Positive solutions for a system with p-Laplace-like operators, via blow-up

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Raúl Manásevich

Centro de Modelamiento Matemático and
Departamento de Ingeniería Matemática
Universidad de Chile

In this talk we will consider the system

$$(\tilde{S}_r) \quad \begin{cases} -(r^{N-1}\phi(u'(r)))' = r^{N-1}a(u(r))f(v(r)) \\ \\ -(r^{N-1}\psi(v'(r)))' = r^{N-1}b(v(r))g(u(r)) \\ \\ u'(0) = v'(0) = u(R) = v(R) = 0, \end{cases}$$

where ϕ and ψ are (odd) increasing homeomorphisms of $\mathbb{R}$, $R > 0$. $a, b, f, g : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions, such that $a(0) = 0$, $b(0) = 0$, $f(0) = 0$, $g(0) = 0$. Furthermore $f(t) > 0$, $g(t) > 0$, $a(t) > 0$, $b(t) > 0$, for all $t > 0$. We will be interested in existence of positive solutions.

By a solution to this system we mean a pair (u, v), $u, v \in C^1[0, R]$, $R > 0$, $\phi(u'(r)), \psi(v'(r)) \in C^1[0, R]$. such that u, v satisfy $(\tilde{S}_r)$.

In case $\phi(t) = m(t)t$, and $\phi(t) = n(t)t$, system $(\tilde{S}_r)$ provides radial solutions to the problem

$$
(S) \quad \begin{cases} -\Delta_\phi u = a(u)f(v) & \text{in } \Omega, \\ -\Delta_\psi v = b(u)g(u) & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $\Omega = B(0, R) \subset \mathbb{R}^N$, $N \geq 3$, and

$$\Delta_\phi u = \mathrm{div}(m(|\nabla u|)\nabla u),$$

and

$$\Delta_\psi v = \mathrm{div}(n(|\nabla v|)\nabla v),$$

called p-Laplace like operators. They are nonlinear, nonhomogeneous operators which generalize the p-Laplace operator

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-1}\nabla u), \quad p > 1.$$

System (S) does not necessarily have a variational structure. This a first motivation to study system $(\tilde{S}_r)$.

By a positive solution to system $(\tilde{S}_r)$, we mean a solution that satisfies $u(r) > 0, v(r) > 0$, for all $r \in [0, R)$. To study these solutions we will consider the following system

$$(S_r) \quad \begin{cases} -(r^{N-1}\phi(u'(r)))' = r^{N-1}a(u(r))f(|v(r)|) \\ -(r^{N-1}\psi(v'(r)))' = r^{N-1}b(v(r))g(|u(r)|) \\ u'(0) = v'(0) = u(R) = v(R) = 0, \end{cases}$$

Let $(u(r), v(r))$ be a non trivial solution of (S_r). Then, $u(r) > 0$, $v(r) > 0$ and are non increasing on $[0, R)$.

Hypotheses

Let us set

$$\Phi(s) = \int_0^s \phi(t)dt, \quad \Psi(s) = \int_0^s \psi(t)dt, \quad F(s) = \int_0^s f(t)dt,$$

$$G(s) = \int_0^s g(t)dt, \quad A(s) = \int_0^s a(t)dt, \quad B(s) = \int_0^s b(t)dt.$$

$(H1)$ There exists $1 < p_\infty < N$, $1 < q_\infty < N$, $a_\infty \geq 0$, and $b_\infty \geq 0$, such that

$$\lim_{s\to+\infty} \frac{s\phi(s)}{\Phi(s)} = p_\infty \qquad \lim_{s\to+\infty} \frac{s\psi(s)}{\Psi(s)} = q_\infty,$$

$$\lim_{s\to+\infty} \frac{sa(s)}{A(s)} = a_\infty + 1 \qquad \lim_{s\to+\infty} \frac{sb(s)}{B(s)} = b_\infty + 1.$$

Hypotheses

$(H2)$ There exists $p_0 > 1$, $q_0 > 1$, $a_0 \geq 0$, and $b_0 \geq 0$, such that

$$\lim_{s\to 0} \frac{s\phi(s)}{\Phi(s)} = p_0 \qquad \lim_{s\to 0} \frac{s\psi(s)}{\Psi(s)} = q_0,$$

$$\lim_{s\to 0} \frac{sa(s)}{A(s)} = a_0 + 1 \qquad \lim_{s\to 0} \frac{sb(s)}{B(s)} = b_0 + 1.$$

