Bounded Solutions for Nonlinear Difference Equations

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Dedication

Best wishes and congratulations to four musketeers

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(Svatoslav-Irena-František-Štefan’s inequality)

Motivation

- Model : Duffing difference equation :

$$\boxed{\Delta^2 x_{m-1} + c\Delta x_m + a \arctan x_m = p_m} \qquad (m \in \mathbb{Z})$$

- $c > 0, \quad a \neq 0$

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- bounded means belonging to $l^\infty_{\mathbb{Z}}$

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- Partially joint work with J.B. Baillon

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- bounded solution : $(x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$

A limiting lemma

- Lemma. *Assume that, for each* $n \in \mathbb{N}^*$, *there exists* $x^n = (x^n_m)_{-n-1 \le m \le n+1}$ *such that*

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and such that $\alpha_m \le x^n_m \le \beta_m \quad (|m| \le n+1)$ *for some* $\alpha = (\alpha_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$, $\beta = (\beta_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$.
Then there exists $\widehat{x} = (\widehat{x}_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ *such that*

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- Same result for $\boxed{\Delta^2 \widehat{x}_{m-1} + c\Delta \widehat{x}_{m-1} + f_m(\widehat{x}_m) = 0}$
- Proof. Borel-Lebesgue lemma, Cantor diagonalization

Bounded lower and upper solutions – 1

- $f_m \in C(\mathbb{R}, \mathbb{R}) \quad (m \in \mathbb{Z})$
- $\alpha = (\alpha_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ (resp. $\beta = (\beta_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$)
  bounded lower solution (resp. *upper solution*) for

$$\boxed{\Delta^2 x_{m-1} + c\Delta x_m + f_m(x_m) = 0 \quad (m \in \mathbb{Z})}$$

if $\quad \Delta^2 \alpha_{m-1} + c\Delta \alpha_m + f_m(\alpha_m) \geq 0$
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Bounded lower and upper solutions – 2

- Theorem. *If* $c \geq 0$ *(resp.* $c \leq 0$*) and*

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$$\left(\text{resp.} \quad \boxed{\Delta^2 x_{m-1} + c\Delta x_{m-1} + f_m(x_m) = 0 \quad (m \in \mathbb{Z})}\right)$$

has a lower solution $\alpha = (\alpha_m)_{m\in\mathbb{Z}}$ *and an upper solution* $\beta = (\beta_m)_{m\in\mathbb{Z}}$ *such that* $\alpha_m \leq \beta_m$ $(m \in \mathbb{Z})$, *then it has a solution* $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ *such that* $\alpha_m \leq x_m \leq \beta_m \quad (m \in \mathbb{Z})$.

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- Proof. LUS for Dirichlet

$$\Delta^2 x_{m-1} + c\Delta x_m + f_m(x_m) = 0 \quad (-n \leq m \leq n)$$
$$x_{-n-1} = \alpha_{-n-1}, \quad x_{n+1} = \alpha_{n+1}$$

for each n and limiting lemma

Constant lower and upper solutions

- Corollary. *If* $c \geq 0$ *(resp.* $c \leq 0$*)* *and if there exist real numbers* $\alpha \leq \beta$ *such that* $f_m(\beta) \leq 0 \leq f_m(\alpha)$ $(m \in \mathbb{Z})$*, then equation*

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has at least one solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ *such that* $\alpha \leq x_m \leq \beta \quad (m \in \mathbb{Z})$.

Constant lower and upper solutions

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- continuous case : $\boxed{x'' + cx' + f(t,x) = 0}$
corresponding result holds for all $c \in \mathbb{R}$
(Barbalat (1958), Opial (1958))

Interlude

Joint work

First order linear equations

- Lemma 1. *If* $c \notin \{0,2\}$ *equation*

$$\boxed{\Delta x_m + c x_m = h_m \quad (m \in \mathbb{Z})}$$

has a unique bounded solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ *for each* $h = (h_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$

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- continuous case : $\boxed{x' + cx = h(t)}$ $h \in BC(\mathbb{R})$
  result holds for each $c \neq 0$ (Perron (1930))

