On Solvability and Unsolvability of the Problem on Transitional Solutions for Second Order Nonlinear Differential Equations

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We consider the boundary value problem

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}, \tag{2}$$

where $f : \mathbb{R} \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(t, 0, 0) = 0, \quad f(t, 1, 0) = 0 \ \text{ for } \ t \in \mathbb{R}. \tag{3}$$

Since $u_0(t) \equiv 0$ and $u_1(t) \equiv 1$ are the solutions of Eq. (1), a solution of this equation satisfying conditions (2) is said to be a **transitional solution**.

In the paper,

G. Ja. Ljubarskiĭ, On solutions of "smoothed shock wave" type of nonlinear equations. (Russian) *Uspekhi Mat. Nauk* **17 (1962), No. 1, 183–189,**

such a kind of solutions is also called *smoothed shock wave* type.

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

Problems of the type (1), (2) arise in the investigation of transitional processes when a physical system transits from an unstable equilibrium state into a stable one. See

K. Zoller, Zur stuktur des verdichtungsstosses. (German) *Z. Physik* **130 (1951), 1–38.**

They also meet in the study of travelling wave solutions of reaction-diffusion-convection equations of the type

$$v_\tau + (h(v))v_x = (D(v)v_x)_x + g(v)\ , \quad \tau \ge 0, \ \ x \in \mathbb{R},$$

where $g(u)$ is a positive nonlinear term in $]0, 1[$, vanishing at 0 and 1, more precisely, in searching for solutions satisfying $v(\tau, x) = u(x + c\tau)$ for some constant $c \in \mathbb{R}$ (the wave speed) and function $u \in C^2(\mathbb{R})$ (the wave profile), connecting the stationary states 0 (the unstable equilibrium) and 1 (the stable one). See, e.g.,

A. Volpert, V. Volpert and V. Volpert, Travelling wave solutions of parabolic systems. In *Trans. of Math. Monogr.,* **Vol. 140,** *Amer. Math. Soc., Providence, Rhode Island,* **1994,**

for an extensive treatment and a wide bibliography on the subject, and

L. Malaguti and C. Marcelli, Travelling wavefronts in reaction-diffusion equations with convection effects and non-regular terms. *Math. Nachr.* **242 (2002), 1–17,**

for a recent contribution about equations with convection effects.

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

A simple but interesting particular case of Eq. (1) is the differential equation

$$u'' = p_0(t)u' + p(t)g(u), \tag{4}$$

arising from the mathematical biology. Here $p_0, p : \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \to [0,1]$ are continuous functions, and

$$g(0) = g(1) = 0, \quad 0 < g(x) < 1 \ \text{ for } \ 0 < x < 1.$$

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

In the monograph

Yu. A. Klokov, Boundary value problems with a condition at infinity for equations of mathematical physics. (Russian) Riga, 1963,

and in the papers

1. M. A. Krasnosel'skiĭ and G. Ja. Ljubarskiĭ, Transitional solutions of non-linear equations. (Russian) *Izv. Vyssh. Uchebn. Zaved., Matematika* **1962, No. 4(29), 81–85;**

2. G. Ja. Ljubarskiĭ, On solutions of "smoothed shock wave" type of nonlinear equations. (Russian) *Uspekhi Mat. Nauk* **17 (1962), No. 1, 183–189;**

3. G. Ja. Ljubarskiĭ, A boundary-value problem on the axis for an *n*th-order non-linear equation. (Russian) *Dokl. Akad. Nauk SSSR* **149 (1963), No. 3, 521–524;**

4. L. Malaguti and C. Marcelli, Heteroclinic orbits in plane dynamical systems. *Arch. Math.* (*Brno*) **38 (2002), No. 3, 183–200,**

the existence of transitional solutions was studied in the autonomous case when Eq. (1) has the form

$$u'' = g_1(u, u')u' - g_2(u) \qquad \text{or} \qquad u'' = g(u, u').$$

In the non-autonomous case, problem (1), (2) was investigated in the paper

L. Malaguti, C. Marcelli, and N. Partsvania, On transitional solutions of second order nonlinear differential equations. *J. Math. Anal. Appl.* **303 (2005), No. 1, 258-273.**

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}; \tag{2}$$

$$u'' = p_0(t)u' + p(t)g(u), \tag{4}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

In this report, we give new results on the solvability and unsolvability of problem (1), (2), and namely of problem (4), (2).

