On the Conditional Stability of Solutions of Nonlinear Differential Systems

I. Kiguradze

A. Razmadze Mathematical Institute, Tbilisi, Georgia

Introduction

In our talk, on the infinite interval I we consider the nonlinear differential system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1}$$

where $I = R_+$ or $I = R$, $f_i : I \times R^n \to R$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions. In the case $I = R_+$, for the system (1) we investigate the boundary value problem

$$x_i(0) = c_i \quad (i = 1, \dots, m), \qquad \limsup_{t \to +\infty} |x_i(t)| < +\infty \quad (i = m+1, \dots, n), \tag{2}$$

and in the case $I = R$ – the boundary value problem

$$\limsup_{t \to -\infty} |x_i(t)| < +\infty \quad (i = 1, \dots, m), \; \limsup_{t \to +\infty} |x_i(t)| < +\infty \; (i = m+1, \dots, n). \tag{3}$$

The theorems containing unimprovable sufficient conditions of solvability and unique solvability of the problems (1), (2) and (1), (3) are formulated in first part in our talk. Under these conditions, solutions of the above-mentioned problems are, generally speaking, unstable in the Ljapunov sense. Therefore there naturally arises the question on their conditional stability in one or another sense. In the second part of the talk there are given the notions of $(m, n-m)$**-stability** and **asymptotic** $(m, n-m)$**-stability** of solutions of the system (1), which in a certain sense make more precise the well-known definition of conditional stability (see [1], [2]).

[1] **L. Cesari, Asymptotic behavior and stability problems in ordinary differential equations. Springer-Velag, Berlin-Göttingen-Heidelberg, 1959.**
[2] **E. A. Coddington,** N. **Levinson, Theory of ordinary differential equations.** *McGraw-Hill Book Company, Inc., New York-Toronto-London,* **1955.**

Moreover, the optimal conditions guaranteeing, respectively, the $(m, n-m)$-stability and asymptotic $(m, n-m)$-stability of a trivial solution of the problems (1),(2) and (1), (3) are found.

Throughout the talk, the use will be made of the following notation:

$$R = \,]-\infty, +\infty[, \qquad R_+ = [0, +\infty[, \quad R_- = \,]-\infty, 0];$$

R^n is the n-dimensional real Euclidian space;

$x = (x_i)_{i=1}^n \in R^n$ is the vector with components x_i $(i = 1, \dots, n)$;

$R^n(g) = \{x = (x_i)_{i=1}^n \in R^n : |x_1| \le g, \dots, |x_n| \le g\}$;

d_{ik} is Kronecker's symbol;

$X = (x_{ik})_{i,k=1}^n$ is the $n \times n$-matrix with components $x_{ik} \in R$ $(i, k = 1, \dots, n)$;

$r(X)$ is the spectral radius of X;

A_s is the set of asymptotically stable, quasi-nonnegative $n \times n$-matrices, i.e. $H = (h_{ik})_{i,k=1}^n \in A_s$ if and only if $h_{ik} \ge 0$ for $i \ne k$ and real parts of eigenvalues of H are negative;

$\tilde{C}_{loc}(I)$ is the space of functions $x : I \to R$, absolutely continuous on every compact interval containing in I;

$L_{loc}(I)$ is the space of functions $x : I \to R$, Lebesgue integrable on every compact interval containing in I;

$L^\infty(I)$ is the space of essentially bounded functions $x : I \to R$ with the norm

$$\|x\|_{L^\infty} = \operatorname{ess\,sup}\{|x(t)| : t \in I\};$$

$\mathcal{K}_{loc}(I \times D)$, where $D \subset R^n$, is the set of functions $f : I \times D \to R$, satisfying the local Carathéodory conditions.

