New solvability conditions for a Nonlocal Boundary Value Problem for Nonlinear Functional Differential Equations

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Notation

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions.

$\widetilde{C}([a,b];\mathrm{D})$ is the set of absolutely continuous functions.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell:\ C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ for which there exists a function $\eta_\ell \in L([a,b];\mathbb{R}_+)$ such that $|\ell(v)(t)| \leq \eta_\ell(t)\|v\|_C \quad \text{for} \quad t \in [a,b]$.

P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

F_{ab} is the set of linear bounded functionals.

PF_{ab} is the set of functionals $h \in F_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.

$\mathcal{B}^i_{hc} = \{u \in C([a,b];\mathbb{R}) :\ (-1)^{i+1}(u(a) - h(u)) \operatorname{sgn} u(a) \leq c\}$, where $h \in F_{ab}$, $c \in \mathbb{R}$, $i = 1, 2$.

$\mathcal{K}_{ab}$ is the set of continuous operators $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R}$ satisfying the Carathéodory conditions.

$K([a,b] \times A; B)$, where $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}$, is the set of function $f : [a,b] \times A \to B$ satisfying the Carathéodory conditions.

Consider the boundary value problem

$$u'(t) = F(u)(t), \tag{1}$$

$$u(a) = h(u) + \varphi(u), \tag{2}$$

where $F \in \mathcal{K}_{ab}$, $h \in F_{ab}$ and $\varphi : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a continuous functional such that, for every $r > 0$, there exists $M_r \in \mathbb{R}_+$ satisfying

$$|\varphi(v)| \leq M_r \quad \text{for } v \in C([a,b];\mathbb{R}),\ ||v||_C \leq r.$$

We will assume that the functional h admits the representation

$$h(v) \overset{def}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}), \tag{3}$$

where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$.

By a solution of the problem (1), (2) we understand a function $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfying the equality (1) almost everywhere in $[a,b]$ and the boundary condition (2).

Results are concretized for the differential equation with deviating arguments of the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))), \tag{4}$$

where $p, g \in L([a,b];\mathbb{R}_+)$, τ, μ, ν are measurable functions and $f \in K([a,b] \times \mathbb{R}^2;\mathbb{R})$.

Linear problem

$$u'(t) = \ell(u)(t) + q(t), \tag{5}$$

$$u(a) = h(u) + c, \tag{6}$$

where $\ell \in \mathcal{L}_{ab}$, $q \in L([a,b];\mathbb{R})$, $h \in F_{ab}$, and $c \in \mathbb{R}$.

$$u'(t) = p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t), \quad \int\limits_a^b u(s)\,ds = c, \tag{7}$$

where $p, q \in L([a,b];\mathbb{R})$, $\tau : [a,b] \to [a,b]$ is a measurable function and $c \in \mathbb{R}$.

[1.] A. Lomtatidze, Z. Opluštil On nonnegative solutions of a certain boundary value problem for first order linear functional differential equations *Electronic Journal of Qualitative Theory of Differential Equations 2003.*

[2.] A. Lomtatidze, Z. Opluštil and J. Šremr On a nonlocal boundary value problem for first order linear functional differential equations, *Mem. Differential Equations Math. Phys.*(to appear)

$$h(v) \overset{def}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}) \tag{3}$$

Main Results

Let introduce some notations. Having $\lambda > 0$ and h_0, $h_1 \in PF_{ab}$, we put

$$\alpha(\lambda, h_0) = \big(1 - h_0(1)\big)\min\Big\{1, \frac{1}{\lambda}\Big\}, \qquad \beta(\lambda, h_1) = \big(\lambda - h_1(1)\big)\min\Big\{1, \frac{1}{\lambda}\Big\} \tag{8}$$

