ON EXISTENCE AND UNIQUENESS OF SOLUTION OF OPTIMAL CONTROL PROBLEMS FOR DISTRIBUTED SYSTEMS UNSOLVED WITH RESPECT TO THE TIME DERIVATIVE

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$\mathcal{X}$, $\mathcal{Y}$ – Hilbert spaces, $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $\ker L \neq \{0\}$, $M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$, $B \in \mathcal{L}(\mathcal{U};\mathcal{Y})$.

$$N(x(0) - x_0) = 0, \tag{1}$$

$$L\dot{x}(t) = Mx(t) + y(t) + Bu(t), \tag{2}$$

$$u \in \mathfrak{U}_\partial, \tag{3}$$

$$J(x,u) = \frac{1}{2}\|x - w\|^2_{H^1(0,T;\mathcal{X})} + \frac{K}{2}\|u - u_0\|^2_{H^r(0,T;\mathcal{U})} \to \inf, \tag{4}$$

where $r \in \{0,1,2,\dots\}$, $x_0 \in \mathcal{X}$, $\mathfrak{U}_\partial \subset H^r(0,T;\mathcal{U})$,

$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{X};\mathcal{Y})\}$

$R^L_{(\mu,p)}(M) = \prod_{k=0}^{p} (\mu_k L - M)^{-1} L, \quad L^L_{(\mu,p)}(M) = \prod_{k=0}^{p} L(\mu_k L - M)^{-1}$

Operator M is *strongly* (L,p)*-radial*, if

(i) $\exists a \in \mathbb{R} \quad (a,+\infty) \subset \rho^L(M)$;

(ii) $\exists K > 0 \quad \forall \mu_k \in (a,+\infty), \quad k = \overline{0,p}, \quad \forall n \in \mathbb{N}$

$$\max\{\|(R^L_{(\mu,p)}(M))^n\|_{\mathcal{L}(\mathcal{X})}, \|(L^L_{(\mu,p)}(M))^n\|_{\mathcal{L}(\mathcal{Y})}\} \le \frac{K}{\prod\limits_{k=0}^{p}(\mu_k - a)^n};$$

(iii) there exists a dense subspace $\overset{\circ}{\mathcal{Y}}$ in $\mathcal{Y}$ such that

$$\|M(\lambda L - M)^{-1} L^L_{(\mu,p)}(M) y\|_{\mathcal{Y}} \le \frac{\text{const}(y)}{(\lambda - a)\prod\limits_{k=0}^{p}(\mu_k - a)} \quad \forall y \in \overset{\circ}{\mathcal{Y}}$$

for all $\lambda, \mu_0, \mu_1 ..., \mu_p \in (a,+\infty)$;

(iv) for all $\lambda, \mu_0, \mu_1, ..., \mu_p \in (a,+\infty)$

$$\|R^L_{(\mu,p)}(M)(\lambda L - M)^{-1}\|_{\mathcal{L}(\mathcal{Y};\mathcal{X})} \le \frac{K}{(\lambda - a)\prod\limits_{k=0}^{p}(\mu_k - a)}.$$

$$\mathcal{X}^0 = \ker R^L_{(\mu,p)}(M), \quad \mathcal{Y}^0 = \ker L^L_{(\mu,p)}(M),$$
$$\mathcal{X}^1 = \overline{\mathrm{im} R^L_{(\mu,p)}(M)}, \quad \mathcal{Y}^1 = \overline{\mathrm{im} L^L_{(\mu,p)}(M)}$$
$$L_k = L\Big|_{\mathcal{X}^k}, \quad \mathrm{dom} M_k = \mathcal{X}^k \cap \mathrm{dom} M, \quad M_k = M\Big|_{\mathrm{dom} M_k}$$

Theorem 1. Let operator M be strongly (L,p)-radial. Then

(i) $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$;

(ii) $L_k \in \mathcal{L}(\mathcal{X}^k; \mathcal{Y}^k)$, $M_k \in \mathcal{C}l(\mathcal{X}^k; \mathcal{Y}^k)$, $k = 0, 1$;

(iii) there exists operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$ и $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$;

(iv) operator $G = M_0^{-1} L_0 \in \mathcal{L}(\mathcal{X}^0)$ is nilpotent of the power $\leq p$;

(v) there exists strongly continuous semigroup $\{X^t \in \mathcal{L}(\mathcal{X}) : t \geq 0\}$ of the equation $L\dot{x}(t) = Mx(t)$;

(vi) operator $S = L_1^{-1} M_1 \in \mathcal{C}l(\mathcal{X}^1)$ generates C_0-continuous semigroup $\{X_1^t = X^t\Big|_{\mathcal{X}^1} \in \mathcal{L}(\mathcal{X}^1) : t \in \overline{\mathbb{R}}_+\}$.

$$x(0) = x_0, \tag{5}$$

$$L\dot{x}(t) = Mx(t) + y(t) \tag{6}$$

Function $x \in W_q^1(0, T; \mathcal{X})$ is called *strong solution* of the problem (5), (6) if it satisfies the condition $\lim\limits_{t \to 0+} \|x(t) - x_0\|_{\mathcal{X}} = 0$ and satisfies the equations (6) almost everywhere on $(0, T)$.