$(H3)$ There exists $\underline{\delta}_\infty > 0$, $\overline{\delta}_\infty > 0$, $\underline{\mu}_\infty > 0$, and $\overline{\mu}_\infty > 0$, such that

$$\underline{\delta}_\infty + 1 = \liminf_{s\to+\infty} \frac{sf(s)}{F(s)} \leq \limsup_{s\to+\infty} \frac{sf(s)}{F(s)} = \overline{\delta}_\infty + 1,$$

$$\underline{\mu}_\infty + 1 = \liminf_{s\to+\infty} \frac{sg(s)}{G(s)} \leq \limsup_{s\to+\infty} \frac{sg(s)}{G(s)} = \overline{\mu}_\infty + 1.$$

Hypotheses

$(H4)$ There exists $\underline{\delta}_0 > 0$, $\overline{\delta}_0 > 0$, $\underline{\mu}_0 > 0$, and $\overline{\mu}_0 > 0$, such that

$$\underline{\delta}_0 + 1 = \liminf_{s\to 0} \frac{sf(s)}{F(s)} \le \limsup_{s\to 0} \frac{sf(s)}{F(s)} = \overline{\delta}_0 + 1,$$

$$\underline{\mu}_0 + 1 = \liminf_{s\to 0} \frac{sg(s)}{G(s)} \le \limsup_{s\to 0} \frac{sg(s)}{G(s)} = \overline{\mu}_0 + 1.$$

$(H5)$ It holds that

$$\underline{\delta}_\infty \underline{\mu}_\infty > (p_\infty - a_\infty - 1)(q_\infty - b_\infty - 1),$$

and

$$\underline{\delta}_0 \underline{\mu}_0 > (p_0 - a_0 - 1)(q_0 - b_0 - 1).$$

Hypotheses

Let us consider the algebraic system for the unknowns (E_1, E_2),

$$
\begin{aligned}
(p_\infty - a_\infty - 1)E_1 - \underline{\delta}_\infty E_2 &= -p_\infty, \\
-\underline{\mu}_\infty E_1 + (q_\infty - b_\infty - 1)E_2 &= q_\infty,
\end{aligned}
\tag{1}
$$

with solution

$$
E_1 = \frac{p_\infty(q_\infty - b_\infty - 1) + q_\infty \underline{\delta}_\infty}{\underline{\delta}_\infty \underline{\mu}_\infty - (p_\infty - a_\infty - 1)(q_\infty - b_\infty - 1)},
$$

and

$$
E_2 = \frac{q_\infty(p_\infty - a_\infty - 1) + p_\infty \underline{\mu}_\infty}{\underline{\delta}_\infty \underline{\mu}_\infty - (p_\infty - a_\infty - 1)(q_\infty - b_\infty - 1)}.
$$

We will assume
$(H6)$

$$
\max_{i=1,\dots,n} \{(E_1 - \frac{N - p_\infty}{p_\infty - 1}), (E_2 - \frac{N - q_\infty}{q_\infty - 1})\} > 0.
$$

Main Theorem and examples

Theorem (Joint work with Marc Henrard)

Under hypotheses (H_1)-(H_6), problem $(\tilde{S}_r)$ has a solution (u,v) such that $u(r)>0$, $v(r)>0$, for all $r\in[0,R)$.

We will show some consequences of this theorem.

Example 1. Consider the problem

$$(S_{p,q})\quad \begin{cases} -(r^{N-1}\phi_p(u'(r)))' = r^{N-1}|v(r)|^{\delta-1}v(r) \\ -(r^{N-1}\phi_q(v'(r)))' = r^{N-1}|(u(r)|^{\mu-1}u(r) \\ u'(0)=v'(0)=u(R)=v(R)=0. \end{cases}$$

where $\phi_p(t)=|t|^{p-2}t$, $p>1$.

Main Theorem and examples

In Communications in PDE 18 (1993), 2071-2106, Clément, myself and Mitidieri, proved the following result.