and define the function $\omega_0(\cdot\,; h)$ by the formula

$$\omega_0(x;h) = \begin{cases} \frac{\left(x+\frac{1}{\lambda}h_0(1)\right)\left(1-h_0(1)\right)}{1-h_0(1)-x} - \left(\frac{1}{\lambda}h_1(1) + \frac{1-\lambda}{\lambda}\right) \\ \qquad \text{if } \lambda \le 1, \ \ \big(1-\lambda+h_1(1)\big)x < \big(1-h(1)\big)\big(1-h_0(1)\big) \\ \frac{\left(x+h_0(1)\right)\left(1-h_0(1)\right)}{1-h_0(1)-x} - \big(h_1(1)+1-\lambda\big) \\ \qquad \text{if } \lambda \le 1, \ \ \big(1-\lambda+h_1(1)\big)x \ge \big(1-h(1)\big)\big(1-h_0(1)\big) \\ \frac{\left(x+\lambda-1+h_0(1)\right)\left(1-h_0(1)\right)}{1-h_0(1)-\lambda x} - h_1(1) \\ \qquad \text{if } \lambda > 1, \ \ \lambda h_1(1)x < \big(1-h(1)\big)\big(1-h_0(1)\big) \\ \frac{\left(x+\frac{\lambda-1}{\lambda}+\frac{1}{\lambda}h_0(1)\right)\left(1-h_0(1)\right)}{1-h_0(1)-\lambda x} - \frac{1}{\lambda}h_1(1) \\ \qquad \text{if } \lambda > 1, \ \ \lambda h_1(1)x \ge \big(1-h(1)\big)\big(1-h_0(1)\big) \end{cases} . \tag{9}$$

$$u'(t) = F(u)(t), \qquad (1)$$

$$u(a) = h(u) + \varphi(u), \qquad (2)$$

$$h(v) \overset{def}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}) \qquad (3)$$

THEOREM 1. Let functional h be defined by the formula (3), where $\lambda > 0$ and h_0, $h_1 \in PF_{ab}$ are such that $h(1) \le 1$, $h_0(1) < 1$, $h_1(1) \le \lambda$. Let, moreover, the condition

$$\varphi(v) \operatorname{sgn} v(a) \le c \quad \text{for } v \in C([a,b];\mathbb{R})$$

be satisfied with some $c \in \mathbb{R}_+$ and there exist ℓ_0, $\ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^1_{hc}([a,b];\mathbb{R})$, the inequality

$$(F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)) \operatorname{sgn} v(t) \le q(t, ||v||_C) \text{ for } t \in [a,b].$$

holds. If, in addition,

$$||\ell_0(1)||_L < \alpha(\lambda, h_0),$$

$$\omega_0(||\ell_0(1)||_L, h) < ||\ell_1(1)||_L < 2\sqrt{\alpha(\lambda, h_0) - ||\ell_0(1)||_L} - h_1(1)\ \min\left\{1, \frac{1}{\lambda}\right\},$$

then the problem (1), (2) has at least one solution.

$$h(v) \stackrel{def}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}) \quad (3)$$

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))) \quad (4)$$

$$u(a) = h(u) + \varphi(u) \qquad (2)$$

Corollary 1. Let functional h be defined by the formula (3), where $\lambda > 0$ and $h_0,\ h_1 \in PF_{ab}$ are such that conditions $h(1) \leq 1$, $h_0(1) < 1$, $h_1(1) \leq \lambda$,

$$\varphi(v) \operatorname{sgn} v(a) \leq c \quad \text{for } v \in C([a,b];\mathbb{R})$$

are fulfilled and $f(t,x,y) \operatorname{sgn} x \leq q(t)$, for $t \in [a,b]$, $x, y \in \mathbb{R}$, where $q \in L([a,b];\mathbb{R}_+)$. Let moreover

$$\int_a^b p(s)\,ds < \alpha(\lambda, h_0)$$

$$\omega_0\Big(\int_a^b p(s)\,ds\Big) < \int_a^b g(s)\,ds < 2\sqrt{\alpha(\lambda, h_0) - \int_a^b p(s)\,ds} \;-\; h_1(1)\min\{1, \frac{1}{\lambda}\}.$$

Then the problem (4), (2) has at least one solution.

$$u'(t) = F(u)(t), \qquad (1)$$

$$u(a) = h(u) + \varphi(u), \qquad (2)$$

$$h(v) \stackrel{def}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}) \qquad (3)$$

THEOREM 2. Let functional h be defined by the formula (3), where $\lambda > 0$ and h_0, $h_1 \in PF_{ab}$ satisfy the condition $h(1) \leq 1$, $h_0(1) < 1$, $h_1(1) \leq \lambda$. Let, moreover, the condition

$$\varphi(v) \operatorname{sgn} v(b) \geq -c \quad \text{for } v \in C([a,b];\mathbb{R})$$

be satisfied with some $c \in \mathbb{R}_+$ and there exist $\ell_0, \ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$, the inequality

$$(F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)) \operatorname{sgn} v(t) \geq -q(t, ||v||_C) \text{ for } t \in [a,b].$$

holds. If, in addition,

$$||\ell_1(1)||_L < \beta(\lambda, h_1),$$

$$\frac{\alpha(\lambda, h_0)}{\beta(\lambda, h_1) - ||\ell_1(1)||_L} - 1 < ||\ell_0(1)||_L < 2\sqrt{\beta(\lambda, h_1) - ||\ell_0(1)||_L} - h_0(1) \min\left\{1, \frac{1}{\lambda}\right\}$$

then the problem (1), (2) has at least one solution.

