F. NEUMAN:

MATHEMATICAL STRUCTURES GENERATED BY DIFFERENTIAL EQUATIONS

HEJNICE, September 2007

I. History

Individual objects: matrices, surfaces, diff. eqs., etc.

Connections between objects, deformations, transformations

For linear ODE

1834 E. E. Kummer 2nd order

$$y'' + p_1(x)y' + p_0(x)y = 0 \qquad y(x) \downarrow$$

$$z'' + q_1(t)z' + q_0(t)z = 0 \qquad z(t) = f(t)\cdot y(h(t))$$

nth order

E. Laguerre, A. R. Forsyth, F. Brioschi, G. H. Halphen, ...: invariants, canonical form

$$y^{(n)} + 0.y^{(n-1)} + 0.y^{(n-2)} + p_{n-3}(x)y^{(n-3)} + \dots + p_0(x)y = 0$$

1892/93 P. Stäckel & S. Lie: the most general tr.

1910 G. D. Birkhoff LOCAL

1931 G. Mammana: factorization ($L_3 = L_1 \circ L_2$)

1948 G. Sansone: 3rd order - all oscillatory

1950 O. Borůvka 2nd order GLOBAL

analytic, geometric, algebraic approach

M. Greguš, M. Ráb, V. Šeda, M. Švec,

E. Barvínek, M. Zlámal, Z. Hustý, S. Staněk, ...

Theory of global transformations of linear ODE:

ANALYSIS: effective criterion of global equivalence explicit description of stationary groups

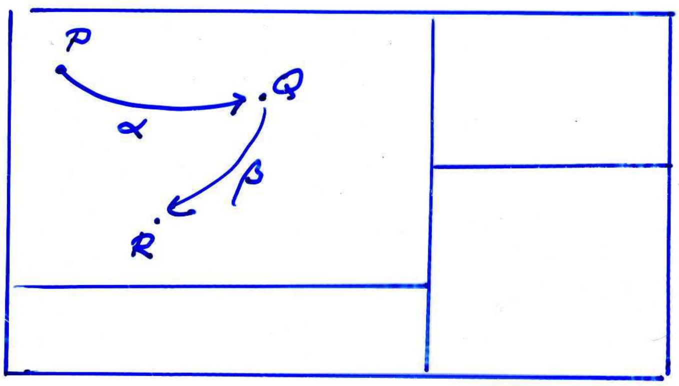
ALGEBRA: categorial approach, Brand and Ehresmann groupoids, coordinate approach

GEOMETRY: representation of solution spaces, zeros, Cartan's moving-frame-of-reference

FUNCTIONAL EQUATIONS: iteration theory, Abel and Euler functional equations

II. Algebraic structures

Objects $P, Q, R, \dots$



morphism $\alpha \in Hom(P,Q)$

$P\alpha = Q$

composition $\beta \in Hom(Q,R)$

$\alpha\beta \in Hom(P,R)$

Axioms:

identity: $\iota_P \alpha = \alpha$

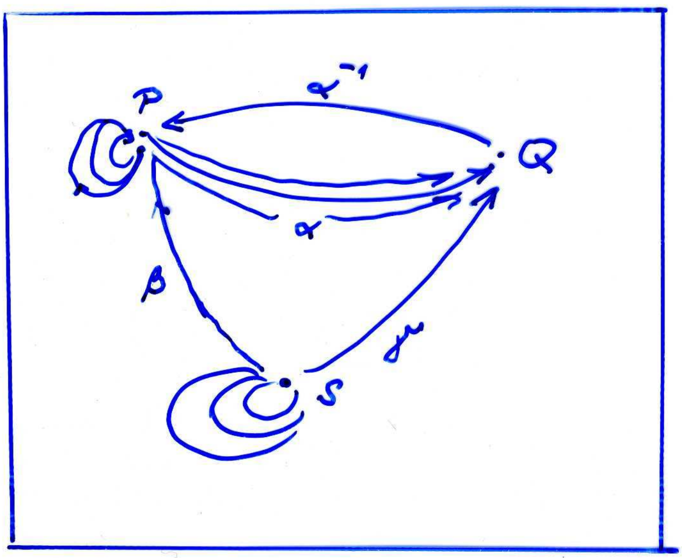
associativity: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

i.e. CATEGORY

inverse element α^{-1}: $\alpha\alpha^{-1} = \iota_P$, $\alpha^{-1}\alpha = \iota_Q$

EHRESMANN GROUPOID

$Hom(P,Q) \neq \emptyset$ BRANDT GROUPOID



S special object

canonical form

Stationary groups (conjugate)

Problems: 1. Criterion of equivalence

2. Canonical forms

3. Stationary groups

4. Special problems (zeros of solutions)

III. Global transformations

$$P \equiv y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \dots + p_0(x)\,y = 0 \quad \text{on } I \subset \mathbb{R},\ n \geq 2$$

$$Q \equiv z^{(n)} + q_{n-1}(t)\,z^{(n-1)} + \dots + q_0(t)\,z = 0 \quad \text{on } J \subset \mathbb{R}$$