Theorem (CMM)

Under the assumptions

$$N > \max\{p, q\}, \quad \delta\mu > (p-1)(q-1), \quad \delta > 0, \quad \mu > 0, \tag{2}$$

and

$$\max\{\alpha_1 - \frac{N-p}{p-1}, \alpha_2 - \frac{N-q}{q-1}\} > 0, \tag{3}$$

where

$$\alpha_1 = \frac{p(q-1)+q\delta}{\delta\mu-(p-1)(q-1)} \quad \text{and} \quad \alpha_2 = \frac{q(p-1)+p\mu}{\delta\mu-(p-1)(q-1)}\}, \tag{4}$$

then problem $(S_{p,q})$ has a solution (u, v), such that $u(r) > 0$, $v(r) > 0$, for all $r \in [0, R)$.

Main Theorem and examples

The proof of this theorem uses a nonexistence of positive solutions to obtain a-priori bounds, combined with Leray-Schauder degree results.

In few words the argument goes like this. One begins assuming by contradiction that solutions are not a-priori bounded and hence there is a sequence of positive solutions (u_k, v_k) such that either

$$t_k = ||u_k|| = u_k(0) \to \infty \quad \text{or} \quad s_k = ||v_k|| = v_k(0) \to \infty,$$

and define a change of variable, rescaling, of the form $y = \gamma_k r$ and

$$w_k(y) = \frac{u_k(r)}{\gamma_k^{\alpha_1}}, \quad \text{and} \quad z_k(y) = \frac{v_k(r)}{\gamma_k^{\alpha_2}},$$

where

$$\gamma_k = t_k^{\frac{1}{\alpha_1}} + s_k^{\frac{1}{\alpha_2}}.$$

Then, (w_k, z_k) satisfies

$$\begin{cases} -(y^{N-1}\phi_p(w_k'(y)))' = y^{N-1}z_k(y)^{\delta}\gamma_k^{\epsilon_1} & y \in (0, R\gamma_k), \\ -(y^{N-1}\phi_q(z_k'(y)))' = y^{N-1}w_k(y)^{\mu}\gamma_k^{\epsilon_2} & y \in (0, R\gamma_k), \\ w_k'(0) = z_k'(0) = w_k(R\gamma_k) = v(R\gamma_k) = 0, \end{cases}$$

and $' = \frac{d}{dy}$. Also

$$\epsilon_1 = \delta\alpha_2 - \alpha_1(p-1) - p, \quad \text{and} \quad \epsilon_1 = \mu\alpha_1 - \alpha_2(q-1) - q.$$

It turns out that $\epsilon_1 = 0$, and $\epsilon_2 = 0$, and thus the equations for (w_k, z_k) look the same as those for (u_k, v_k), but in a different interval that gets larger with k. For $T > 0$, we let k large so that $R\gamma_k > T$.
Then one can show that there is a subsequence, renamed the same, such that

Main Theorem and examples

$(w_k, z_k) \to (\hat{w}, \hat{z})$, that satisfies

$$(Lim) \quad \begin{cases} -(y^{N-1}\phi_p(\hat{w}'(y)))' = y^{N-1}\hat{z}(y)^{\delta} & y \in (0, T), \\ -(y^{N-1}\phi_q(\hat{z}'(y)))' = y^{N-1}\hat{w}(y)^{\mu} & y \in (0, T), \\ \hat{w}_k'(0) = \hat{z}_k'(0) = 0, \end{cases}$$

and such that $\hat{w}_k(y) > 0$, $\hat{z}_k(y) > 0$, $y \in [0, T]$.

A diagonal argument then shows that $(\hat{w}, \hat{z})$ can be extended to $[0, \infty)$ and satisfy (Lim) for all $y \in (0, \infty)$. Furthermore $\hat{w}_k(y) > 0$, $\hat{z}_k(y) > 0$, $y \in [0, \infty)$.

But this in contradiction with the following nonexistence of positive solutions result that was proved in Communications in PDE 18 (1993), 2071-2106, by Clément, myself and Mitidieri.

Main Theorem and examples

Theorem (Non Existence)

Consider the following system

$$(Liu)\quad \begin{cases} -(y^{N-1}\phi_p(w'(y)))' \geq y^{N-1}z(y)^{\delta} & y\in(0,\infty),\\ -(y^{N-1}\phi_q(z'(y)))' \geq y^{N-1}w(y)^{\mu} & y\in(0,\infty),\\ \hat{w}'(0)=\hat{z}'(0)=0. \end{cases}$$

If (2), (3), and (4) hold, then system (Liu) does not have a positive solution.

