(0) \geq A, \varphi_2(T) \geq B,$$
$$\varphi_1(t) \leq \sigma_i'(t) \leq \varphi_2(t) \quad \text{on } [0,T],\ i = 1,2.$$

Furthermore, let φ_1, φ_2 satisfy inequalities

$$f(t,x,\varphi_1(t)) \leq (\phi(\varphi_1(t)))', \quad f(t,x,\varphi_2(t)) \geq (\phi(\varphi_2(t)))'$$

for a.e. $t \in [0,T]$ and for all $x \in [\sigma_1(t), \sigma_2(t)]$. Then the problem (Eq), (NC) has a solution $u \in C^1[0,T]$ such that

$$\sigma_1 \leq u \leq \sigma_2, \quad \varphi_1 \leq u' \leq \varphi_2 \qquad \text{on } [0,T]. \qquad (*)$$

Basic ideas of proof

- $$(\phi(u'))' = f(t, u, u'), \quad \text{(Eq)}$$
  $$u'(0) = A, \quad u'(T) = B \quad \text{(NC)}$$
- $$F_1(t,u,v) = \begin{cases} f(t,u,\varphi_2) + \frac{v-\varphi_2}{v-\varphi_2+1}, & v > \varphi_2, \\ f(t,u,v), & \varphi_1 \le v \le \varphi_2, \\ f(t,u,\varphi_1) + \frac{v-\varphi_1}{|v-\varphi_1|+1}, & v < \varphi_1 \end{cases}$$
- auxiliary problem
  $$(\phi(x'))' = F_1(t, x, x'),$$
  $$x'(0) = A, \quad x'(T) = B$$
- solution u of auxiliary problem satisfies the inequality
  $$\sigma_1 \le u \le \sigma_2 \text{ on } [0,T]$$
- it can be proved that
  $$\varphi_1 \le u' \le \varphi_2 \text{ on } [0,T]$$

Example

$$\begin{cases} \left(|x'|^{p-2}x'\right)' = (x'^k - \operatorname{sign} x') \cdot t + x^q + t^r - 1, \\ \quad x'(0) = 1, \quad x'(1) = -1, \end{cases}$$

$k, q \in \mathbb{N}$ are odd, $p > 2$, $r \geq 0$.

Functions

$$\sigma_1(t) = -t^2 + t - 2p, \quad \sigma_2(t) = -t^2 + t + 2$$

are respectively lower and upper functions of the problem. For some $C > \left(\frac{2.25^q+1}{p-1}\right)^{\frac{1}{p-2}} + 2$ and $D > \left(\frac{(2p)^q+2}{p-1}\right)^{\frac{1}{p-2}} + 1$, for $t \in [0, 1]$, let us define

$$\varphi_1(t) = t - C, \quad \varphi_2(t) = -t + D.$$

Then the right-hand side of the equation satisfies the conditions of the previous theorem and therefore it follows the existence of the solutions satisfying inequalities $(*)$.

Preliminary lemma

Lemma

Let σ_1, σ_2 be respectively lower and upper functions of (Eq), (NC) and let $\sigma_1 \le \sigma_2$ on $[0, T]$. Let there exist a continuous function $\omega : [0, \infty) \to (0, \infty)$ such that

$$\int_{-\infty}^{\phi(-1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \quad \int_{\phi(1)}^{\infty} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \qquad (**)$$

and let $k \in L_1[0, T]$ be nonnegative a.e. on $[0, T]$. Then there exists $\mu_* > 0$ such that for each function $u \in C^1[0, T]$ fulfiling (NC) and inequalities

$$\sigma_1 \le u \le \sigma_2 \quad on\ [0, T],$$
$$(\phi(u'(t)))' \le \omega(|u'(t)|)(k(t) + |u'(t)|) \quad \text{for a.e. } t \in [0, T],$$

the following estimate holds

$$|u'(t)| < \mu_* \quad \text{for all } t \in [0, T].$$

Basic ideas of proof

- $|u'(0)| \leq 1 + \max\{|A|, |B|\} = c_0$
  $|u'(T)| \leq 1 + \max\{|A|, |B|\} = c_0$
- $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty, \ \exists\, \mu_1, \mu_* \in (c_0, \infty), \mu_1 < \mu_*$:

$$\int_{\phi(c_0)}^{\phi(\mu_1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} \ > \ r + \|k\|_{L_1[0,T]}$$

$$\int_{\phi(-\mu_*)}^{\phi(-c_0)} \frac{ds}{\omega(|\phi^{-1}(s)|)} \ > \ r + \|k\|_{L_1[0,T]}$$