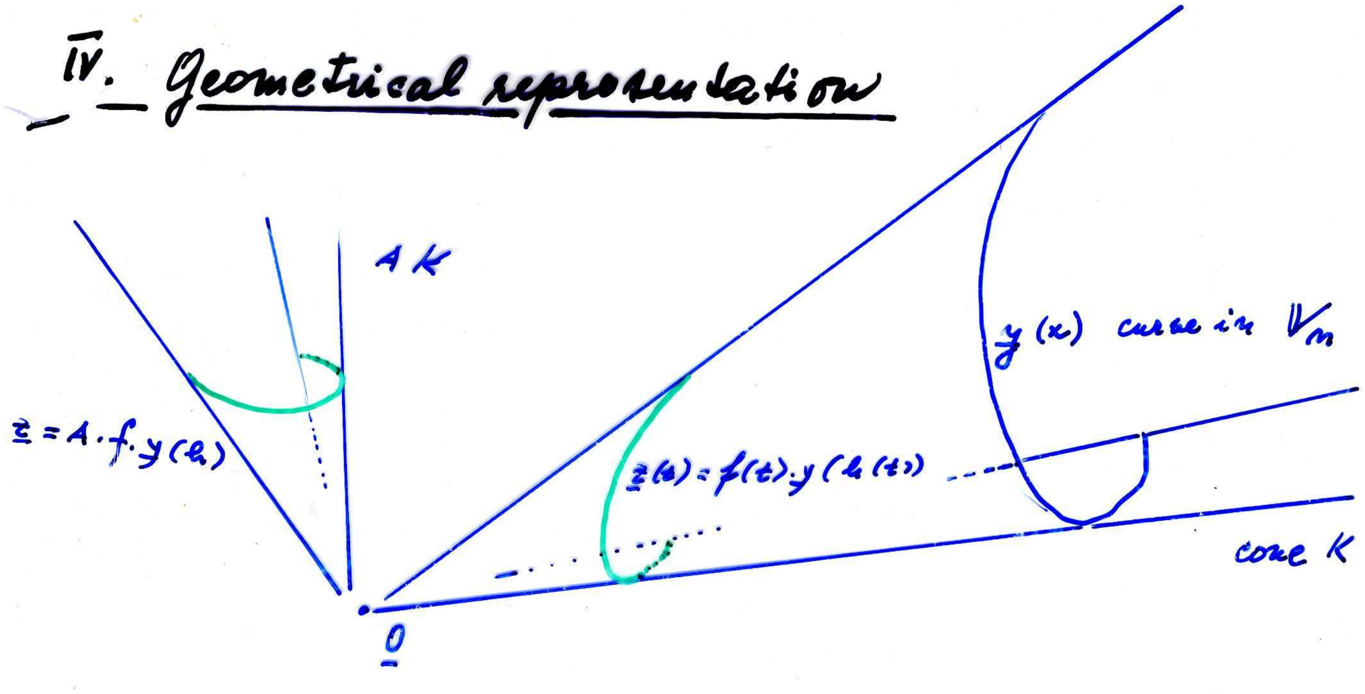
$$\alpha \equiv \quad z(t) = f(t)\,y(h(t)), \quad \text{GLOBAL}: h(J) = I,\ f \neq 0,\ f \in C^n,\ h' \neq 0,\ h \in C^n$$

$$P \equiv \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \text{ lin. ind. sols}, \quad Q \equiv \underline{z} = \begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix}$$

$n \times n$ const. reg.

$$\alpha \equiv \quad \underline{z}(t) = A \cdot f(t) \cdot \underline{y}(h(t))$$

IV. Geometrical representation



V. Criterion of global equivalence

2nd order: Borůvka 1967: eqs P and Q are gl. equiv. $\Leftrightarrow$ the same number of zeros of solutions (noneffective)

nth order ($n \geq 3$): 1984: in general effective conditions on coefficients of P and Q

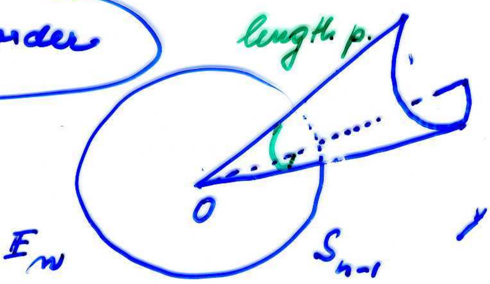
VI. Canonical forms - global

2nd order: Borůvka: $y'' + y = 0$ on

$(0, \pi/2)$, $(0, \pi)$, $(0, 3\pi/2)$, ..., $(0, k\pi/2)$, ..., $(0, \infty)$, $(-\infty, \infty)$

1 zero, 1 zero, 2 zeros, one-side, both-side oscill.

nth order



$y'' + y = 0$ on I

$$y''' - \frac{k'(x)}{k(x)} y'' + (1 + k^2) y' - \frac{k'}{k} y = 0$$

$k > 0$, $k \in C^1(I)$

...

or

$$y^{(n)} + 0 \cdot y^{(n-1)} + 1 \cdot y^{(n-2)} + p_{n-3}(x) y^{(n-3)} + \dots + p_0(x) y = 0 \quad \text{on } I$$

1 0 $\cancel{0}$ 1

VII. Stationary groups

2nd order: Borůvka 1967: dispersions

nth order: 1984: 10 types (with countable many subtypes) from 3 parameter groups .... to identity

