Periodic BVP for second-order ODE

Alexander Lomtatidze, Bedřich Půža

$$u'' = f(t, u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

$$f \in Car\big([0, \omega] \times \mathbb{R}; \mathbb{R}\big)$$

$$u'' + f(t,u) = 0 \tag{1'}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Theorem A. *Let α and β be lower and upper solutions of* $(1')$, (2). *Let, moreover,*

$$a_\pm(t) \le \liminf_{u\to\pm\infty} \frac{f(t,u)}{u}, \quad \limsup_{u\to\pm\infty} \frac{f(t,u)}{u} \le b_\pm(t) \quad \textit{uniformly on } [0,\omega].$$

Assume further that the box $[a_+, b_+] \times [a_-, b_-]$ is admissible. Then the problem $(1')$, (2) *is solvable and solution is localized.*

$$u'' + f(t,u) = 0 \tag{1'}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Theorem B. *Let α and β be lower and upper solutions of $(1')$, (2). Let, moreover, for some functions $a_\pm \le 0$, $b_\pm \ge 0$ in $L\big([0,\omega]\big)$,*

$$a_-(t) \le \liminf_{u\to-\infty} \frac{f(t,u)}{u}, \quad \limsup_{u\to-\infty} \frac{f(t,u)}{u} \le b_-(t) \quad \textit{uniformly on } [0,\omega],$$

$$a_+(t) \le \liminf_{u\to+\infty} \frac{f(t,u)}{u} \quad \textit{uniformly on } [0,\omega].$$

Assume further that, for any $g \in L\big([0,\omega]\big)$ with

$$g(t) \le b_-(t) \quad \textit{for } t \in [0,\omega]$$

and $\bar{t} \in [0,\omega[$, the problem

$$u'' + g(t)u\,; \quad u(\bar{t}) = 0, \quad u(\bar{t}+\omega) = 0$$

has only the trivial solution. Then the problem $(1')$, (2) is solvable and solution is localized.

Remark. Mention that no restriction is required on a_- and a_+.

C. De Coster and M. Tarallo,
Foliations, associated reductions and lower and upper functions,
Cal. Var. **15** (2003), 25–44.

see also

C. De Coster and P. Habets,
Two-point boundary value problems, lower and upper functions,
Mathematics in Science and Engineering, Volume 205, Elsevier, 2006.

$$u'' = f(t, u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Theorem A. *Let α and β be lower and upper functions of (1), (2). Let, moreover,*

$$f(t,x)\,\mathrm{sgn}\,x \le p_0(t)[x]_+ + p_1(t)[x]_- + q(t,|x|) \qquad \text{on } [0,\omega]\times\mathbb{R},$$
$$f(t,x)\,\mathrm{sgn}\,x \ge -g_0(t)[x]_+ - g_1(t)[x]_- - q(t,|x|) \quad \text{on } [0,\omega]\times\mathbb{R},$$

where $q \in Car\big([0,\omega]\times\mathbb{R}_+;\mathbb{R}\big)$ is a sublinear function. Assume further that the box $[-g_0, p_0]\times[-g_1, p_1]$ is admissible. Then the problem (1), (2) is solvable and solution is localized.

$$u'' = f(t,u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Theorem B. *Let α and β be lower and upper functions of (1), (2). Let, moreover,*

$$\begin{aligned} f(t,x) &\le p_0(t)[x]_+ + p(t)[x]_- + q(t,|x|) && \text{on } [0,\omega]\times\mathbb{R}, \\ f(t,x) &\ge -g_0(t)|x| - q(t,|x|) && \text{on } [0,\omega]\times\,]-\infty,0], \end{aligned}$$

where $q \in Car\big([0,\omega]\times\mathbb{R}_+;\mathbb{R}\big)$ is a sublinear function,

$$p_0(t) \ge 0, \quad p(t) \ge 0, \quad g_0(t) \ge 0 \quad \text{for } t \in [0,\omega]$$

and

$$-p_0 \in V_\omega\,.$$

Then the problem (1), (2) is solvable and solution is localized.

Remark. No additional assumptions are required on functions p and g_0.

Theorem A.

$$f(t,x) \sim p(t,x)x + q(t,x)$$

Theorem B.

$$f(t,x) \sim p(t,x)x + q_0(t,x) \quad \textit{for } x < 0,$$

$$f(t,x) \leq p_0(t)x + q_1(t,x) \quad \textit{for } x > 0$$

$$u'' = f(t,u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Theorem 1. *Let α and β be lower and upper functions of (1), (2). Let, moreover,*

$$f(t,x)\,\mathrm{sgn}\,x \geq -p_0(t)|x| - q(t,|x|) \quad \text{on } [0,\omega]\times\mathbb{R},$$

where

$$-p_0 \in V_\omega$$

and $q \in Car\big([0,\omega]\times\mathbb{R}_+;\mathbb{R}\big)$ is sublinear. Then the problem (1), (2) is solvable and solution is localized.