In this form we have sketched a proof for the existence of positive solutions for system $(S_{p,q})$.

With the notation of the general case, we have

$$\phi(s)=|s|^{p-2}s \quad \psi(s)=|s|^{q-2}s,$$

Main Theorem and examples

with corresponding primitives

$$\Phi(s) = \frac{|s|^p}{p} \quad \Psi(s) = \frac{|s|^q}{q}$$

$$F(s) = \frac{|s|^{\delta+1}}{\delta+1} \quad G(s) = \frac{|s|^{\mu+1}}{\mu+1}, \quad A(s) = s, \quad B(s) = s.$$

Thus, conditions (H_1), (H_2), become

$$p_\infty = p_0 = p > 1, \quad q_\infty = q_0 = q > 1, \quad a_\infty = a_0 = 0, \quad b_\infty = b_0 = 0.$$

Conditions (H_3), (H_4), results as

$$\frac{sf(s)}{F(s)} = \delta + 1, \quad \frac{sg(s)}{G(s)} = \mu + 1,$$

while (H_5), is given by

$$\delta\mu > (p-1)(q-1).$$

Finally, the solution for system (1), is given by $E_1 = \alpha_1$ and $E_2 = \alpha_2$, and hence condition (H_6) is exactly (3).

Is plain clear that Theorem 2 is just a particular case of Theorem 1.

Example 2. Following essentially the same ideas, Clement, Fleckinger, Mitidieri and de Thelin, JDE 166, 455-477 (2000), extended Theorem 2, to the case

$$(\tilde{S}_{p,q}) \quad \begin{cases} -(r^{N-1}\phi_p(u'(r)))' = r^{N-1}|u|^{a-1}u|v(r)|^{\delta-1}v(r) \\ -(r^{N-1}\phi_q(v'(r)))' = r^{N-1}|v|^{b-1}v|(u(r)|^{\mu-1}u(r) \\ u'(0) = v'(0) = u(R) = v(R) = 0, \end{cases}$$

Main Theorem and examples

They obtained existence of positive solutions under the basic conditions

$$N > \max\{p, q\}, \quad \delta\mu > (p-1-a)(q-1-b), \quad \delta > 0, \quad \mu > 0, \tag{5}$$

one of the following assumptions hold

$$p - 1 - a > 0, \quad q - 1 - b > 0, \tag{6}$$

and

$$\max\{\alpha_1 - \frac{N-p}{p-1}, \alpha_2 - \frac{N-q}{q-1}\} \geq 0, \tag{7}$$

where now

$$\alpha_1 = \frac{p(q-1-b) + q\delta}{\delta\mu - (p-1-a)(q-1-b)} \quad \text{and} \quad \alpha_2 = \frac{q(p-1-a) + p\mu}{\delta\mu - (p-1-a)(q-1-b)}.$$

Example 3. Garcia-Huidobro, Guerra, and myself in AAA 3(1998) 997-1014, considered a version of system $(\tilde{S}_r)$, indeed the case of a cyclic system of the form

$$(D) \quad \begin{cases} -(r^{N-1}\phi_i(u_i'(r)))' = r^{N-1} f_i(u_{i+1}(r)) & r \in (0, R) \\ \quad u_i'(0) = 0 = u_i(R), \end{cases}$$

where it is understood that $u_{n+1} = u_1$.

Here for $i = 1, \ldots, n$, the functions ϕ_i are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and the $f_i : \mathbb{R} \mapsto \mathbb{R}$ are odd continuous functions such that $sf_i(s) > 0$ for $s \neq 0$. Also $' = \frac{d}{dr}$.

The case $n = 2$, is a particular case of system $(\tilde{S}_r)$. With the notation for the general case the conditions we used (for problem (D)), translate as follows: conditions (H_1), (H_2), become

$$p_\infty > 1, p_0 > 1, \quad q_\infty > 1, q_0 > 1, \quad a(t) = 1, \quad b(t) = 1,$$

and hence $a_\infty = a_0 = 0, \quad b_\infty = b_0 = 0.$

Main Theorem and examples

Conditions (H_3), (H_4), become respectively

$$\lim_{s\to\infty}\frac{sf(s)}{F(s)}=\delta_\infty+1,\quad \lim_{s\to\infty}\frac{sg(s)}{G(s)}=\mu_\infty+1, \tag{9}$$