Sequences with bounded primitive

- Δ-primitive H^{Δ} of $h = (h_m)_{m \in \mathbb{Z}}$
- $H^{\Delta} = (H^{\Delta}_m)_{m \in \mathbb{Z}}$: $\Delta H^{\Delta}_m = h_m \quad (m \in \mathbb{Z})$

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$$H^\Delta_m = \begin{cases} \sum_{k=0}^{m-1} h_k & \text{if} \quad m \geq 1 \\ 0 & \text{if} \quad m = 0 \\ -\sum_{k=m}^{-1} h_k & \text{if} \quad m \leq -1 \end{cases} \qquad (m \in \mathbb{Z})$$

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- $BP_\mathbb{Z}$:= $\{h = (h_m)_{m\in\mathbb{Z}} : H^\Delta \in l^\infty_\mathbb{Z}\}$
- $BP_\mathbb{Z} \subsetneq l^\infty_\mathbb{Z}$

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- continuous case : $BP(\mathbb{R}) \not\subset BC(\mathbb{R})$, $BC(\mathbb{R}) \not\subset BP(\mathbb{R})$

Second order linear equations

- Proposition 1. *If* $c \notin \{-2, 0\}$, *equation*

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has a bounded solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ *if and only if* $h \in BP_\mathbb{Z}$.

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- continuous case : $\boxed{x'' + cx' = h(t) \quad h \in BC(\mathbb{R})}$

result holds when $c \neq 0$ if and only if $h \in BP(\mathbb{R})$ (Ortega (1995))

Generalized mean values

- *lower (resp. upper) mean value of* $p = (p_j)_{j\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$
  $\widehat{p} := \lim_{n\to\infty} \inf_{m-k\geq n} \left(\frac{1}{m-k} \sum_{j=k+1}^{m} p_j \right)$
  $\Big($resp. $\widetilde{p} := \lim_{n\to\infty} \sup_{m-k\geq n} \left(\frac{1}{m-k} \sum_{j=k+1}^{m} p_j \right)\Big)$

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- Lemma. *The following statements are equivalent :*
  (i) $\alpha < \widehat{p} \leq \widetilde{p} < \beta.$
  (ii) there exists $p^* \in BP_{\mathbb{Z}}$, $p^{**} \in l^\infty_{\mathbb{Z}}$ *such that*
  $p = p^* + p^{**}$ *and* $\alpha < \inf_{k\in\mathbb{Z}} p^{**}_k \leq \sup_{k\in\mathbb{Z}} p^{**}_k < \beta$

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- Special case. *If* $\widetilde{p} = \widetilde{p} = 0,$ *then, for each* $\epsilon > 0$ *there exists* $p^* \in BP_\mathbb{Z}$, $p^{**} \in l^\infty_\mathbb{Z}$ *such that* $p = p^* + p^{**}$ *and* $\sup_{k\in\mathbb{Z}} |p^{**}_k| < \epsilon$

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  $p = p^* + p^{**}$ *and* $\sup_{k\in\mathbb{Z}} |p^{**}_k| < \epsilon$
- continuous case : replace sums by integrals
  (Ortega-Tineo (1996))

Interlude

Mathematical anxiety

Duffing difference equations

Theorem. *Assume*

1. $c > 0, \quad g \in C(\mathbb{R}, \mathbb{R}), \quad p \in l^\infty_\mathbb{Z}$
2. *There exists* $r_0 > 0$ *and* $\delta_- < \delta_+$ *such that*

$$g(y) \geq \delta_+ \quad \text{for} \quad y \leq -r_0, \quad g(y) \leq \delta_- \quad \text{for} \quad y \geq r_0.$$

3. $\delta_- < \widehat{p} \leq \widetilde{p} < \delta_+.$

Then equation

$$\boxed{\Delta^2 x_{m-1} + c\Delta x_m + g(x_m) = p_m \quad (m \in \mathbb{Z})}$$

has at least one solution $x = (x_m)_{m \in \mathbb{Z}} \in l^\infty_\mathbb{Z}$

Sketch of the proof

- write $p = p^* + p^{**}$ with $p^* \in BP_{\mathbb{Z}}$, $p^{**} \in l^\infty_{\mathbb{Z}}$ and $\delta_- < \inf_{k\in\mathbb{Z}} p^{**}_k \leq \sup_{k\in\mathbb{Z}} p^{**}_k < \delta_+$