VIII. Special problems

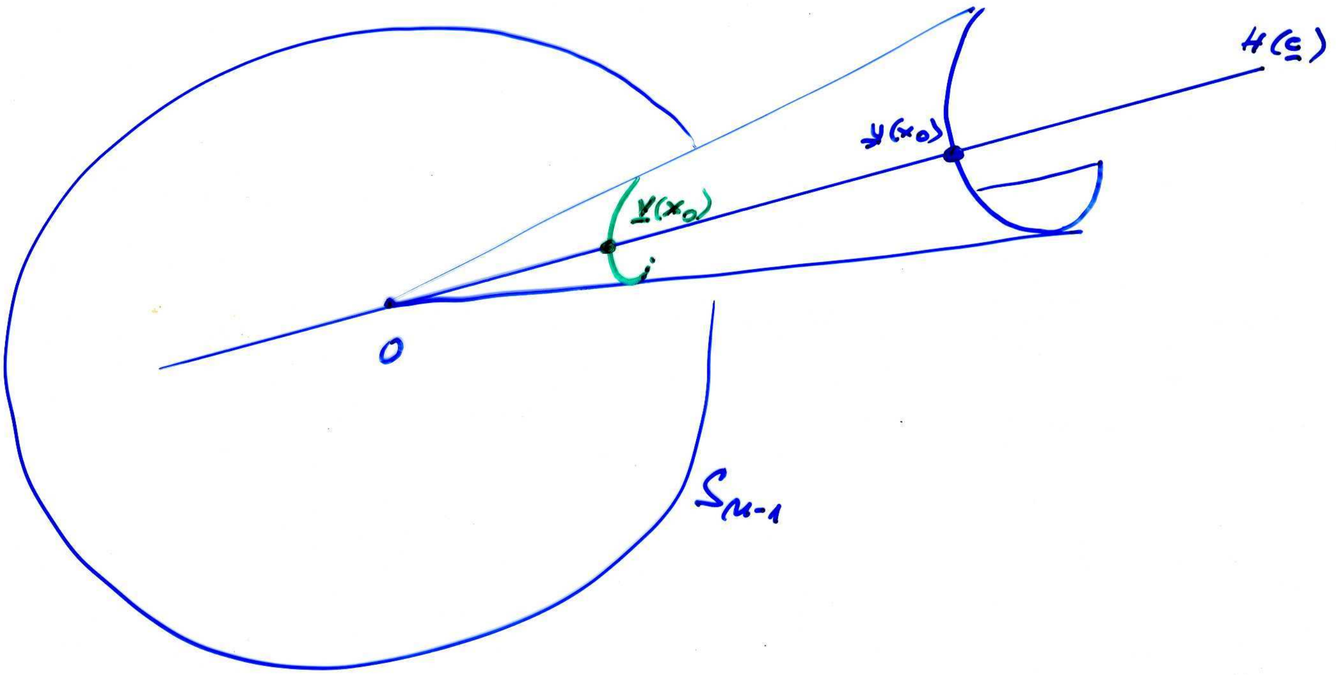
A. zeros of solutions

$$y(x_0) = c_1 y_1(x_0) + \dots + c_n y_n(x_0) = 0$$

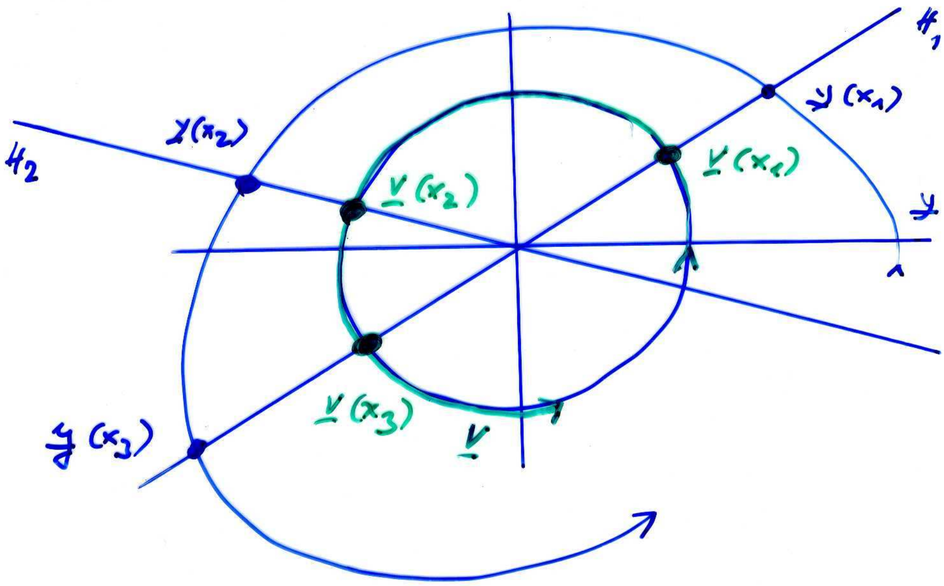
x_0 zero of $y(x)$

curve $y(x)$ intersects the hyperplane

$H(\underline{c}) \equiv c_1 \xi_1 + \dots + c_n \xi_n = 0$ at the point of parameter x_0



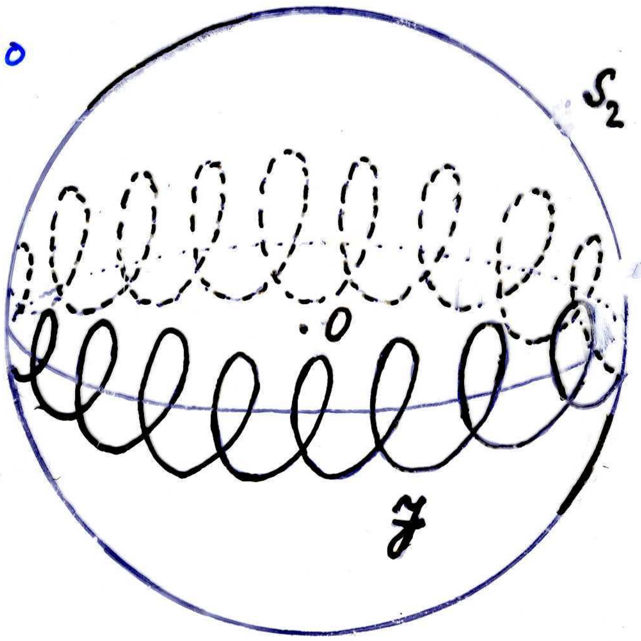
$n = 2$ $\quad \det(y(x), y'(x)) \neq 0$