1. Existence and Uniqueness Theorems

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1.1}$$

$$x_i(0) = c_i \quad (i = 1, \dots, m), \ \limsup_{t\to+\infty} |x_i(t)| < +\infty \quad (i = m+1, \dots, n), \tag{1.2}$$

$$\limsup_{t\to-\infty} |x_i(t)| < +\infty \ (i = 1, \dots, m), \ \limsup_{t\to+\infty} |x_i(t)| < +\infty \ (i = m+1, \dots, n) \tag{1.3}$$

Everywhere below, when we deal with the problems (1.1), (1.2) and (1.1), (1.3), we assume, respectively, that

$$f_i \in \mathcal{K}_{loc}(R_+ \times R^n) \quad (i = 1, \dots, n) \text{ and } f_i \in \mathcal{K}_{loc}(R \times R^n) \quad (i = 1, \dots, n).$$

By a solution of the system (1.1), defined on the interval I, is understood the vector function $(x_i)_{i=1}^n : I \to R^n$ with components $x_i \in \widetilde{C}_{loc}(I)$ $(i = 1, \dots, n)$, which almost everywhere on I satisfies this system.

A solution $(x_i)_{i=1}^n$ of the system (1.1), defined on R_+ (defined on R) and satisfying the boundary conditions (1.2) (the boundary conditions (1.3)), is called a solution of the problem (1.1), (1.2) (of the problem (1.1), (1.3)).

The problems (1.1),(1.2) and (1.1),(1.3) were earlier considered in

[3] I. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh.* **30 (1987), 3-103;** English transl.: *J. Sov. Math*. **43 (1988), No. 2, 2259-2339.**

Unlike theorems proved in [3] the results of our talk cover the cases in which the right-hand sides of the system (1.1) are functions rapidly increasing with respect to the phase variables.

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n), \tag{1.1}$$

$$x_i(0) = c_i \quad (i = 1, \ldots, m), \quad \limsup_{t \to +\infty} |x_i(t)| < +\infty \quad (i = m+1, \ldots, n). \tag{1.2}$$

Theorem 1.1. *Let there exist nonnegative functions* $g_i \in \mathcal{K}_{loc}(R_+ \times R^n)$ $(i = 1, \ldots, n)$, $h \in L^\infty(R_+)$ *and a constant matrix* $H = (h_{ik})_{i,k=1}^n$ *such that*

$$H \in A_s, \tag{1.4}$$

and on the set $R_+ \times R^n$ *the inequalities*

$$s_i f_i(t, x_1, \ldots, x_n) \operatorname{sgn}(x_i) \le g_i(t, x_1, \ldots, x_n) \left(\sum_{k=1}^n h_{ik} |x_k| + h(t) \right) (i = 1, \ldots, n) \tag{1.5}$$

where $s_1 = \cdots = s_m = 1$ *and* $s_{m+1} = \cdots = s_n = -1$, *are satisfied. Then for arbitrary* $c_i \in R$ $(i = 1, \ldots, m)$ *the problem* (1.1), (1.2) *has at least one solution, and every solution of that problem is bounded on* R_+.

It is known (see **I.T. Kiguradze, Initial and boundary value problems for systems of ordinary differential equations, I.** (Russian) *Metsniereba, Tbilisi*, **1997,** Theorem 1.18) that the quasi-nonnegative matrix $H = (h_{ik})_{i,k=1}^n$ satisfies the condition (1.4) iff

$$h_{ii} < 0 \quad (i = 1, \ldots, n) \quad \text{and} \quad r(H_0) < 1, \tag{1.6}$$

where $$H_0 = \left((1 - d_{ik}) \frac{h_{ik}}{|h_{ii}|} \right)_{i,k=1}^n .$$

Thus the condition (1.4) in Theorem 1.1 can be replaced by the equivalent condition (1.6).