In contrast to the theorems proven in the paper

L. Malaguti, C. Marcelli, and N. Partsvania, On transitional solutions of second order nonlinear differential equations. *J. Math. Anal. Appl.* **303 (2005), No. 1, 258-273,**

these results solve the question on the solvability of problem (4), (2) even in the case where either the condition

$$p_0(t)p(t) < 0 \quad \text{for} \quad t \in \mathbb{R}$$

is violated, or the above condition is satisfied and

$$\inf\left\{\left|\frac{p(t)}{p_0(t)}\right| : \ \ t \in \mathbb{R}\right\} = 0.$$

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \;\text{ for }\; t \in \mathbb{R}. \tag{2}$$

Note that problem (1), (2) is closely related to the well-known results on the existence of solutions of Eq. (1) satisfying the condition

$$\gamma_1(t) \le u(t) \le \gamma_2(t) \quad \text{for} \quad t \in \mathbb{R},$$

where $\gamma_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are prescribed continuous functions such that

$$\gamma_1(t) \le \gamma_2(t) \quad \text{for} \quad t \in \mathbb{R},$$

obtained in the papers

1. Z. Opial, Sur les intégrales bornées de l'équation $u'' = f(t, u, u')$**. (French)** *Ann. Polon. Math.* **4 (1958), No. 3, 314–324;**

2. I. T. Kiguradze, Some singular boundary value problems for second order nonlinear ordinary differential equations. (Russian) *Differentsial'nye Uravnenija* **4 (1968), No. 10, 1753–1773;**

3. I. T. Kiguradze and B. L. Shekhter, Singular boundary value problems for second-order ordinary differential equations. (Russian) *Itogi Nauki i Tekhniki, Sovrem. Probl. Mat., Noveĭshie Dostizh.* **30 (1987), 105–201; translated in** *J. Soviet Math.* **43 (1988), No. 2, 2340–2417.**

$$u'' = f(t,u,u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}; \tag{2}$$

$$f(t,0,0) = 0, \quad f(t,1,0) = 0 \ \text{ for } \ t \in \mathbb{R}. \tag{3}$$

The existence theorems obtained by us are based on Proposition 1 below which, in its turn, is a certain modification of Theorem 5.1 proven in the work

I. T. Kiguradze and B. L. Shekhter, Singular boundary value problems for second-order ordinary differential equations. (Russian) *Itogi Nauki i Tekhniki, Sovrem. Probl. Mat., Noveĭshie Dostizh.* **30 (1987), 105–201; translated in** *J. Soviet Math.* **43 (1988), No. 2, 2340–2417.**

Precisely, we essentially use the following

Proposition 1. *Let along with* (3) *the condition*

$$\sigma f(t,x,y)\operatorname{sgn} y \ge -h(t)(1+y^2) \ \text{ for } \ t \in \mathbb{R}, \ \ 0 \le x \le 1, \ \ y \in \mathbb{R} \tag{5}$$

be fulfilled, where $\sigma \in \{-1, 1\}$, *and* $h : \mathbb{R} \to [0,+\infty[$ *is a continuous function. Let, moreover, there exist numbers* t_i $(i = 1,2)$ *and twice continuously differentiable functions* $\gamma_1 : [t_1,+\infty[\to [0,1]$ *and* $\gamma_2 :]-\infty, t_2] \to [0,1]$ *such that* $t_2 \le t_1$,

$$\gamma_1(t_1) = 0, \ \ \gamma_1'(t_1) \ge 0, \ \ f(t,\gamma_1(t),\gamma_1'(t)) \le \gamma_1''(t) \ \text{ for } \ t \ge t_1, \tag{6_1}$$

$$\gamma_2(t_2) = 1, \ \ \gamma_2'(t_2) \ge 0, \ \ f(t,\gamma_2(t),\gamma_2'(t)) \ge \gamma_2''(t) \ \text{ for } \ t \le t_2, \tag{6_2}$$

$$\lim_{t\to+\infty} \gamma_1(t) = 1, \quad \lim_{t\to-\infty} \gamma_2(t) = 0. \tag{7}$$