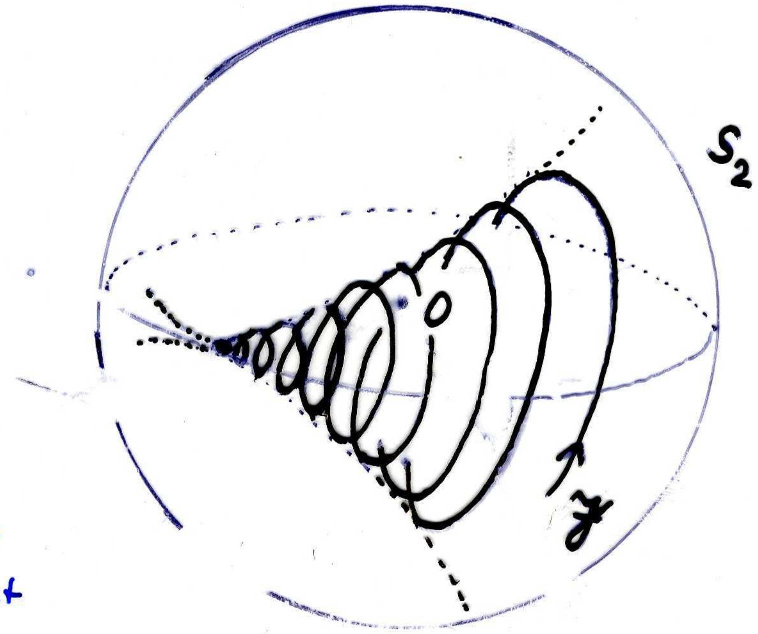
$n = 3$

$\det(y(x), y'(x), y''(x)) \neq 0$

no inflexions



all oscillatory solutions

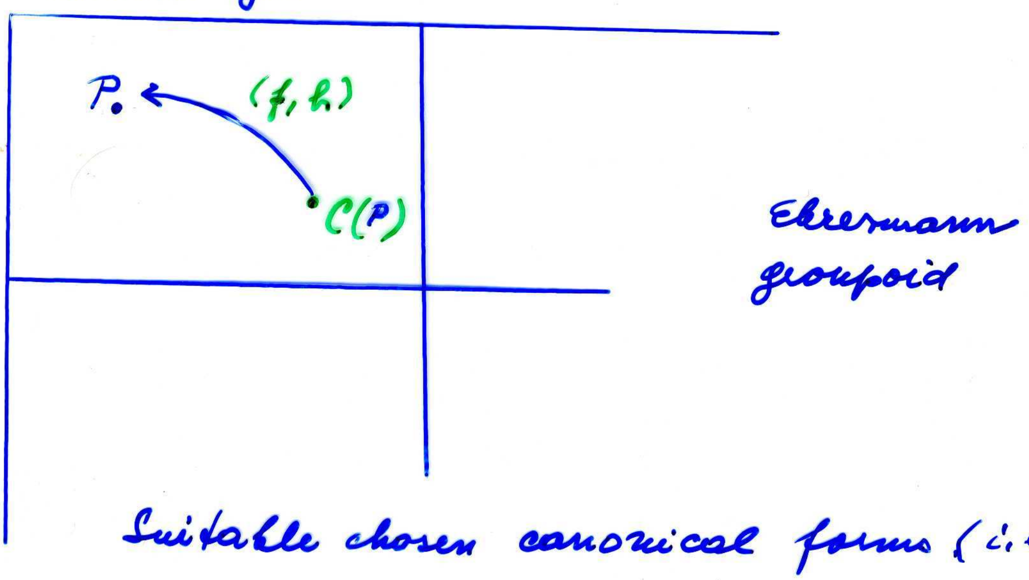


1 oscill. sol.
not constant coefficients

B. Coordinate approach

eg. P = its canonical form & transformation
$C(P)$ (f, h)

Brandt gr.



Suitable chosen canonical forms (i.e. projection on S_{n-1} and length parametrization):
property of solutions of P <=>
conditions on f and h

Explicit constructions of all equations with solutions of prescribed properties : L_2, L_p, boundedness, tending to zero, etc.

Zero properties of solutions and their asymptotic behavior lead to ABEL functional equation

IX. Further research - open problems

Basic approach:

objects   functional diff. eqs - linear
- nonlinear
- 1st, 2nd, ... order

delays: 1, 2, ...

systems

difference eqs. & systems

morphisms   defining better by properties:
the most general form keeping ....

NICE: geometric representation of solutions,
of objects,
of morphisms

correspondence between objects and solution spaces

ALWAYS PROBLEMS:

Criterion of equivalence
Canonical forms
Stationary groups
Invariants
Construction of objects with given properties

I WISH YOU FULL SUCCESS

Thank you for your attention

MATHEMATICAL STRUCTURES GENERATED BY DIFFERENTIAL EQUATIONS

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A survey based on presentations at several recent meetings

I. Historical remarks

The early beginning of the study of differential equations is closely connected with the discovery and development of the infinitesimal calculus by G. W. Leibnitz (1646 – 1716) and I. Newton (1643 – 1727). The integration factor and the method of variation of parameters were introduced by J. Bernoulli in 1691 and 1693, the special equation

$$y' = a(x)y^2 + b(x)y + c(x)$$

was studied by J. P. Riccati in 1724, solutions of linear equations with constant coefficients were discovered by L. Euler in 1750. Among significant contributors of this period there were Ch. Huygens, J. L. d'Alembert, A. C. Clairaut, J. Wallis, B. Taylor, J. Stirling, C. MacLaurin, P. S. Laplace, J. L. Lagrange, G. Monge, J. and D. Bernoulli, J. Liouville, E. Weyr, A. Cauchy, and S. Lie. The study of differential equations was often connected with problems in physics, astronomy, engineering and also with the development of other parts of mathematics, especially geometry.