Theorem 2. *Let operator M be strongly (L, p)-radial. Then for all $y \in H^{p+1}(\mathcal{Y})$,*

$$x_0 \in \mathcal{M}_y = \left\{ x \in \mathrm{dom} M : (I - P)x = -\sum_{k=0}^{p} G^k M_0^{-1}(I - Q)y^{(k)}(0) \right\}$$

the problem (5), (6) has a unique strong solution of the form

$$x(t) = -\sum_{q=0}^{p} G^q M_0^{-1}(I - Q)y^{(q)}(t) + \int\limits_0^t X^{t-s} L_1^{-1} Q y(s) ds + X^t x_0,$$

where Q is projector along $\mathcal{Y}^0$ on $\mathcal{Y}^1$.

$$Px(0) = Px_0, \tag{7}$$

Function $x \in W_q^1(0,T;\mathcal{X})$ is called *strong solution* of the problem (6), (7) if it satisfies the condition $\lim\limits_{t\to 0+} \|Px(t) - Px_0\|_{\mathcal{X}} = 0$ and satisfies the equations (6) almost everywhere on $(0,T)$.

Theorem 3. *Let operator M be strongly (L,p)-radial. Then for all $y \in H^{p+1}(0,T;\mathcal{Y}$ and $x_0 \in \mathrm{dom} M_1 \dot{+} \mathcal{X}^0$ the problem (6), (7) has a unique strong solution of the same form as in the theorem 2.*

Consider the control problem (3), (4) for the system

$$x(0) = x_0, \quad L\dot{x}(t) = Mx(t) + y(t) + Bu(t). \tag{8}$$

$$\mathcal{Z}_r = \{z \in H^1(0,T;\mathcal{X}) : L\dot{z} - Mz \in H^r(0,T;\mathcal{Y})\},$$

где $r \in \{0, 1, \ldots, p+1\}$.

The set $\mathfrak{W}$ of pairs $(x,u) \in \mathcal{Z}_r \times H^r(0,T;\mathcal{U})$ is called *the set of addmissible pairs* of the problem (3), (4), (8), if pairs (x,u) satisfies the conditions (3), (8).

A solution of the problem (3), (4), (8) is a pair $(\hat{x},\hat{u}) \in \mathfrak{W}$ such that

$$J(\hat{x},\hat{u}) = \inf_{(x,u)\in\mathfrak{W}} J(x,u).$$

A functional $J(x,u)$ is called *coercive*, if for all $R > 0$ the set $\{(x,u) \in \mathfrak{W} : J(x,u) \leq R\}$ is bounded in the space $\mathcal{Z}_r \times H^r(0,T;\mathcal{U})$.

Denote for $x_0 \in \mathrm{dom}M$, $y \in H^{p+1}(0,T;\mathcal{Y})$ the set of control functions $u \in H^{p+1}(0,T;\mathcal{U})$ satisfying the condition

$$(I-P)x_0 = -\sum_{k=0}^{p+1} G^k M_0^{-1}(I-Q)(Bu^{(k)}(0) + y^{(k)}(0))$$

by $H_\partial(x_0, y)$. Following result is obtained.

Theorem 4. *Let M be strongly (L,p)-radial, $\mathfrak{U}_\partial \cap H_\partial(x_0,y) \neq \emptyset$. Then there exists a unique solution $(\hat{x},\hat{u}) \in \mathcal{Z}_r \times H^r(0,T;\mathcal{U})$ of the problem* (3), (4), (8).

For the control problem (3), (4) with general Showalter condition

$$Px(0) = Px_0, \quad L\dot{x}(t) = Mx(t) + y(t) + Bu(t) \tag{9}$$

Theorem 5. *Let the condition $\mathfrak{U}_\partial \cap H^{p+1}(0,T;\mathcal{U}) \neq \emptyset$ is satisfied. Then the control problem* (3), (4), (9) *has a unique solution $(\hat{x},\hat{u}) \in \mathcal{Z}_r \times H^r(0,T;\mathcal{U})$.*

$$w(x,0) = w_0(x), \quad x \in \Omega, \tag{10}$$

$$w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{11}$$

$$w_t(x,t) = \nu \Delta w(x,t) - q(x,t) + y(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{12}$$

$$\nabla \cdot w = 0, \quad (x,t) \in \Omega \times (0,T). \tag{13}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded region with boundary $\partial\Omega$ of the class C^∞. Here constant ν is positive and $q = \nabla p$ is gradient of pressure. $w(x,t)$ and $q(x,t)$ are unknown vector-functions.