$$\lim_{s\to 0}\frac{sf(s)}{F(s)}=\delta_0+1,\quad \lim_{s\to 0}\frac{sg(s)}{G(s)}=\mu_0+1, \tag{10}$$

where $\delta_\infty>0$, $\mu_\infty>0$, $\delta_0>0$, and $\mu_0>0$. Condition (H_5), is given by the two expressions

$$\delta_\infty\mu_\infty>(p_\infty-1)(q_\infty-1),$$

and

$$\delta_0\mu_0>(p_0-1)(q_0-1).$$

Then in AAA, we proved the following result

Theorem

Suppose the following additional conditions $p_\infty < N$, $q_\infty < N$, and that condition (H_6) holds, i.e.,

$$\max_{i=1,\dots,n}\{(E_1 - \frac{N-p_\infty}{p_\infty - 1}), (E_2 - \frac{N-q_\infty}{q_\infty - 1})\} > 0,$$

where now

$$E_1 = \frac{p_\infty(q_\infty - 1) + q_\infty \delta_\infty}{\delta_\infty \mu_\infty - (p_\infty - 1)(q_\infty - 1)},$$

and

$$E_2 = \frac{q_\infty(p_\infty - 1) + p_\infty \mu_\infty}{\delta_\infty \mu_\infty - (p_\infty - 1)(q_\infty - 1)}.$$

Then problem (D) has a positive solution.

The proof of this theorem uses a blow-up technique to get the a-priori bound combined with the same nonexistence of positive solutions theorem of example 1 with $\delta = \delta_\infty$ and $\mu = \mu_\infty$.

The blow up is done with a sort of implicit rescaling an improvement of which we will explain later.

Example 4. Examples 1, 2 have in common that the functions on the right hand side are increasing, while Example 3 uses conditions (9) and (10).

In this form for functions like

$$f(s) = (\sin s + 2)s^{\delta}, \quad g(s) = (\cos s + 2)s^{\mu}, \tag{11}$$

the theorems in those examples do not apply. In this case both f and g are nonincreasing and do not satisfy (9) and (10). Indeed, one can prove that

$$\frac{\delta+1}{2} = \liminf_{s\to+\infty} \frac{sf(s)}{F(s)} < \limsup_{s\to+\infty} \frac{sf(s)}{F(s)} = \frac{3(\delta+1)}{2}.$$

$$\frac{\mu+1}{2} = \liminf_{s\to+\infty} \frac{sg(s)}{G(s)} < \limsup_{s\to+\infty} \frac{sg(s)}{G(s)} = \frac{3(\mu+1)}{2}.$$

Main Theorem and examples

Hence

$$\underline{\delta}_\infty = \frac{\delta - 1}{2}, \quad \underline{\mu}_\infty = \frac{\mu - 1}{2}, \quad \bar{\delta}_\infty = \frac{3\delta + 1}{2}, \quad \bar{\mu}_\infty = \frac{3\mu + 1}{2}.$$

Furthermore

$$\lim_{s\to 0} \frac{sf(s)}{F(s)} = \delta + 1 \quad \text{and} \quad \lim_{s\to 0} \frac{sg(s)}{G(s)} = \mu + 1.$$

Then from our main theorem we get the following result. Consider the system

$$(P) \quad \begin{cases} -(r^{N-1}\phi_p(u'(r)))' = r^{N-1}f(v(r)) \\ -(r^{N-1}\phi_q(v'(r)))' = r^{N-1}g(u(r)) \\ u'(0) = v'(0) = u(R) = v(R) = 0. \end{cases}$$

where f and g are as in (11).

Main Theorem and examples

Theorem

Under the assumptions

$$N > \max\{p, q\}, \quad (\delta - 1)(\mu - 1) > 4(p-1)(q-1), \quad \delta > 1, \quad \mu > 1,$$

and

$$\max\{E_1 - \frac{N-p}{p-1}, E_2 - \frac{N-q}{q-1}\} > 0,$$

then problem (P) *has a solution* (u, v)*, such that* $u(r) > 0$*,* $v(r) > 0$*, for all* $r \in [0, R)$*.*

Here

$$E_1 = \frac{4p(q-1) + 2q(\delta - 1)}{(\delta - 1)(\mu - 1) - 4(p-1)(q-1)},$$

$$E_2 = \frac{4q(p-1) + 2p(\mu - 1)}{(\delta - 1)(\mu - 1) - 4(p-1)(q-1)}.$$

A-priori bounds

In extending the blow up method to our situation it turns out that a key step is to find a solution (x, y) in terms of s (for s near $+\infty$) to the system

$$F(y)A(x) = y\Phi(xs) \qquad G(x)B(y) = x\Psi(ys)$$

In this respect we can prove the following.