In each area of mathematics there is a significant step consisting in investigating not only particular, single, individual objects, but in considering connections among these objects, such as transformations, deformations of the objects one into another.

For linear differential equations this step was done in 1834 by E. E. Kummer [**4**], who was the first who considered a transformation, a substitution of the form

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$$z(t) = f(t)y(h(t)) \tag{1}$$

converting solutions $y = y(x)$ of a second order linear differential equation

$$y'' + p_1(x)y' + p_0(x)y = 0$$

into solutions $z = z(t)$ of another equation of the same kind,

$$z'' + q_1(t)z' + q_0(t)z = 0.$$

Nonlinear 3rd order equations expressing the relations among the coefficients of the equations and involving functions f and h from the transformation are now called the Kummer equations as well as the transformation itself.

Also higher order linear differential equations, their invariants and canonical forms were studied by F. Brioschi, A. R. Forsyth, E. Laguerre, E. Forsyth [**3**], just to mention only some of them. They considered the transformation (1) still involving two functions as already introduced by Kummer. Perhaps the best known result from the second half of the XIXth century is the so-called Laguerre-Forsyth canonical form of linear differential equations of the n-th order

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_0(x)y = 0$$

characterized by vanishing of the coefficients of the $(n-1)$st and $(n-2)$nd derivatives of the independent variables, i.e. equations of the form

$$y^{(n)} + p_{n-3}(t)y^{(n-3)} + \cdots + p_0(t)y = 0.$$

However, it was not until 1893 that P. Stächel [**13**] and independently S. Lie [**5**] proved that the transformation considered by Kummer and all his successors is in fact the most general pointwise transformation that under a differentiability condition converts any linear homogeneous differential equation of the nth order, $n \geq 2$, into an equation of the same kind. Only their result gave the justification to all the previous investigations because, basically, Kummer and others exploited only the fact that linearity and homogeneity of equations are preserved. But, still differentiability conditions remained.

At the beginning of the last century, in 1910, G. D. Birkhoff [**1**] presented an example of a third order equation not transformable into the Laguerre-Forsyth canonical form on its whole interval of definition. In fact he pointed out that the previous investigations were of local character as a whole. This was not very encouraging, since many important questions require global investigations. Local methods and results are not sufficient when studying problems of a global nature, such as boundedness, periodicity, asymptotic and oscillatory behavior, nonvanishing solutions, and consequently the factorization of linear differential operators, as well as many other questions.

Of course, there were isolated results of a global character, like Sturm Separation and Comparison Theorems on zeros of solutions of the second order linear differential equations, and other particular results. However, there was not a unified theory offering sufficiently general methods and dealing systematically with global behavior of solutions. To demonstrate it, let us mention G. Sansone's [**12**] example of a third order linear differential equation with all oscillatory solutions that occurred as late as in 1948, 17 years after G. Mammana [**6**] in 1931 showed that the non-existence of such an equation would have been a basic (sufficient and necessary) condition for factorization of linear differential operators (of the third order).

On two examples, namely on the canonical equations and on the description of distribution of zeros of solutions, we want to show that there was indeed an absence of a general method for solving global problems of linear differential equations, whose answers were not hidden in old papers.

In the fifties O. Borůvka started the systematic study of global properties of the second order linear differential equations,

$$y'' + q(x)y = 0, \qquad q \in C^0(a,b), \qquad -\infty \le a < b \le \infty,$$

the simplest ones from those whose solutions are not available in a "closed form", still having an extensive literature. He carried out an in-depth investigation and summarized his original methods and results in his monograph [**2**].

From then an intensive research of linear differential equations of an arbitrary order was carried out and resulted in developing sufficiently general methods and results describing global properties of these equations. It is important to mention that not only analytic methods were involved in those investigations, but also algebraic, topological and geometric tools, including differential geometric ones, together with methods and results of the theory of dynamic systems and of functional equations played an essential role.

II. Definition of global transformations

Consider a linear differential equation

$$P \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \qquad \text{on} \qquad I,$$

I being an open interval of the reals, p_i are real continuous functions defined on I for $i = 0, 1, \ldots, n-1$, i.e. $p_i \in C^0(I), p_i : I \to \mathbb{R}, n \ge 2$. Denote also

$$Q \equiv z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0 \qquad \text{on} \qquad J,$$

another example of an equation of this form.

Definition 1. We say that the equation P is *globally transformable* into the equation Q if there exist two functions,

$$f \in C^n(J), f(t) \neq 0 \ \text{ and } \ h \in C^n(J), h'(t) \neq 0 \ \text{ for each } \ t \in J, \quad \text{and} \quad h(J) = I,$$

such that whenever $y : I \to \mathbb{R}$ is a solution of P, then

$$z : J \to \mathbb{R}, \quad z(t) := f(t) \cdot y(h(t)), \quad t \in J, \tag{1}$$

is a solution of Q. This transformation is *global* in the sense that solutions are transformed on their whole intervals of definition, $h(J) = I$.