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- use Corollary on constant lower and upper solutions

Landesman-Lazer condition

- Corollary. *If*

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  $-\infty < \limsup_{y \to +\infty} g(y) < g(x) < \liminf_{y \to -\infty} g(y) < +\infty$

  for all $x \in \mathbb{R}$, *then* $p \in l^\infty_{\mathbb{Z}}$ *and (*) is necessary for the existence of a bounded solution*

Examples

1. *Equation*

$$\boxed{\Delta^2 x_{m-1} + c\Delta x_m - \frac{b x_m}{1+|x_m|} = p_m \quad (m \in \mathbb{Z})}$$

with $c > 0$ *and* $b > 0$ *has a solution* $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ *if and only if* $p = (p_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ *and* $-b < \widehat{p} \leq \widetilde{p} < b$

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2. *Equation*

$$\Delta^2 x_{m-1} + c\Delta x_m - \frac{b x_m}{1+|x_m|^a} = p_m \quad (m \in \mathbb{Z})$$

with $c > 0,$ $b > 0$ *and* $0 \leq a < 1$ *has a solution* $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ *if and only if* $p = (p_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$

Remarks

- *similar results hold for equation*

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- Open problem. Prove or disprove that *equation*

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($c > 0, \quad b > 0$) *has a solution* $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$
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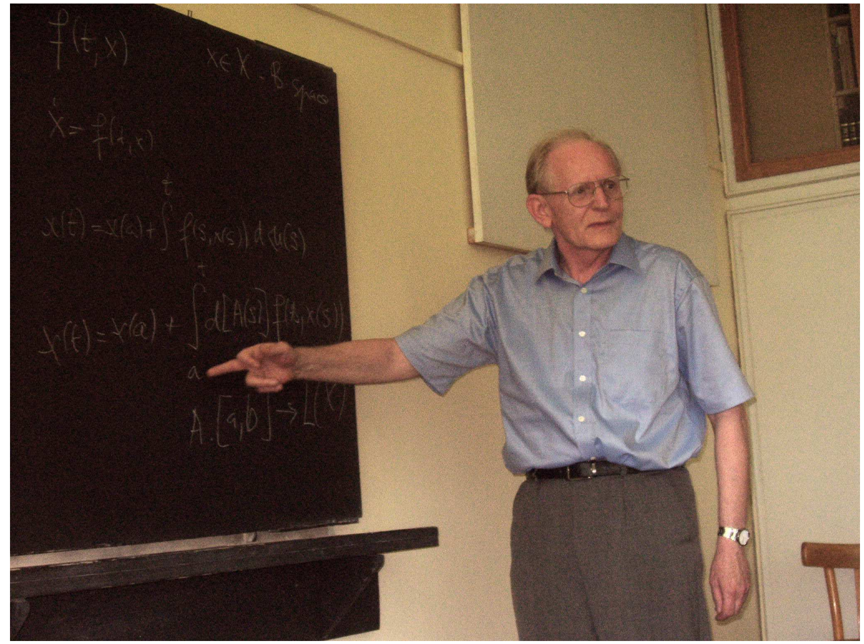
Interlude



Identity problems : is it me ?

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- corresponding result for difference system
  $\boxed{x_{n+1} - x_n = f_n(x_n)}$ or equivalently for discrete dynamical system $\boxed{x_{n+1} = g_n(x_n)}$?