Note that the condition (1.4) both in Theorem 1.1 and in other theorems below is unimprovable and it cannot be weakened. In particular, the condition (1.4) cannot be replaced by the condition

$$h_{ii} < 0 \quad (i = 1, \ldots, n), \quad r(H_0) \le 1. \tag{1.6'}$$

Indeed, consider the problem

$$\frac{dx_i}{dt} = (-1)^i (x_1 + x_2) + i - 1 \quad (i = 1,2), \tag{1.7}$$

$$x_i(0) = c_1, \quad \limsup_{t \to +\infty} |x_2(t)| < +\infty. \tag{1.8}$$

For this problem all the conditions of Theorem 1.1 are hold except (1.4), instead of which the condition (1.6’) holds, since

$$H = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Nevertheless the problem (1.7),(1.8) does not have a solution since general solution of the system (1.7) has the form

$$x_1(t) = a_1 - (a_1 + a_2)t - \frac{t^2}{2}, \quad x_2(t) = a_2 + (a_1 + a_2 + 1)t + \frac{t^2}{2},$$

where a_1 and a_2 are arbitrary real numbers.

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1.1}$$

$$x_i(0) = c_i \quad (i = 1, \dots, m), \quad \limsup_{t \to +\infty} |x_i(t)| < +\infty \quad (i = m+1, \dots, n). \tag{1.2}$$

Corollary 1.1. *Let the conditions of Theorem* 1.1 *be fulfilled and*

$$\int_0^{+\infty} p_i(s)\, ds = +\infty \quad (i = m+1, \dots, n), \tag{1.9}$$

where

$$p_i(t) = \inf\left\{ g_i(t, x_1, \dots, x_n) : (x_k)_{k=1}^n \in R^n \right\}. \tag{1.10}$$

Then every solution of the problem (1.1), (1.2) *admits the estimate*

$$\sum_{k=1}^{m} |x_k(t)| \le r \left(\sum_{k=1}^{m} |c_k| + \|h\|_{L^\infty} \right) \quad \text{for} \quad t \in R_+, \tag{1.11}$$

where r *is a positive constant, depending only on* H.

From the estimates (1.11) it, in particular, follows that if the conditions of Corollary 1.1 are fulfilled, then an arbitrary solution of the system (1.1), satisfying the conditions

$$x_i(0) = c_i \quad (i = 1, \dots, m) \quad \text{and} \quad \sum_{k=1}^{n} |x_k(0)| > r \left(\sum_{k=1}^{m} |c_k| + \|h\|_{L^\infty} \right),$$

is either unbounded or blowing-up.

Corollary 1.2. *Let the conditions of Theorem* 1.1 *be fulfilled and*

$$\lim_{t \to +\infty} h(t) = 0, \qquad \int_0^{+\infty} p_i(s)\, ds = +\infty \quad (i = 1, \dots, n), \tag{1.12}$$

where each p_i *is the function given by the equality* (1.10). *Then an arbitrary solution of the problem* (1.1), (1.2) *satisfies the equalities*

$$\lim_{t \to +\infty} x_i(t) = 0 \quad (i = 1, \dots, n). \tag{1.13}$$

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1.1}$$

$$x_i(0) = c_i \quad (i = 1, \dots, m), \ \limsup_{t \to +\infty} |x_i(t)| < +\infty \quad (i = m+1, \dots, n), \tag{1.2}$$

$$\int_0^{+\infty} p_i(s)\,ds = +\infty \quad (i = m+1, \dots, n), \tag{1.9}$$

$$\sum_{k=1}^{m} |x_k(t)| \le r\left(\sum_{k=1}^{m} |c_k| + \|h\|_{L^\infty}\right) \quad for \quad t \in R_+, \tag{1.11}$$

$$\lim_{t \to +\infty} h(t) = 0, \qquad \int_0^{+\infty} p_i(s)\,ds = +\infty \quad (i = 1, \dots, n), \tag{1.12}$$

$$\lim_{t \to +\infty} x_i(t) = 0 \quad (i = 1, \dots, n). \tag{1.13}$$

Let us formulate the theorem on unique solvability of the problem (1.1), (1.2).