$\mathbb{H}^2 = (H^2(\Omega))^n$, $\overset{\circ}{\mathbb{H}}{}^1 = (\overset{\circ}{H}{}^1(\Omega))^n$, $\mathbb{L}_2 = (L_2(\Omega))^n$, $\mathcal{L} = \{w \in (C_0^\infty(\Omega))^n : \nabla \cdot w = 0\}$. Closure of subspace $\mathcal{L}$ in the sense of the norm of the space $\mathbb{L}_2$ denote as $\mathbb{H}_\sigma$. $\mathbb{L}_2 = \mathbb{H}_\sigma \oplus \mathbb{H}_\pi$, where $\mathbb{H}_\pi$ is orthogonal compliment to $\mathbb{H}_\sigma$. Denote by $\Pi : \mathbb{L}_2 \to \mathbb{H}_\pi$ corresponding orthoprojector. $\Pi_1 : \overset{\circ}{\mathbb{H}}{}^1 \cap \mathbb{H}^2 \to \overset{\circ}{\mathbb{H}}{}^1 \cap \mathbb{H}^2$. $\Sigma = I - \Pi$. $\overset{\circ}{\mathbb{H}}{}^1 \cap \mathbb{H}^2 = \mathbb{H}^2_\sigma \oplus \mathbb{H}^2_\pi$, where $\mathbb{H}^2_\sigma = \ker \Pi_1$, $\mathbb{H}^2_\pi = \mathrm{im}\Pi_1$.

$A = \mathrm{diag}\{\Delta, \ldots, \Delta\}$, $A : \mathrm{dom}A \to \mathbb{L}_2$, $\mathrm{dom}A = \mathbb{H}^2$. Besides, $A_\sigma = A\Big|_{\mathbb{H}^2_\sigma}$, $A_\pi = A\Big|_{\mathbb{H}^2_\pi}$. $A_\sigma : \mathbb{H}^2_\sigma \to \mathbb{H}_\sigma$, $A_\pi : \mathbb{H}^2_\pi \to \mathbb{H}_\pi$.

Function $z \in \overset{\circ}{\mathbb{H}}{}^1 \cap \mathbb{H}^2$ satisfies the condition (13) if and only if $z \in \mathbb{H}^2_\sigma$ or, in other words, $\Pi_1 z = 0$.

$$\Pi_1 w(x,t) = y_q(x,t), \quad (x,t) \in \Omega \times (0,T). \tag{14}$$

$\mathcal{X} = \mathcal{Y} = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \mathbb{H}^2_q$; $\mathbb{H}^2_q = \mathbb{H}^2_\pi$;

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \nu A_\sigma & 0 & 0 \\ 0 & \nu A_\pi & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$L : \mathcal{X} \to \mathcal{Y}$, $\ker L = \{0\} \times \{0\} \times \mathbb{H}^2_q$, $\mathrm{im}L = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \{0\}$, $M : \mathcal{X} \to \mathcal{Y}$ has a domain $\mathrm{dom}M = \mathbb{H}^2_\sigma \times \mathbb{H}^2_\pi \times \mathbb{H}^2_q$.

Theorem 6. *Operator M is strongly $(L,1)$-radial.*

Replace the initial condition (10) by the condition

$$\Sigma w(x,0) = w_0(x), \quad x \in \Omega \tag{15}$$

and get the general Showalter problem. For any fixed $t \in (0,T)$ denote $\Sigma y(x,t) = y_\sigma(x,t)$, $\Pi y(x,t) = y_\pi(x,t)$.

Theorem 7. *For any $w_0 \in \mathbb{H}^2_\sigma$ and $y \in H^1(\mathbb{L}_2)$, $y_q \in H^2(\mathbb{H}^2_q)$ there exists a unique strong solution of the problem* (11), (12), (14), (15), *that has the form*

$$w_\sigma(x,t) = \sum_{k=1}^{\infty} e^{\lambda_k t}\langle w_0, \varphi_k\rangle \varphi_k(x) + \int_0^t \sum_{k=1}^{\infty} e^{\lambda_k (t-s)}\langle y_\sigma(\xi,s), \varphi_k(\xi)\rangle \varphi_k(x) ds,$$