Guiding function – discrete case

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- this result is false

Counterexample – 1

$$g_m(x) = \begin{cases} 1 & \text{if} \quad x \le -2 \\ mx + 2m + 1 & \text{if} \quad -2 < x < -1 \\ m + 1 & \text{if} \quad -1 \le x \le 1 \\ -mx + 2m + 1 & \text{if} \quad 1 < x < 2 \\ 1 & \text{if} \quad x \ge 2 \end{cases} \qquad (m \in \mathbb{Z})$$

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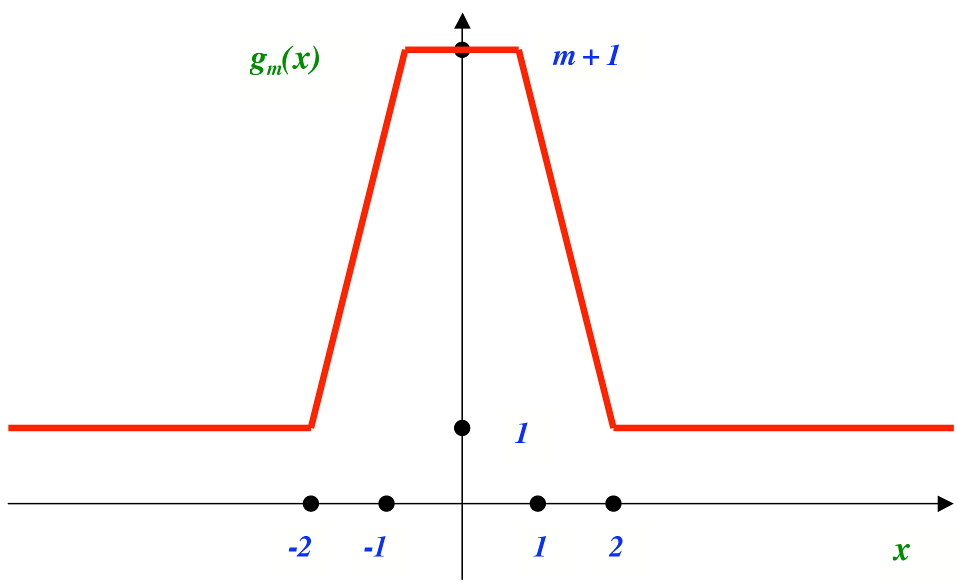
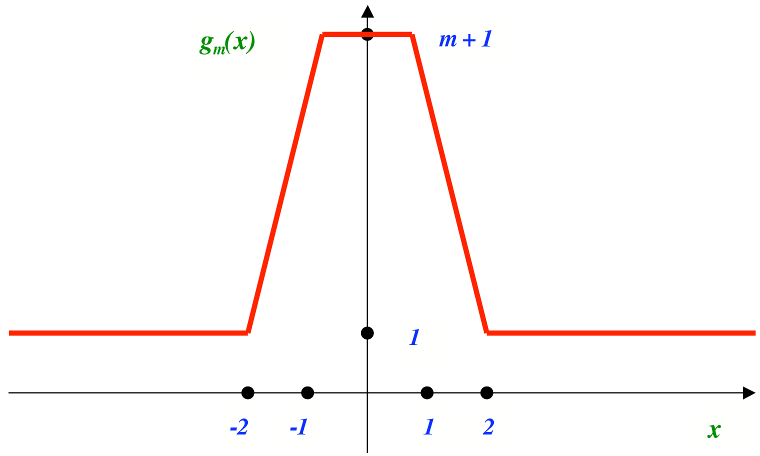


Counterexample – 2



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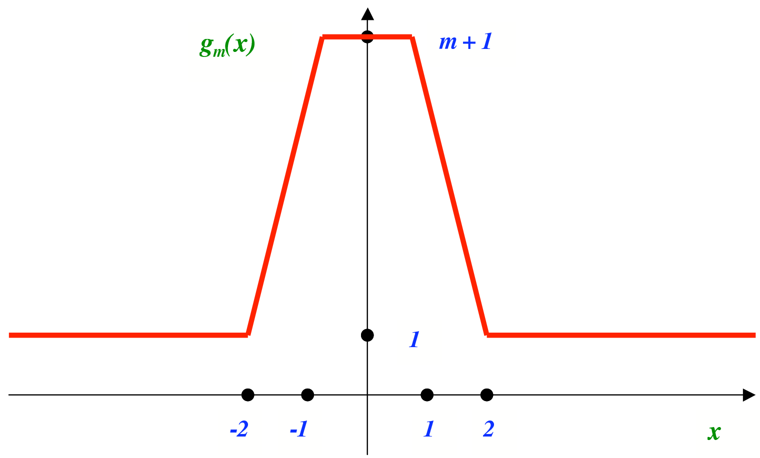