Example.

$$u'' = -p_0(t)u + u^3 - 2\sqrt{|u|} + 1 \tag{3}$$

$$f(t,x) = -p_0(t)x + x^3 - 2\sqrt{|x|} + 1$$

$$f(t,x)\,\mathrm{sgn}\,x \geq -p_0(t)|x| - 2\sqrt{|x|} - 1, \qquad q(t,x) = 1 + 2\sqrt{|x|}$$

On the other hand,

$$\alpha \equiv 1 \quad \text{and} \quad \beta \equiv 0 \quad \text{are lower and upper functions.}$$

$$u'' = f(t,u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Corollary 1. *Let the function f is monotone in the second variable and*

$$f(t,x)\,\mathrm{sgn}\,x \geq -p_0(t)|x| - q(t,|x|) \quad \text{on } [0,\omega]\times\mathbb{R},$$

where $q \in Car\big([0,\omega]\times\mathbb{R}_+;\mathbb{R}\big)$ is sublinear and

$$-p_0 \in V_\omega\,.$$

Then the problem (1), (2) is solvable if and only if there exists $\gamma \in C\big([0,\omega]\big)$ such that

$$\int_0^\omega f(t,\gamma(t))dt = 0\,.$$

Remark. If f is nondecreasing then the problem (1), (2) is uniquely solvable.

$$u'' = f(t, u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Theorem 2. *Let α and β be lower and upper functions of* (1), (2)*. Let, moreover,*

$$f(t, x) \geq -p_0(t)[x]_+ - p(t)[x]_- - q(t, |x|) \quad \text{on } [0, \omega] \times \mathbb{R},$$

where $q \in Car\big([0, \omega] \times \mathbb{R}_+; \mathbb{R}\big)$ is sublinear and $-p_0 \in V_\omega$. Then the problem (1), (2) *is solvable and solution is localized.*

Theorem 3. *Let α and β be lower and upper functions of* (1), (2)*. Let, moreover,*

$$f(t, x) \leq p_0(t)[x]_- + p(t)[x]_+ + q(t, |x|) \quad \text{on } [0, \omega] \times \mathbb{R},$$

where $q \in Car\big([0, \omega] \times \mathbb{R}_+; \mathbb{R}\big)$ is sublinear and $-p_0 \in V_\omega$. Then the problem (1), (2) *is solvable and solution is localized.*

Remark. Function $p \in L\big([0, \omega]\big)$ may be arbitrary.

Remark. Theorem 3 generalized Theorem B.

Remark. If $p_0 \equiv 0$, $p \equiv 0$ and $q(t, x) = h(t)$ then we get results of I. Rachůnková, S. Staněk, M. Tvrdý.

Example.

$$u'' = -p_0(t)|u| + u^4 - 2\sqrt{|u|} + 1$$

$\alpha \equiv 1$, $\beta \equiv 0$ are lower and upper functions. Theorem 2 implies solvability.

Example.

$$u'' = p_0(t)|u| - u^4 + 1$$

p_0 is bounded

$\beta \equiv 0$ is an upper function, $\alpha \equiv Const.$ large enough is a lower function. Theorem 3 implies solvability.

Massera's type results

The system

$$x' = A(t)x + B(t),$$

where A is an ω-periodic matrix function and B is an ω-periodic vector function, has a periodic solution iff it possesses a bounded solution.

Massera's type results

The system

$$x' = A(t)x + B(t),$$

where A is an ω-periodic matrix function and B is an ω-periodic vector function, has a periodic solution iff it possesses a bounded solution.

For nonlinear systems this result is not true in general.

Massera's type results

The system

$$x' = A(t)x + B(t),$$

where A is an ω-periodic matrix function and B is an ω-periodic vector function, has a periodic solution iff it possesses a bounded solution.

For nonlinear systems this result is not true in general.

However, for first-order scalar equation

$$u' = f(t, u)$$

we have

the existence of a bounded solution $\iff$ the existence of an ω-periodic solution

This result is usually referred as Massera's theorem.

In the same paper it was proved

$$u' = g(t, u, v),$$
$$v' = h(t, u, v)$$

the existence of a bounded solution $\iff$ the existence of an ω-periodic solution

provided

1. every Cauchy problem is uniquely solvable
2. all solutions are global (right)

Remark. Condition 2 is essential and cannot be omitted.