Then problem (1), (2) *is solvable.*

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

We investigate the question of solvability of problem (1), (2) in the case where the function f satisfies either the inequalities

$$f(t,x,y) \ge cy - \frac{c^2}{4}x \ \text{ for } \ t \le -a, \quad 0 \le x \le 1, \quad 0 \le y \le \frac{c}{2}, \tag{8}$$

$$-h(t)(1+y^2) \le f(t,x,y) \le c_1 y - f_0(t,x) \tag{9}$$

$$\text{for } \ t \in \mathbb{R}, \quad 0 \le x \le 1, \quad y \ge 0,$$

or the inequalities

$$f(t,x,y) \le -cy + \frac{c^2}{4}(1-x) \ \text{ for } \ t \ge a, \quad 0 \le x \le 1, \quad 0 \le y \le \frac{c}{2}, \tag{8$'$}$$

$$-c_1 y + f_0(t,x) \le f(t,x,y) \le h(t)(1+y^2) \tag{9$'$}$$

$$\text{for } \ t \in \mathbb{R}, \quad 0 \le x \le 1, \quad y \ge 0,$$

where a, c, and c_1 are positive numbers, and $f_0 : \mathbb{R} \times [0,1] \to [0,+\infty[$ and $h : \mathbb{R} \to [0,+\infty[$ are continuous functions.

Everywhere below, we use the following notation

$$f_{0*}(t,x) = \min\{f_0(t,s) : \ x \le s \le 1-x\} \ \text{ for } \ t \in \mathbb{R}, \quad 0 < x < \frac{1}{2}. \tag{10}$$

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}; \tag{2}$$

$$f(t, 0, 0) = 0, \quad f(t, 1, 0) = 0 \ \text{ for } \ t \in \mathbb{R}. \tag{3}$$

$$f(t,x,y) \ge cy - \frac{c^2}{4}x \ \text{ for } \ t \le -a, \ \ 0 \le x \le 1, \ \ 0 \le y \le \frac{c}{2}, \tag{8}$$

$$-h(t)(1+y^2) \le f(t,x,y) \le c_1 y - f_0(t,x) \tag{9}$$
$$\text{for } \ t \in \mathbb{R}, \ \ 0 \le x \le 1, \ \ y \ge 0.$$

Theorem 1. *Let along with* (3), (8), *and* (9) *the condition*

$$\int_{-\infty}^{0} f_{0*}(s,x)\,ds = +\infty \quad \textit{for} \quad 0 < x < 1/2 \tag{11_1}$$

be fulfilled. Let, moreover, either

$$\int_{0}^{+\infty} f_{0*}(s,x)\,ds = +\infty \quad \textit{for} \quad 0 < x < 1/2, \tag{11_2}$$

or there exist $\lambda \in]0,1[$ *,* $\delta \in]0,1[$ *, and* $a_i \in \mathbb{R}$ $(i = 1,2)$ *such that* $a_1 < a_2$ *and*

$$f_0(t,x) \ge \delta(1-x)^\lambda \quad \textit{for} \quad a_1 \le t \le a_2, \quad 1-\delta \le x \le 1. \tag{12}$$

Then problem (1), (2) *has at least one solution satisfying the condition*

$$u'(t) > 0 \quad \textit{for} \quad t \in \{s \in \mathbb{R} : \ \ u(s) < 1\}. \tag{13}$$

$$u'' = f(t, u, u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}; \tag{2}$$

$$f(t, 0, 0) = 0, \quad f(t, 1, 0) = 0 \ \text{ for } \ t \in \mathbb{R}. \tag{3}$$

$$f(t,x,y) \le -cy + \frac{c^2}{4}(1-x) \ \text{ for } \ t \ge a, \quad 0 \le x \le 1, \quad 0 \le y \le \frac{c}{2}, \tag{$8'$}$$

$$-c_1 y + f_0(t,x) \le f(t,x,y) \le h(t)(1+y^2) \tag{$9'$}$$

$$\text{for } \ t \in \mathbb{R}, \quad 0 \le x \le 1, \quad y \ge 0,$$

$$\int\limits_{-\infty}^{0} f_{0*}(s,x)\,ds = +\infty \quad \text{for} \quad 0 < x < 1/2, \tag{11_1}$$

$$\int\limits_{0}^{+\infty} f_{0*}(s,x)\,ds = +\infty \quad \text{for} \quad 0 < x < 1/2. \tag{11_2}$$

Theorem 1$'$. *Let conditions* (3), ($8'$), ($9'$), *and* (11_2) *be fulfilled. Let, moreover, either condition* (11_1) *hold or there exist* $\lambda \in]0,1[$, $\delta \in]0,1[$, *and* $a_i \in \mathbb{R}$ $(i = 1,2)$ *such that* $a_1 < a_2$ *and*

$$f_0(t,x) \ge \delta x^\lambda \quad \textit{for} \quad a_1 \le t \le a_2, \quad 0 \le x \le \delta. \tag{$12'$}$$