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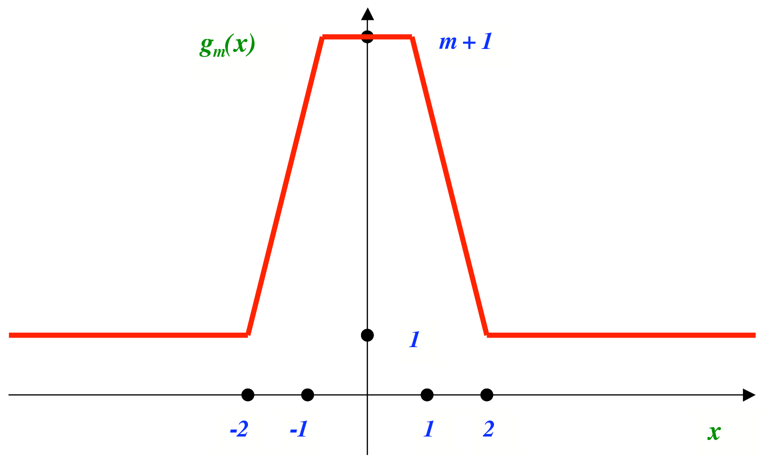

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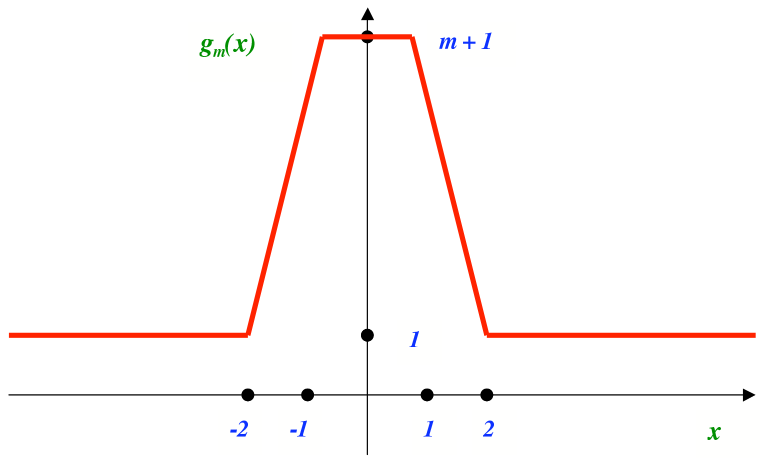

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  all solutions are unbounded in the future
- $V(x) = |x|, \quad |x| \geq \rho_0 = 2 \quad \Rightarrow \quad |g_m(x)| \leq |x|$

A discrete guiding function thm

- Theorem. *Let* $g_m \in C(\mathbb{R}^n, \mathbb{R}^n) \quad (m \in \mathbb{Z})$. *If*

$$\boxed{x_{m+1} = g_m(x_m) \quad (m \in \mathbb{Z})} \quad (*)$$

admits a guiding function V *with constant* ρ_0 *and if*

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- Remark. (**) trivially holds if $g_m = g \ (m \in \mathbb{Z})$

Sketch of the proof

- take $\rho_1 > \max\{\rho_0, \sup_{m\in\mathbb{Z}} \max_{\|x\|\leq\rho_0} \|g_m(x)\|\}$
- define $V_1 := \max_{\|x\|\leq\rho_1} V(x)$
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- use the following limiting lemma to obtain a solution $x = (x_m)_{m\in\mathbb{Z}} \in (l^\infty_{\mathbb{Z}})^n$

Another limiting lemma

- Lemma. *Assume that* $g_m \in C(\mathbb{R}^n, \mathbb{R}^n) \quad (m \in \mathbb{Z})$ *and that there exists* $\rho > 0$ *such that, for each* $k \in \mathbb{N}^*,$ *the system*

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$$\|x_m^k\| \le \rho \quad (m \in \mathbb{Z})$$

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- continuous case : Krasnosel'skii (1966)

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