III. Algebraic approach to global transformations

A class is called a *category* if to each pair of its elements P, Q, *objects,* a set $\mathrm{Hom}(P, Q)$ of *morphisms* is assigned such that the following axioms are satisfied:

1. The sets $\mathrm{Hom}(P, Q)$ are disjoint for different pairs (P, Q).

2. A composition $\alpha\beta \in \mathrm{Hom}(P, T)$ is defined for each $\alpha \in \mathrm{Hom}(P, Q)$ and $\beta \in \mathrm{Hom}(Q, T)$ such that

a) the associativity $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds for each $\gamma \in \mathrm{Hom}\ (T, U)$,

b) there exists an identity ι_P for each object P, ι_Q for Q:

$$\iota_P\alpha = \alpha, \qquad \alpha\iota_Q = \alpha \text{ for each } \alpha \in \mathrm{Hom}\ (P, Q).$$

A category is an *Ehresmann groupoid* if each morphism α has an inverse $\alpha^{-1} \in \mathrm{Hom}(Q, P)$:

$$\alpha\alpha^{-1} = \iota_P, \qquad \alpha^{-1}\alpha = \iota_Q.$$

Moreover, an Ehresmann groupoid is a *Brandt groupoid* if $\mathrm{Hom}(P, Q)$ is not empty for any pair (P, Q) of its objects P, Q.

An Ehresmann groupoid is a collection of connected components, Brandt groupoids, also called *classes of equivalent objects.* The set $\mathrm{Hom}(P, P)$ is a group, a *stationary group* of the object P.

Historically one may observe that the following problems were studied when Ehresmann groupoids occurred:

Criterion of equivalence.

Canonical objects in Brandt groupoids.

Structure of stationary groups.

Here we would like to mention that also a new feature has recently been considered in this general algebraic approach, namely

Coordinate approach
involving analytic constructions of objects with some prescribed properties. We will explain it in more detail in Par. IX. There are also

Special problems depending on particular objects.

Proposition 1. *From this algebraic point of view we may consider global transformations* (1) *attached to equation* (P) *as morphisms* $Hom(P, Q)$ *of the Ehresmann groupoid with linear differential equations greater than 2 as its objects.*

Notation. We use the notation $\alpha = (f, h)$ in (1) and write also $P\alpha = Q$.

IV. Geometric approach to global transformations

Let $\mathbf{y}(x) = (y_1(x), \ldots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation P considered as a column vector function or as a curve in n-dimensional Euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \ldots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t)^T$ denotes an n-tuple of linearly independent solutions of the equation Q, then the global transformation (1) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(t))$$

or, for an arbitrary regular constant $n \times n$ matrix A ,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(t)),$$

expressing that another n-tuple of linearly independent solutions of the *same* equation Q is taken. We may consider the following geometric representation of classes of equivalent equations, i.e. Brandt groupoids of the Ehresmann groupoid of linear differential equations:

Proposition 2. *All curves representing all n-tuples of linearly independent solutions of all equations globally equivalent to an equation P with an n-tuple of linearly independent solutions (a curve)* $\mathbf{y}$ *are obtained as sections (given by f) in a certain parameterization (given by h) of a centroaffine image (determined by A) of a fixed cone K formed by half-lines going from the origin and the curve* $\mathbf{y}$.

V. Criterion of global equivalence

For the **second order equations** this criterion was proved by O. Borůvka [**2**] in 1967:

Proposition 3. *Two second order equations are globally equivalent if and only if their solutions have the same number of zeros.*

In general this criterion is not effective, i.e. it cannot be expressed by quadratures from coefficients of equations under consideration.

In 1984 we derived the criterion for the **n-th order equations, n≥3,** which is in general effective, see [**8, 9**].

VI. Global canonical forms

For the **second order equations** the following canonical forms were considered by O. Borůvka [**2**] as (a countable set of) equations

Now, we can formulate Borvka's criterion of a global equivalence of the second order linear differential equations:

Equations (p) *and* (q) (always considered with their intervals of definition) *are globally equivalent if and only if they are of the same type and at the same moment of the same character.*

Canonical equations representing all classes of globally equivalent second order differential equations are chosen as follows:

$y'' + y = 0$	on	$(0, \pi/2)$	type 1,	general
$y'' + y = 0$	on	$(0, \pi)$	type 1,	special
$y'' + y = 0$	on	$(0, 3\pi/2)$	type 2,	general
$y'' + y = 0$	on	$(0, 2\pi)$	type 2,	special
...			...	...
$y'' + y = 0$	on	$(0, (m-1/2)\pi)$	type m,	general
$y'' + y = 0$	on	$(0, m\pi)$	type m,	special
...			...	...
$y'' + y = 0$	on	$(0, \infty)$	type ∞,	one-side oscillatory
$y'' + y = 0$	on	$(-\infty, \infty)$	type ∞,	both-side oscillatory.

Remember that equations are considered globally, they are taken together with their intervals of definition.

These intervals of definition express the precise meaning of the number of zeros of solutions mentioned in Proposition 3.

Canonical forms for **higher order equations** may be chosen as

$$\mathbf{1} \cdot y^{(n)} + \mathbf{1} \cdot y^{(n-2)} + p_{n-3}(x) y^{(n-3)} + \cdots + p_0(x) y = 0$$

on certain set of intervals. These equations are characterized by the first three coefficients : $(\mathbf{1}, \mathbf{0}, \mathbf{1})$. If Laguerre and Forsyth had taken 1 instead of their 0 as the coefficient of $y^{(n-2)}$, they would have obtained global canonical forms.

Proof is based on **algebraic** properties of iterative differential equations, for details see [**9**].

Another approach exploits the **geometric** representation as in Par. IV., and the Cartan moving-frame-of-reference method.