Then problem (1), (2) *has at least one solution satisfying the condition*

$$u'(t) > 0 \quad \textit{for} \quad t \in \{s \in \mathbb{R} : \ \ u(s) > 0\}. \tag{$13'$}$$

Suppose

$$f(t,x,y) = p_0(t)y + p(t)g(x), \quad f_0(t,x) = |p(t)|g(x),$$

and

$$h(t) = |p_0(t)| + |p(t)|.$$

Evidently, if the inequalities

$$g(1) = 0, \quad 0 < g(x) \le x \quad \text{for} \quad 0 < x < 1, \tag{14}$$

$$p_0(t) \le c_1, \ \ p(t) \le 0 \ \ \text{for} \ \ t \in \mathbb{R}, \ \ p_0(t) \ge c, \ \ p(t) \ge -\frac{c^2}{4} \ \ \text{for} \ \ t \le -a \tag{15}$$

hold, where $a > 0$, $c > 0$, and $c_1 \in [c, +\infty[$, then f satisfies conditions

$$f(t,0,0) = 0, \ \ f(t,1,0) = 0 \ \ \text{for} \ \ t \in \mathbb{R}, \tag{3}$$

$$f(t,x,y) \ge cy - \frac{c^2}{4}x \ \ \text{for} \ \ t \le -a, \ \ 0 \le x \le 1, \ \ 0 \le y \le \frac{c}{2}, \tag{8}$$

$$-h(t)(1+y^2) \le f(t,x,y) \le c_1 y - f_0(t,x) \tag{9}$$

$$\text{for} \ \ t \in \mathbb{R}, \ \ 0 \le x \le 1, \ \ y \ge 0.$$

In the case, where

$$g(0) = 0, \quad 0 < g(x) \le 1 - x \quad \text{for} \quad 0 < x < 1, \tag{14$'$}$$

$$p_0(t) \ge -c_1, \ \ p(t) \ge 0 \ \ \text{for} \ \ t \in \mathbb{R}, \ \ p_0(t) \le -c, \ \ p(t) \le \frac{c^2}{4} \ \ \text{for} \ \ t \ge a \tag{15$'$}$$

with $a > 0$, $c > 0$, and $c_1 \in [c, +\infty[$, the function f satisfies conditions (3),

$$f(t,x,y) \le -cy + \frac{c^2}{4}(1-x) \ \ \text{for} \ \ t \ge a, \ \ 0 \le x \le 1, \ \ 0 \le y \le \frac{c}{2}, \tag{8$'$}$$

$$-c_1 y + f_0(t,x) \le f(t,x,y) \le h(t)(1+y^2) \tag{9$'$}$$

$$\text{for} \ \ t \in \mathbb{R}, \ \ 0 \le x \le 1, \ \ y \ge 0.$$

$$u'' = p_0(t)u' + p(t)g(u), \tag{4}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

Thus Theorems 1 and Theorem 1′, respectively, imply the following corollaries.

Corollary 1. *Let along with* (14) *and* (15) *the condition*

$$\int_{-\infty}^{0} p(s)\,ds = -\infty \tag{16_1}$$

be fulfilled. Let, moreover, either

$$\int_{0}^{+\infty} p(s)\,ds = -\infty, \tag{16_2}$$

or there exist $\lambda \in]0,1[$ *such that*

$$\liminf_{x\to 1} (1-x)^{-\lambda} g(x) > 0. \tag{17}$$

Then (4), (2) *has at least one solution satisfying the condition*

$$u'(t) > 0 \quad \textit{for} \quad t \in \{s \in \mathbb{R} : \ \ u(s) < 1\}. \tag{13}$$

Corollary 1′. *Let along with* (14′) *and* (15′) *the condition*

$$\int_{0}^{+\infty} p(s)\,ds = +\infty \tag{18_1}$$

be fulfilled. Let, moreover, either

$$\int_{-\infty}^{0} p(s)\,ds = +\infty, \tag{18_2}$$

or there exist $\lambda \in]0,1[$ *such that*

$$\liminf_{x\to 0} x^{-\lambda} g(x) > 0. \tag{17′}$$

Then (4), (2) *has at least one solution satisfying the condition*

$$u'(t) > 0 \quad \textit{for} \quad t \in \{s \in \mathbb{R} : \ \ u(s) > 0\}. \tag{13′}$$

$$u'' = f(t,u,u'), \tag{1}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

Now we give theorems on the unsolvability of problem (1), (2).