In the same paper it was proved

$$u' = g(t, u, v),$$
$$v' = h(t, u, v)$$

the existence of a bounded solution $\iff$ the existence of an ω-periodic solution

provided

1. every Cauchy problem is uniquely solvable
2. all solutions are global (right)

Remark. Condition 2 is essential and cannot be omitted.

Massera's result from 1950 generalizes

N. Levinson,
Transformation theory of non-linear differential equations of the second order.
Annals of Mathematics (2), **45** (1944), 723–737

Notation of "Dissipative".

$$u'' = f(t, u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{2}$$

$$\sup \left\{ |u(t)| + |u'(t)| : t \geq 0 \right\} < +\infty \tag{4}$$

Theorem C. *Solvability of* (1), (4) *implies solvability of* (1), (2) *if*

1. *every Cauchy problem is uniquely solvable,*
2. *all solutions are (right) global.*

$$u'' = f(t, u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{2}$$

$$\sup\left\{|u(t)| : t \geq 0\right\} < +\infty \tag{5}$$

Theorem 4. *Let the existence of lower and upper functions guarantees solvability of* (1), (2)*. Then solvability of* (1), (5) *guarantees solvability of* (1), (2)*, as well.*

$$u'' = f(t, u) \tag{1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2}$$

Corollary. *Let one of the next inequalities holds on* $[0, \omega] \times \mathbb{R}$

$$f(t, x) \operatorname{sgn} x \geq -p_0(t)|x| - q(t, |x|)$$

or

$$f(t, x) \geq -p_0(t)[x]_+ - p(t)[x]_- - q(t, |x|)$$

or

$$f(t, x) \leq p_0(t)[x]_- + p(t)[x]_+ + q(t, |x|),$$

where $q \in Car\big([0, \omega] \times \mathbb{R}_+; \mathbb{R}\big)$ *is sublinear and*

$$-p_0 \in V_\omega \, .$$

Then the existence of a bounded (right) solution of (1) *implies the solvability of* (1), (2).

Example.

$$u'' = -2\big[u - \sin t\big]_-^3 + 6\big[u - \sin t\big]_+^{1/3} - \sin t \tag{6}$$

$f(t,x) \le 6\sqrt[3]{|x|} + 6$

$\Big(f(t,x) \le p_0(t)[x]_- + p(t)[x]_+ + q(t,|x|)\Big)$

1. There exist a bounded solution.
2. There exist a 2π-periodic solution.
3. Not every solution is global.
4. Not every global solution is bounded.
5. Cauchy problem is not uniquely solvable.

if, for any $p, g \in L\big([0,\omega]\big)$ with

$$a_+(t) \le p(t) \le b_+(t),$$
$$a_-(t) \le g(t) \le b_-(t),$$

the nontrivial solution of

$$u'' + p(t)[u]_+ - g(t)[u]_- = 0,$$
$$u(0) = u(\omega), \quad u'(0) = u'(\omega)$$

do not have zeros.

if the problem (1′), (2) has at least one solution u such that, for some $t_0 \in [0, \omega[$,

$$\min\{\alpha(t_0), \beta(t_0)\} \le u(t_0) \le \max\{\alpha(t_0), \beta(t_0)\}.$$

Definition. We say that a function $\gamma \in C\big([0,\omega]\big)$ is a lower (upper) function of (1), (2) if

1. $\gamma \in AC\big([0,\omega]\big)$ and γ' can be written in the form
$$\gamma'(t) = \gamma_0(t) + \gamma_1(t),$$
where $\gamma_0 \in AC\big([0,\omega]\big)$ and $\gamma_1; [0,\omega] \to \mathbb{R}$ is nondecreasing (nonincreasing) and
$$\gamma_1'(t) = 0 \quad \text{a.e. in } [0,\omega];$$
2. $\gamma(0) = \gamma(\omega)$, $\gamma'(0) \geq \gamma'(\omega)$ $\Big(\gamma'(0) \leq \gamma'(\omega)\Big)$;
3. for a.e. $t \in [0,\omega]$
$$\gamma''(t) \geq f(t,\gamma(t)) \quad \Big(\gamma''(t) \leq f(t,\gamma(t))\Big).$$

I. Kiguradze, *Some singular BVP for second order nonlinear ODE*. Differential Equations **4**(1968), No. 10, 1753–1773.

$$\lim_{x\to+\infty} \frac{1}{x} \int\limits_0^{\omega} q(t,x)dt = 0$$

Definition. We say that $-p_0 \in V_\omega$ if the equation

$$u'' = -p_0(t)u$$

is disconjugate on every interval of length ω, i.e., distance from two consecutive zeros of each nontrivial solution is greater than ω.

$$\Updownarrow$$

Problem

$$u'' = -p_0(t)u\,; \quad u(a) = 0, \quad u(a+\omega) = 0$$

has only the trivial solution for any $a \in [0, \omega[$.