For $\mathbf{y}(x)$ define the n-tuple $\mathbf{v} = (v_1, \ldots, v_n)^T$ in the Euclidean space $\mathbb{E}_n$,

$$\mathbf{v}(x) := \mathbf{y}(x)/\|\mathbf{y}(x)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm. It was shown ([**9**]), that $\mathbf{v} \in C^n(I), \mathbf{v} : I \to \mathbb{E}_n$, and the Wronski determinant of $\mathbf{v}$, $W[\mathbf{v}] := \det(\mathbf{v}, \mathbf{v}', \ldots, \mathbf{v}^{(n-1)})$, is different from zero on I. Of course, $\|\mathbf{v}(x)\| = 1$, i.e. $\mathbf{v}(x) \in \mathbb{S}_{n-1}$, where $\mathbb{S}_{n-1}$ is the unit sphere in $\mathbb{E}_n$. Evidently, the differential equation which has this $\mathbf{v}$ as its n-tuple of linearly independent solutions is globally equivalent to P. Moreover, if

$$\mathbf{u}(s) := \mathbf{v}(g(s)),$$

where the function g satisfies

$$g(s) : K \to I \subseteq \mathbb{R}, \quad g(K) = I, \quad |(g^{-1}(x))'| = \|\mathbf{v}'(x)\|$$

$$\text{for the inverse } g^{-1} \text{ to } g, \text{ and hence } g \in C^n(K), g'(s) \neq 0 \quad \text{on} \quad K,$$

then we have $\|\mathbf{u}'(s)\| = 1$, i.e. this $\mathbf{u}$ is the *length* reparameterization of the curve $\mathbf{v}$. Of course, $\|\mathbf{u}(s)\| = \|\mathbf{v}(g(s))\| = 1$.

Canonical equations (from this geometric point of view) are characterized (and defined) as those linear differential equations admitting n-tuples of linearly independent solutions $\mathbf{u}$ satisfying

$$\|\mathbf{u}(s)\| = 1, \quad \|\mathbf{u}'(s)\| = 1.$$

The explicit expression for canonical equations can be obtained by the following procedure. The vector function $\mathbf{u} : K \to \mathbb{R}^n$ satisfies the Frenet system of differential equations for the frame $(\mathbf{u}_1, \cdots, \mathbf{u}_n)$, where $\mathbf{u}_1$ stands instead of $\mathbf{u}$:

Definition 2. *Canonical equations* are characterized and defined as those linear differential equations admitting n-tuples of linearly independent solutions $\mathbf{u}$ satisfying

$$\|\mathbf{u}(s)\| = 1, \quad \|\mathbf{u}'(s)\| = 1. \tag{1}$$

The explicit expression for canonical equations can be obtained by the following procedure. The vector function $\mathbf{u} : K \to \mathbb{R}$ satisfies the Frenet system of differential equations, if we write $\mathbf{u}_1$ instead of $\mathbf{u}$:

$$\begin{aligned}
\mathbf{u}_1' &= \mathbf{u}_2 \\
\mathbf{u}_2' &= -\mathbf{u}_1 + k_1(s)\mathbf{u}_3 \\
\mathbf{u}_3' &= -k_1(s)\mathbf{u}_2 + k_2(s)\mathbf{u}_4 \\
&\vdots \\
\mathbf{u}_{n-1}' &= -k_{n-3}(s)\mathbf{u}_{n-2} + k_{n-2}(s)\mathbf{u}_n \\
\mathbf{u}_n' &= -k_{n-2}(s)\mathbf{u}_{n-1} \ ,
\end{aligned} \tag{2}$$

$s \in K$, with (curvatures) $k_i \in C^{n-i-1}$, $\ k_i(s) \neq 0$ on K for $i = 1, \dots, n-2$.

The canonical differential equation corresponding to $\mathbf{u}$ is the nth order linear differential equation for $\mathbf{u}_1$ obtained by eliminating other $\mathbf{u}_i$ from the above system (2). The coefficients of this equation are formed from the curvatures k_i.

The canonical differential equation corresponding to $\mathbf{y}$ is the n-th order linear differential equation, the solutions of which are n linearly independent components of the vector function $\mathbf{u}_1$, obtained by eliminating other $\mathbf{u}_i$'s from the above system. The coefficients of this equation are formed from the curvatures k_i. E.g., if we write u instead of $\mathbf{u}_1 (= \mathbf{u})$, or any component of it, we get

for $n = 2$

$$u'' + u = 0 \ ,$$

for $n = 3$

$$u''' - \frac{k_1'(s)}{k_1(s)} u'' + (1 + k_1^2(s))u' - \frac{k_1'(s)}{k_1(s)} u = 0 \quad ,$$

$$\dots,$$

on (different) intervals $K \subset \mathbb{R}$, for details and proofs to the above considerations see [**9**].

VII. Stationary groups

These groups for the second order equations were described in 1967 by O. Borůvka [**2**]. He called their elements *dispersions.*

A characterization of stationary groups for the n-th order equations was given in 1984 by using a criterion of global equivalence for these equations, see [**8**]. There are, up to conjugacy, 10 types of these groups (with countable many subtypes), ranging from 3-parameter groups to the trivial one. Problems of this sort initiated extensive research on simultaneous solutions of systems of Abel functional equations, iteration groups of continuous functions, and dynamical systems, [**7, 9, 15**].

VIII. Zeros of solutions

The essence of our approach to distribution of zeros is based on two readings of the following relation

$$\mathbf{c}^T \cdot \mathbf{y}(x_0) = c_1 y_1(x_0) + \cdots + c_n y_n(x_0) = 0,$$

$\mathbf{y}$ being as in Par. IV. The first meaning of the relation is:

a solution $\mathbf{c}^T \cdot \mathbf{y}(x)$ has a zero at x_0.
The second, equivalent one is:

a hyperplane $c_1\eta_1 + \cdots + c_n\eta_n = 0$ intersects the curve $\mathbf{y}(x)$ at a point of the parameter x_0.