Theorem 1.2. *Let there exist nonnegative functions* $p_i \in L_{loc}(R_+)$ $(i = 1, \dots, n)$, $h \in L^\infty(R_+)$, *and a constant matrix* $H = (h_{ik})_{i,k=1}^{n} \in A_s$ *such that, respectively, on* $R_+ \times R^n$ *and* R_+ *the conditions*

$$s_i(f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n))\operatorname{sgn}(x_i - y_i) \le$$

$$\le p_i(t) \sum_{k=1}^{n} h_{ik} |x_k - y_k| \quad (i = 1, \dots, n), \tag{1.14}$$

$$|f_i(t, 0, \dots, 0)| \le h(t) p_i(t) \quad (i = 1, \dots, n), \tag{1.15}$$

where $s_1 = \dots = s_m = 1$, $s_{m+1} = \dots = s_n = -1$, *are satisfied. If, moreover, the equalities* (1.9) (*the equalities* (1.12)) *hold, then for arbitrary* $c_i \in R$ $(i = 1, \dots, m)$ (1.1),(1.2) *has a unique solution satisfying the condition* (1.11) (*the conditions* (1.11) *and* (1.13)), *where* r *is a positive constant, depending only on* H.

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1.1}$$

$$\limsup_{t\to-\infty} |x_i(t)| < +\infty \ (i = 1, \dots, m), \ \limsup_{t\to+\infty} |x_i(t)| < +\infty (i = m+1, \dots, n) \tag{1.3}$$

$$s_i f_i(t, x_1, \dots, x_n) \operatorname{sgn}(x_i) \le g_i(t, x_1, \dots, x_n) \left(\sum_{k=1}^{n} h_{ik} |x_k| + h(t) \right) (i = 1, \dots, n). \tag{1.5}$$

Theorem 1.3. *Let there exist nonnegative functions* $g_i \in \mathcal{K}_{loc}(R \times R^n)$ $(i = 1, \dots, n)$ *and a constant matrix* $H = (h_{ik})_{i,k=1}^{n} \in A_s$ *such that on the set* $R \times R^n$ *the inequalities* (1.5) *are satisfied, where* $s_1 = \dots = s_m = 1$ *and* $s_{m+1} = \dots = s_n = -1$. *Then the problem* (1.1), (1.3) *has at least one solution, and every solution of that problem is bounded on* R.

Corollary 1.3. *Let the conditions of Theorem* 1.3 *be fulfilled and*

$$\int_{-\infty}^{0} p_i(s)ds = +\infty \quad (i = 1, \dots, m), \quad \int_{0}^{+\infty} p_i(s)ds = +\infty \ (i = m+1, \dots, n), \tag{1.16}$$

where each p_i *is the function given by the equality* (1.10). *Then every solution of the problem* (1.1), (1.3) *admits the estimates*

$$|x_i(t)| \le r \|h\|_{L^\infty} \quad \text{for} \quad t \in R \quad (i = 1, \dots, n), \tag{1.17}$$

where r *is a positive constant, depending only on* H. *If, however, instead of* (1.16) *there are satisfied the conditions*

$$\lim_{t\to-\infty} h_i(t) = \lim_{t\to+\infty} h_i(t) = 0, \ \int_{-\infty}^{0} p_i(s)ds = \int_{0}^{\infty} p_i(s)ds = +\infty \ (i = 1, \dots, n), \tag{1.18}$$

then every solution of the problem (1.1), (1.3) *along with* (1.17) *satisfies the conditions*

$$\lim_{t\to-\infty} x_i(t) = \lim_{t\to+\infty} x_i(t) = 0 \quad (i = 1, \dots, n). \tag{1.19}$$

The following theorem concerns to unique solvability of the problem (1.1),(1.3).