$$w_\pi(x,t) = y_q(x,t),$$

$$q(x,t) = y_\pi(x,t) + \nu A y_q(x,t) - \frac{\partial y_q(x,t)}{\partial t}.$$

$$\Sigma w(x,0) = w_0(x), \quad x \in \Omega, \tag{16}$$

$$w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{17}$$

$$w_t(x,t) = \nu \Delta w(x,t) - q(x,t) + u(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{18}$$

$$\Pi_1 w = u_q(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{19}$$

$$(u, u_q) \in \mathfrak{U}_\partial, \tag{20}$$

$$J(w,q,u) = \frac{1}{2}\|w - v_0\|^2_{H^1(0,T;\mathbb{L}_2)} + \frac{1}{2}\|q - q_0\|^2_{H^1(0,T;\mathbb{H}^2)} + \frac{N}{2}\|u - u_0\|^2_{H^r(0,T;\mathbb{L}_2)} + \frac{K}{2}\|u_q - u_1\|^2_{H^r(0,T;\mathbb{H}^2)} \to \inf, \tag{21}$$

where $v_0 \in H^1(0,T;\mathbb{L}_2)$, $q_0 \in H^1(0,T;\mathbb{H}^2_q)$, $u_0 \in H^r(0,T;\mathbb{L}_2)$, $u_1 \in H^r(0,T;\mathbb{H}^2_q)$, $\mathfrak{U}_\partial \subset H^r(0,T;\mathbb{L}_2) \times H^r(0,T;\mathbb{H}^2_q)$.

$\mathcal{Z}_r = \{(z,k) \in H^1(0,T;\mathbb{L}_2) \times H^1(0,T;\mathbb{H}^2_q) :$
$z_t - \nu\Delta z + k \in H^r(\mathbb{L}_2),\ \Pi_1 z \in H^r(0,T;\mathbb{H}^2_q)\}$.

Theorem 8. *Let* $\mathfrak{U}_\partial \cap H^2(0,T;\mathbb{L}_2) \times H^2(0,T;\mathbb{H}^2_q) \neq \emptyset$. *Then the problem* (16) –(21) *has a unique solution* $(\hat{w}, \hat{q}, \hat{u}, \hat{u}_q) \in \mathcal{Z}_r \times H^r(0,T;\mathbb{L}_2) \times H^2(0,T;\mathbb{H}^2_q)$.

Dzektser equation

Let $\Omega \subset \mathbb{R}^s$ is a bounded domain $\partial\Omega$ of the class C^∞, $\lambda, \alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$. Consider the optimal control problem

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{22}$$

$$\frac{\partial}{\partial n} v(x,t) = \frac{\partial}{\partial n} \Delta v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{23}$$

$$(\lambda - \Delta) v_t(x,t) = \alpha \Delta v(x,t) - \beta \Delta^2 v(x,t) + u(x,t), \tag{24}$$

$$(x,t) \in \Omega \times (0,T), \tag{25}$$

$$u(x,t) \in \mathfrak{U}_\partial, \tag{26}$$

$$J(v,u) = \frac{1}{2} \|v - w\|^2_{H^1(0,T;H^2(\Omega))} + \frac{N}{2} \|u - u_0\|^2_{L_2(0,T;L_2(\Omega))} \to \inf, \tag{27}$$

$\mathfrak{U}_\partial$ is nonempty closed convex set in $L_2(0,T;L_2(\Omega))$. The equation describes an evolution of free surface of filtered fluid.

In the case $\alpha/\beta \neq \lambda$ this problem can be reduced to the problem (3), (4), (8) with strongly $(L, 0)$-radial operator M after choosing.

$$\mathcal{X} = H^2_{\frac{\partial}{\partial n}}(\Omega) = \left\{ v \in H^2(\Omega) : \tfrac{\partial}{\partial n} v(x) = 0,\ x \in \partial\Omega \right\},\ \mathcal{Y} = \mathcal{U} = L_2(\Omega).$$

$\mathcal{Z}_0 = H^1(0, T; H^2_{\frac{\partial}{\partial n}}(\Omega)) \cap L_2(0, T; H^4(\Omega))$, $H_\partial(v_0, 0) = \{u \in H^1(0, T; L_2(\Omega)) :$
$\int\limits_\Omega (u(x, 0) + (\alpha\Delta - \beta\Delta^2)v_0(x))\varphi_k(x)dx = 0,\ k = \overline{1, m}\}$,

THEOREM 9. *Let* $\mathfrak{U}_\partial \cap H_\partial(v_0, 0) \neq \emptyset$. *Then the control problem* (22) – (27) *has a unique solution* $(\hat{v}, \hat{u}) \in \mathcal{Z}_0 \times L_2(0, T; L_2(\Omega))$.

THANK YOU FOR THE ATTENTION