Theorem 2. *Let there exist numbers $i \in \{1,2\}$, $c > 0$, and continuous functions $\ell : \mathbb{R} \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to [0,+\infty[$ such that*

$$(-1)^i f(t,x,y) \le \ell(t,x,y)y \ \ \textit{for} \ \ t \in \mathbb{R}, \ \ 0 \le x \le 1, \ \ y \in \mathbb{R}, \tag{19}$$

$$f(t,x,y) \ge cy - q(t)(1-x) \ \ \textit{for} \ \ t \in \mathbb{R}, \ \ 0 \le x \le 1, \ \ y \ge 0, \tag{20}$$

and

$$\int_0^{+\infty} q(s)\,ds < +\infty. \tag{21}$$

Then problem (1), (2) *has no solution.*

Theorem 2′. *Let there exist numbers $i \in \{1,2\}$, $c > 0$, and continuous functions $\ell : \mathbb{R} \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to [0,+\infty[$ such that along with* (19) *the following conditions are fulfilled:*

$$f(t,x,y) \le -cy + q(t)x \quad \textit{for} \quad t \in \mathbb{R}, \quad 0 \le x \le 1, \quad y \ge 0,$$

and

$$\int_{-\infty}^{0} q(s)\,ds < +\infty.$$

Then problem (1), (2) *has no solution.*

$$u'' = p_0(t)u' + p(t)g(u), \tag{4}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

$$g(1) = 0, \quad 0 < g(x) \le x \quad \text{for} \quad 0 < x < 1, \tag{14}$$

$$\int_{-\infty}^{0} p(s)\,ds = -\infty, \tag{16_1}$$

$$\int_{0}^{+\infty} p(s)\,ds = -\infty. \tag{16_2}$$

Corollary 1 and Theorem 2 imply the following proposition.

Corollary 2. *Let along with* (14) *and* (16_1) *the following inequalities*

$$c \le p_0(t) \le c_1, \ \ p(t) \le 0 \ \ \textit{for} \ \ t \in \mathbb{R}, \ \ p(t) \ge -\frac{c^2}{4} \ \ \textit{for} \ \ t \le -a$$

be fulfilled, where $a > 0$, $c > 0$, *and* $c_1 \in [c, +\infty[$. *Let, moreover,*

$$\limsup_{x\to 1} \frac{g(x)}{1-x} < +\infty.$$

Then for the solvability of problem (4), (2) *the necessary and sufficient condition is* (16_2).

$$u'' = p_0(t)u' + p(t)g(u), \tag{4}$$

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \le u(t) \le 1 \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

$$g(0) = 0, \quad 0 < g(x) \le 1 - x \quad \text{for} \quad 0 < x < 1, \tag{14$'$}$$

$$\int_0^{+\infty} p(s)\,ds = +\infty, \tag{18$_1$}$$

$$\int_{-\infty}^{0} p(s)\,ds = +\infty. \tag{18$_2$}$$

Corollary $1'$ and Theorem $2'$ yield the following proposition.

Corollary 2$'$. *Let along with* $(14')$ *and* (18_1) *the following conditions*

$$-c_1 \le p_0(t) \le -c, \ \ p(t) \ge 0 \ \textit{ for } \ t \in \mathbb{R}, \ \ p(t) \le \frac{c^2}{4} \ \textit{ for } \ t \ge a$$

hold, where $a > 0$, $c > 0$, *and* $c_1 \in [c, +\infty[$. *If, moreover,*

$$\limsup_{x\to 0} \frac{g(x)}{x} < +\infty,$$

then for the solvability of problem $(4), (2)$ *the necessary and sufficient condition is* (18_2).

Remark. According to Corollary 2, it is evident that we cannot assume $\lambda \geq 1$ in the condition

$$\liminf_{x \to 1} (1-x)^{-\lambda} g(x) > 0 \tag{17}$$

of Corollary 1.

Analogously, according to Corollary 2′, it is evident that we cannot assume $\lambda \geq 1$ in the condition

$$\liminf_{x \to 0} x^{-\lambda} g(x) > 0 \tag{17′}$$

of Corollary 1′.