Proposition 4. *If* $\mathbf{y}$ *is considered in the n-dimensional Euclidean space and its central projection* $\mathbf{v}$ *onto the unit sphere is taken without a change of parameterization, then parameters of intersections of* $\mathbf{v}$ *with great circles correspond to zeros of the corresponding solutions; even multiplicity of zeros occur as orders of contacts* $+1$.

By using this approach we can see, simply by drawing a curve $\mathbf{v}$ on a sphere, what is possible and what impossible, without lengthy and sometimes tiresome ϵ, δ calculations. Only $\mathbf{v}$ must be sufficiently smooth, i.e. of the class C^n for the n-th order equations with nonvanishing Wronski $\det(\mathbf{v}, \mathbf{v}', \ldots, \mathbf{v}^{(n-1)}) \neq 0$ at each point. As examples let us mention the Sturm Separation Theorem for the second order equations, equations of the third order with all oscillatory solutions (Sansone's result), or an equation of the third order with just one-dimensional subset of oscillatory solutions that cannot occur for equations with constant coefficients, for details see [**9, 14**]

IX. Coordinate approach – analytic method of constructions of objects with prescribed properties

This coordinate approach is based on the following observation. Consider an Ehresmann groupoid (e.g. linear differential equations as its objects and global transformations as morphisms).

Definition 2. Each object P is expressible by

• its canonical form (equation) R and

• the morphism (global transformation) γ transforming R into P, $R\gamma = P$. Those (R, γ) are considered as *coordinates* of P.

For suitable chosen canonical forms, qualitative behavior of objects (their solution spaces) may be expressible by certain properties of γ. Exactly this is the case, when canonical forms of linear differential equations are chosen as in Par. VI.

If we succeed to express qualitative behavior of solutions spaces equivalently by properties of global transformations γ, we can use this fact for an effective construction of equations of prescribed properties, as demonstrated in the propositions below.

Differential equations with prescribed behavior of their solutions have often been studied in the mathematical literature. Especially, the second order linear differential equations in the Jacobi form

$$y'' + q(x)y = 0 \tag{q}$$

were constructed with only bounded solutions, with all solutions in L^p (for $p = 2$ the so called limit circle case), with all solutions tending to zero as x goes to the right end of the interval of definition of (q). Also equations (q) admitting solutions with a given distribution of zeros (oscillatory or nonoscillatory behavior, disconjugacy) were described. This approach enabled us to answer some open questions from this area of research, [**9**].

Recently we have extended our coordinate approach from the second to an arbitrary order of linear differential equations, for details see [**10, 11**].

Consider a canonical equation R and its $n-$tuple of linearly independent solutions $\mathbf{u}$ satisfying

$$\|\mathbf{u}(s)\| = 1$$

as in Par. VII, and $\gamma = (f, h)$.

Proposition 5. (Bounded solutions) *All linear differential equations P equivalent to R with only bounded solutions are exactly those obtained as $P = R\gamma, \gamma = (f, h)$ with bounded f.*

Boundedness is always meant as bounded from below and above on the whole interval of definition. All linear differential equations with only bounded solution are constructed when all canonical equations are considered, since each equation belongs to a certain class of equivalent equations having always their canonical form.

Proposition 6. (Solutions tending to zero) *All solutions of an equation P equivalent to the canonical equation R tend to zero as they approach the right end of the interval of definition I exactly when equation P is obtained from equation R by means of a transformation $\gamma = (f, h), R\gamma = P$ with $f(x)$ tending to zero as x approaches the right end of I.*

All equations admitting only solutions tending to zero can be constructed in a similar way from canonical equation as mentioned above.

Proposition 7. (Solutions in $\mathbf{L}^p$) *An equation P admits only solutions in the class L^p if and only if it can be obtained from its canonical equation R by a transformation $\gamma = (f, h), P = R\gamma$ with $f \in L^p$.*

Proofs of these propositions are in [**10, 11**]. We have seen that *asymptotic properties* of solutions of a linear differential equation in a class of equivalent equations depend exclusively on the factor f in the global transformation $\gamma = (f, h)$ transforming a canonical equation of the class to a given equation.

On the contrary, *the character of distribution of the zeros* of solutions is the same for all equations in the same class of equivalence (up to metric properties that depend on h in γ). In particular, if an equation is oscillatory to both sides of its interval of definition, then all equivalent equations (including a corresponding canonical one) are oscillatory to the left and to the right. The same is true for the number of the zeros of solutions, the disconjugacy of equations, etc.

It means that for constructing equations with prescribed character of zeros of their solutions it is sufficient to have one equation of this property for the whole class of equivalent equations. By using the geometric approach described in Par. VIII., this required equations can be obtained by considering curves on unit spheres in n-dimensional Euclidean space having specified intersections with main circles on these spheres.

We have seen that problems concerning the existence or nonexistence of linear differential equations and their effective constructions can be converted to questions from the constructive theory of real functions.

X. Comments

Important questions consist in selection of morphisms for given objects. In some cases they are historically given. Another way is to choose properties that should be satisfied and to derive the most general form of morphisms that keep them unchanged, e.g. for functional difference equations.

Another problem is the selection of suitable canonical forms in particular situations. Consider for example *parameterization of functions.*

Problem.

For a nonnegative integer n we define the n-equivalence between functions f_1 and f_2 by the existence of a homeomorphism φ of the class C^n between the definition domain of f_2 and that of f_1, such that

$$f_1 \circ \varphi = f_2$$

holds.

The problem consists in finding a representative, a canonical form in each class of this equivalence. Partial answers have been known only for rather restrictive subsets of continuous functions, see [**9**]. Results for larger sets would have important consequences e.g. in construction of *unique* canonical forms for linear differential equations and their invariants.

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