- these inequalities and the other asssumptions in theorem imply that

$$u'(t) < \mu_1 \ \forall\, t \in [0, T]$$

- similarly it can be proved

$$u'(t) > -\mu_* \ \forall\, t \in [0, T]$$

Onesided growth conditions

Theorem

Let σ_1, σ_2 *be respectively lower and upper functions of the problem (Eq), (NC) and let* $\sigma_1 \le \sigma_2$ *on* $[0,T]$. *Let a continuous function* $\omega : [0,\infty) \to (0,\infty)$ *satisfy* ($**$), $k \in L_1[0,T]$ *be nonnegative a.e. on* $[0,T]$ *and*

$$f(t,x,y) \le \omega(|y|)(k(t) + |y|) \quad \text{for a.e. } t \in [0,T] \text{ and every } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}.$$

Then the problem (Eq), (NC) has a solution u *such that* $\sigma_1 \le u \le \sigma_2$ *on* $[0,T]$.

Basic ideas of proof

- $r^* = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + \max\{\mu_*, \ \|\sigma_1'\|_\infty, \ \|\sigma_2'\|_\infty\}$
- $\chi(s, r^*) = \begin{cases} 1 & \text{if} \quad 0 \le s \le r^* \\ 2 - \frac{s}{r^*} & \text{if} \quad r^* < s < 2r^* \\ 0 & \text{if} \quad s \ge 2r^* \end{cases}$
- $F_2(t, x, y) = \chi(|x| + |y|, r^*) \cdot f(t, x, y)$ has a Lebesgue integrable majorant
- auxiliary problem

$$(\phi(x'(t))) = F_2(t, x, x'), \quad (NC)$$

has a solution $u, \ \sigma_1 \le u \le \sigma_2$ on $[0, T]$

- $(\phi(u'(t)))' = F_2(t, u(t), u'(t)) \le \omega(|u'(t)|)(k(t) + |u'(t)|)$ for a.e $t \in [0, T]$
  $\Rightarrow \quad |u'(t)| \le \mu_* \quad \forall\, t \in [0, T] \quad \Rightarrow \quad \|u\|_\infty + \|u'\|_\infty < \ r^*$
  $\Rightarrow \quad F_2(t, u, u') = f(t, u, u')$ for a.e. $t \in [0, T]$.

Example

$$\begin{cases} \left(|x'|^{p-2}\cdot x'\right)' = \frac{1}{\sqrt{t}}\left((x')^{p-1}-1\right) + x^m + (x')^p, \\ \quad x'(0) = 1, \quad x'(1) = -1, \end{cases}$$

$p, m \in \mathbb{N}$ - odd, $p > 1$

Functions $\sigma_1(t) = -t^2 + t - C, \quad \sigma_2(t) = -t^2 + t + D$ with a large positive $C, D \in \mathbb{R}$ are respectively lower and upper functions of the problem. We have

$$\phi^{-1}(x) = |x|^{\frac{1}{p-1}} \operatorname{sign} x, \;\; \omega(s) = 1 + s^{p-1},$$
$$\int_1^\infty \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \;\; \int_{-\infty}^{-1} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty,$$

$$f(t,x,y) = \frac{1}{\sqrt{t}}\left(y^{p-1}-1\right) + x^m + y^p$$
$$\leq \frac{1}{\sqrt{t}}(|y|^{p-1}+1) + (\sigma_2^m(t) + |y|)(|y|^{p-1}+1) \leq$$
$$\leq (1+|y|^{p-1})(\frac{1}{\sqrt{t}} + \sigma_2^m(t) + |y|) = \omega(|y|)(k(t)+|y|).$$

Summary

- The ϕ-Laplacian generalizes certain operators from applications, in particular the p-Laplacian operator.
- The sufficient conditions of solvability of Neumann boundary value problem with ϕ-Laplacian were presented.

Thank You For Your Attention.