$$h(v) \overset{def}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}) \quad (3)$$

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))) \quad (4)$$

$$u(a) = h(u) + \varphi(u) \qquad (2)$$

COROLLARY 2. Let functional h be defined by the formula (3), where $\lambda > 0$ and $h_0,\ h_1 \in PF_{ab}$ are such that conditions $h(1) \le 1$, $h_0(1) < 1$, $h_1(1) \le \lambda$,

$$\varphi(v) \operatorname{sgn} v(b) \ge -c \quad \text{for } v \in C([a,b];\mathbb{R})$$

are fulfilled and $f(t,x,y) \operatorname{sgn} x \ge -q(t)$, for $t \in [a,b]$, $x, y \in \mathbb{R}$ where $q \in L([a,b];\mathbb{R}_+)$. Let moreover

$$\int_a^b g(s)\,ds < \alpha(\lambda, h_0)$$

$$\frac{\alpha(\lambda, h_0)}{\beta(\lambda, h_1) - \int_a^b g(s)\,ds} - 1 < \int_a^b p(s)\,ds < 2\sqrt{\beta(\lambda, h_1) - \int_a^b p(s)\,ds} - h_0(1) \min\left\{1, \frac{1}{\lambda}\right\}.$$

Then the problem (4), (2) has at least one solution.

$$u'(t) = F(u)(t), \qquad (1)$$

$$u(a) = h(u) + \varphi(u), \qquad (2)$$

$$\mathcal{B}^1_{hc} = \{u \in C([a,b];\mathbb{R}) : \ (u(a) - h(u)) \operatorname{sgn} u(a) \leq c\}, \ \text{where } h \in F_{ab}$$

THEOREM 3. Let $\lambda > 0$,

$$\big(\varphi(v) - \varphi(w)\big) \operatorname{sgn}(v(a) - w(a)) \leq 0 \ for \ v, w \ \in C([a,b];\mathbb{R})$$

and let there exists $\ell_0, \ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^1_{hc}([a,b];\mathbb{R})$, where $c = |h(0)|$, the inequality

$$\big(F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big) \operatorname{sgn}(v(t) - w(t)) \leq 0$$

holds. Let moreover

$$||\ell_0(1)||_L < \alpha(\lambda, h_0)$$

$$\omega_0(||\ell_0(1)||_L, h) < ||\ell_1(1)||_L$$

$$||\ell_1(1)||_L < 2\sqrt{\alpha(\lambda, h_0) - ||\ell_0(1)||_L} - h_1(1) \ \min\left\{1, \frac{1}{\lambda}\right\}.$$

Then the problem (1),(2) is uniquely solvable.

$$u'(t) = F(u)(t), \tag{1}$$

$$u(a) = h(u) + \varphi(u), \tag{2}$$

$$\mathcal{B}^2_{hc} = \{u \in C([a,b];\mathbb{R}) : \ (u(a) - h(u)) \operatorname{sgn} u(a) \geq -c\}, \text{ where } h \in F_{ab}$$

THEOREM 4. Let $\lambda > 0$,

$$\big(\varphi(v) - \varphi(w)\big) \operatorname{sgn}\left(v(b) - w(b)\right) \geq 0 \ for \ v, w \ \in C([a,b];\mathbb{R})$$

and let there exists $\ell_0, \ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$, where $c = |h(0)|$ the inequality

$$\big(F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big) \operatorname{sgn}\left(v(t) - w(t)\right) \geq 0$$

holds. Let moreover

$$||\ell_1(1)||_L < \beta(\lambda, h_1),$$
$$\frac{\alpha(\lambda, h_0)}{\beta(\lambda, h_1) - ||\ell_1(1)||_L} - 1 < ||\ell_0(1)||_L$$
$$||\ell_0(1)||_L < 2\sqrt{\beta(\lambda, h_1) - ||\ell_0(1)||_L} - h_0(1) \min\left\{1, \frac{1}{\lambda}\right\}$$

Then the problem (1),(2) is uniquely solvable.

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