Lemma

Assume (H1-H5), then there exists diffeomorphisms

$$\alpha : [s_0, \infty) \mapsto [x^0, \infty) \quad \textit{and} \quad \beta : [s_0, \infty) \mapsto [y^0, \infty),$$

such that for all $s \in [s_0, \infty)$,

$$F(\beta(s))A(\alpha(s)) = \beta(s)\Phi(\alpha(s)s)$$

$$G(\alpha(s))B(\beta(s)) = \Psi(\beta(s)s)\alpha(s)$$

A-priori bounds

The functions α, β play the role of *rescaling* variables.

In the case of Example 1, that is when $\phi(s) = |s|^{p-2}s$, $\psi(s) = |s|^{q-2}s$, $f(s) = |s|^{\delta-1}s$, and $g(s) = |s|^{\mu-1}s$, for $s > 0$, one can prove that

$$\alpha(s) = C_1 s^{\frac{p(q-1)+\delta\mu}{\delta\mu-(p-1)(q-1)}},$$

and

$$\beta(s) = C_2 s^{\frac{q(p-1)+\delta\mu}{\delta\mu-(p-1)(q-1)}},$$

where C_1, C_2 are constants (independent of s).

We notice that the exponents on the right hand side are exactly α_1 and α_2 in Example 1.

A-priori bounds

Sketch of the proof of the Main Theorem. We consider the following system

$$(\mathcal{S}_r) \quad \begin{cases} -(r^{N-1}\phi(u'(r)))' = r^{N-1}a(u(r))(f(|v(r)| + \lambda h) \\ -(r^{N-1}\psi(v'(r)))' = r^{N-1}b(v(r))g(|u(r)|) \\ u'(0) = v'(0) = u(R) = v(R) = 0, \end{cases}$$

where $\lambda \in [0,1]$, and h is a positive parameter that has been fixed such that $(\mathcal{S}_r)$ does not have solutions in a ball $B(0, R_1) \subset C[0,R]$. This yields that the Leray-Schauder degree of a certain is zero.

The proof continues by contradiction, suppose that there exists a sequence of solutions $\{(u_k, v_k), \lambda_k)\} \in C[0,R] \times C[0,R] \times [0,1]$ such that $\|(u_k, v_k)\| \to \infty$.

A-priori bounds

Then, in the first place, one can prove that $\|u_k\| \to \infty$ and $\|v_k\| \to \infty$, and then by taking a subsequence, if necessary, one can assume $\|u_k\| \geq s_0$ and $\|v_k\| \geq s_0$.

Define

$$\gamma_k = \alpha^{-1}(\|u_k\|) + \beta^{-1}(\|v_k\|)$$

and

$$t_k = \alpha(\gamma_k), \quad s_k = \beta(\gamma_k).$$

Then $\gamma_k \to +\infty$ and

$$\|u_k\| \leq t_k, \quad \|v_k\| \leq s_k.$$

Define the change of variables $y = \gamma_k r$,

$$w_k(y) = \frac{u_k(r)}{t_k}, \quad z_k(y) = \frac{v_k(r)}{s_k}.$$

A-priori bounds

Clearly

$$|w_k(y)| \le 1, \quad |z_k(y)| \le 1,$$

for all $y \in [0, \gamma_k R]$. In terms of these new variables the problem $(\mathcal{S}_r)$ becomes

$$-(y^{N-1}\phi(t_k\gamma_k w_k'(y)))' = y^{N-1}\frac{1}{\gamma_k}\left[a(t_k w_k(y))f(s_k|z_k(y)|) + \lambda_k h\right].$$

$$-(y^{N-1}\psi(s_k\gamma_k z_k'(y)))' = y^{N-1}\frac{1}{\gamma_k}\left[b(s_k z_k(y))g(t_k|w_k(y)|)\right].$$

$$w_k'(0) = 0 = w_k(\gamma_k R), \qquad z_k'(0) = 0 = z_k(\gamma_k R).$$

Let now $T > 0$ be fixed and assume that $\gamma_k R > T$. Then,

$$w_k'(y) < 0, \qquad z_k'(y) < 0, \qquad w_k(y) > 0, \qquad z_k(y) > 0,$$

for all $k \in \mathbb{N}$ and for all $y \in [0, T]$.