$$\Updownarrow$$

For any $g \in L\big([0,\omega]\big)$ with

$$g(t) \le p_0(t)$$

and any $\bar{t} \in [0, \omega]$, the problem

$$u'' = -g(t)u\,; \quad u(\bar{t}) = 0, \quad u(\bar{t}+\omega) = 0$$

has only the trivial solution.

$$\Uparrow$$

$$p_0(t) \overset{\le}{\not\equiv} \left(\frac{\pi}{\omega}\right)^2 \quad \text{for } t \in [0,\omega] \qquad \text{or} \qquad \int_0^\omega p_0(s)ds \le \frac{4}{\omega}$$

this elegant condition was observed by J. Mawhin in

Remark on the preceding paper of Ahmad and Lazer on periodic solutions
Bolletino U.M.I. (6), **3-A** (1984), 229–238

(f - nonincreasing and asymptotically linear)

I. Rachůnková, S. Staněk, M. Tvrdý
Singularities and Laplacians in BVP for nonlinear ODE
In: Handbook of DE, ODE Vol. 3, 607–723, Elsevier, 2006.

J. L. Massera, *The existence of periodic solutions of systems of differential equations*, Duke Math. J. **17** (1950), 457–475.

(with additional assumption: unique solvability of Cauchy problem)

Example.

$$u'' = -2\big[u - \sin t\big]_-^3 + 6\big[u - \sin t\big]_+^{1/3} - \sin t \tag{6}$$

$f(t,x) \le 6\sqrt[3]{|x|} + 6$

$\Big(f(t,x) \le p_0(t)[x]_- + p(t)[x]_+ + q(t,|x|)\Big)$

1. There exist a bounded solution.

$$u(t) = -\frac{1}{t+c} + \sin t, \quad c \ge 0$$

2. There exist a 2π-periodic solution.
3. Not every solution is global.
4. Not every global solution is bounded.
5. Cauchy problem is not uniquely solvable.

Example.

$$u'' = -2\big[u - \sin t\big]_-^3 + 6\big[u - \sin t\big]_+^{1/3} - \sin t \tag{6}$$

$f(t,x) \le 6\sqrt[3]{|x|} + 6$

$\Big(f(t,x) \le p_0(t)[x]_- + p(t)[x]_+ + q(t,|x|)\Big)$

1. There exist a bounded solution.
2. There exist a 2π-periodic solution.

$$u(t) = \sin t$$

3. Not every solution is global.
4. Not every global solution is bounded.
5. Cauchy problem is not uniquely solvable.

Example.

$$u'' = -2\big[u - \sin t\big]_{-}^{3} + 6\big[u - \sin t\big]_{+}^{1/3} - \sin t \tag{6}$$

$f(t,x) \le 6\sqrt[3]{|x|} + 6$

$\Big(f(t,x) \le p_0(t)[x]_{-} + p(t)[x]_{+} + q(t,|x|)\Big)$

1. There exist a bounded solution.
2. There exist a 2π-periodic solution.
3. Not every solution is global.

$$u(t) = -\frac{1}{t-c} + \sin t, \quad c > 0$$

4. Not every global solution is bounded.
5. Cauchy problem is not uniquely solvable.

Example.

$$u'' = -2\big[u - \sin t\big]_-^3 + 6\big[u - \sin t\big]_+^{1/3} - \sin t \tag{6}$$

$f(t,x) \le 6\sqrt[3]{|x|} + 6$

$\Big(f(t,x) \le p_0(t)[x]_- + p(t)[x]_+ + q(t,|x|)\Big)$

1. There exist a bounded solution.
2. There exist a 2π-periodic solution.
3. Not every solution is global.
4. Not every global solution is bounded.

$$u(t) = (t+c)^3 + \sin t$$

5. Cauchy problem is not uniquely solvable.

Example.

$$u'' = -2\big[u - \sin t\big]_-^3 + 6\big[u - \sin t\big]_+^{1/3} - \sin t \tag{6}$$

$f(t,x) \le 6\sqrt[3]{|x|} + 6$

$\Big(f(t,x) \le p_0(t)[x]_- + p(t)[x]_+ + q(t,|x|)\Big)$

1. There exist a bounded solution.
2. There exist a 2π-periodic solution.
3. Not every solution is global.
4. Not every global solution is bounded.
5. Cauchy problem is not uniquely solvable.

$$u_c(t) = \begin{cases} \sin t & \text{for } t \in [0, c[\\ (t-c)^3 + \sin t & \text{for } t \in [c, +\infty[\end{cases}$$

are solutions of (6) satisfying $u(0) = 0$, $u'(0) = 1$