One can prove that the sequences $(w_k')_{k\in\mathbb{N}}$, $(z_k')_{k\in\mathbb{N}}$ are bounded in $C[0,T]$, and then by Arzela-Ascoli theorem, passing to a subsequence, one obtains that

$$w_k \to w, \quad z_k \to z \quad \text{in} \quad C[0,T].$$

Now, from

$$1 = \frac{1}{\gamma_k}(\alpha^{-1}(t_k w_k(0)) + \beta^{-1}(t_k z_k(0))) = \frac{\alpha^{-1}(t_k w_k(0))}{\alpha^{-1}(t_k)} + \frac{\beta^{-1}(t_k z_k(0))}{\beta^{-1}(t_k)},$$

one can prove that

$$1 \le w(0)^{1/E_1} + z(0)^{1/E_2},$$

which implies that (w,z) is not identically zero.

A-priori bounds

One can furthermore prove that w, z are $C^1[0, T]$ and that for a.e. $y \in [0, T]$,

$$C_- y^{N-1} w(y)^{a_\infty} z(y)^{\underline{\delta}_\infty} \leq -(y^{N-1}\phi_{p_\infty}(w'(y)))'$$

$$\leq C_+ y^{N-1} w(y)^{a_\infty} z(y)^{\overline{\delta}_\infty} \tag{12}$$

and

$$\tilde{C}_- y^{N-1} z(y)^{b_\infty} w(y)^{\underline{\mu}_\infty} \leq -(y^{N-1}\phi_{q_\infty}(z'(y)))' \geq$$

$$-(y^{N-1}\phi_{q_\infty}(z'(y)))' \leq \tilde{C}_+ y^{N-1} z(y)^{b_\infty} w(y)^{\overline{\mu}_\infty} \tag{13}$$

where

$$C_- = \left(\frac{\underline{\delta}_\infty + 1}{\overline{\delta}_\infty + 1}\right)\frac{(\underline{\delta}_\infty + 1)(a_\infty + 1)}{p_\infty},$$

$$C_+ = \left(\frac{\overline{\delta}_\infty + 1}{\underline{\delta}_\infty + 1}\right)\frac{(\overline{\delta}_\infty + 1)(a_\infty + 1)}{p_\infty}.$$

$$\tilde{C}_- = \left(\frac{\underline{\mu}_\infty + 1}{\overline{\mu}_\infty + 1}\right)\frac{(\underline{\mu}_\infty + 1)(b_\infty + 1)}{q_\infty},$$

and

$$\tilde{C}_+ = \left(\frac{\overline{\mu}_\infty + 1}{\underline{\mu}_\infty + 1}\right)\frac{(\overline{\mu}_\infty + 1)(b_\infty + 1)}{q_\infty}.$$

A-priori bounds

A diagonal argument shows that (12) and (13) hold for all $y \in [0, \infty)$.

In addition one can show that the functions

$$M_w(y) = yw'(y) + \frac{N - p_\infty}{p_\infty - 1} w(y),$$

and

$$M_z(y) = yz'(y) + \frac{N - q_\infty}{q_\infty - 1} z(y)$$

are nonnegative and nondecreasing for $y > 0$. In particular, the functions

$$y^{\frac{N-p_\infty}{p_\infty - 1}} w(y) \quad \text{and} \quad y^{\frac{N-q_\infty}{q_\infty - 1}} z(y)$$

are nondecreasing on $[0, \infty)$.

A-priori bounds

The argument continues until one gets

$$y^{E_1-\frac{N-p_\infty}{p_\infty-1}} \leq C, \quad \text{for all } y \geq y_0.$$

Since, similarly

$$y^{E_2-\frac{N-q_\infty}{q_\infty-1}} \leq C, \quad \text{for all } y \geq y_0,$$

we find that

$$y^{\max\{(E_1-\frac{N-p_\infty}{p_\infty-1}),(E_2-\frac{N-q_\infty}{q_\infty-1})\}} \leq C, \quad \text{for all } y \geq y_0.$$

By (H6), this is a contradiction, and hence solutions to $(\mathcal{S}_r)$ are a-priori bounded.