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1.1}$$

$$\limsup_{t\to-\infty} |x_i(t)| < +\infty \ (i = 1, \dots, m), \ \limsup_{t\to+\infty} |x_i(t)| < +\infty \ (i = m+1, \dots, n) \tag{1.3}$$

$$s_i(f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n))\operatorname{sgn}(x_i - y_i) \le$$
$$\le p_i(t)\sum_{k=1}^{n} h_{ik}|x_k - y_k| \quad (i = 1, \dots, n), \tag{1.14}$$

$$|f_i(t, 0, \dots, 0)| \le h(t)p_i(t) \quad (i = 1, \dots, n), \tag{1.15}$$

$$\int_{-\infty}^{0} p_i(s)ds = +\infty \quad (i = 1, \dots, m), \quad \int_{0}^{+\infty} p_i(s)ds = +\infty \ (i = m+1, \dots, n), \tag{1.16}$$

$$|x_i(t)| \le r\|h\|_{L^\infty} \quad for \quad t \in R \quad (i = 1, \dots, n), \tag{1.17}$$

$$\lim_{t\to-\infty} h_i(t) = \lim_{t\to+\infty} h_i(t) = 0, \ \int_{-\infty}^{0} p_i(s)ds = \int_{0}^{\infty} p_i(s)ds = +\infty \ (i = 1, \dots, n) \tag{1.18}$$

$$\lim_{t\to-\infty} x_i(t) = \lim_{t\to+\infty} x_i(t) = 0 \quad (i = 1, \dots, n). \tag{1.19}$$

Theorem 1.4. *Let there exist nonnegative functions* $p_i \in L_{loc}(R) \quad (i = 1, \dots, n)$, $h \in L^\infty(R)$ *and a constant matrix* $H = (h_{ik})_{i,k=1}^{n} \in A_s$ *such that, respectively, on* $R \times R^n$ *and* R *the conditions* (1.14) *and* (1.15), *where* $s_1 = \dots = s_m = 1$, $s_{m+1} = \dots = s_n = -1$, *are satisfied. If, moreover, the equalities* (1.16) (*the equalities* (1.18)) *are fulfilled, then the problem* (1.1), (1.3) *has a unique solution satisfying the condition* (1.17) (*the conditions* (1.17) *and* (1.19)), *where* r *is a positive constant, depending only on* H.

2. Theorems on the Conditional Stability

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n). \tag{2.1}$$

Let us consider first the case where for some $g > 0$,

$$f_i \in \mathcal{K}_{loc}\big(R_+ \times R^n(g)\big) \quad (i = 1, \dots, n),$$

and the system (2.1) on R_+ has a trivial solution, i.e. $f_i(t, 0, \dots, 0) \equiv 0$ $(i = 1, \dots, n)$.

A classical result on the conditional stability of system (2.1) concerns the case where this system has the form

$$\frac{dx_i}{dt} = \sum_{i=1}^{n} p_{ik} x_k + q_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{2.1_0}$$

Here $\mathcal{P} = (p_{ik})_{i,k=1}^{n}$ is a constant real matrix, $q_i : R_+ \times R_+ \to R$ are continuous functions having continuous partial derivatives with respect to phase variables such that

$$\lim_{\sum_{k=1}^{n} |x_k| \to 0} \frac{\partial q_i(t, x_1, \dots, x_n)}{\partial x_j} = 0 \quad (i, j = 1, \dots, n)$$

uniformly with respect to $t \in R_+$. If, in addition, the matrix $\mathcal{P}$ has m eigenvalues with negative real parts and $n - m$ eigenvalues with positive real parts, then for any $a \in R_+$ there exists an m-dimensional manifold $D \subset R^m$ containing the origin such that an arbitrary solution $x = (x_i)_{i=1}^{n}$ of the system (2.1_0), satisfying the condition $x(a) \in D$, vanishes at infinity, i.e.

$$\lim_{t \to +\infty} x_i(t) = 0 \quad (i = 1, \dots, n)$$

(see [2], Theorem 4.1).

[2] **E. A. Coddington,** N. **Levinson, Theory of ordinary differential equations.** *McGraw-Hill Book Company, Inc., New York-Toronto-London,* **1955.**

Let us clarify that in the classical case the conditional stability is understood as the existence of a manifold D of the above-mentioned kind.

This definition is applicable only to the case where for the considered differential system the Cauchy problem is uniquely solvable for any initial data.

The definitions of conditional stability, proposed by us, are applicable also to the case where for the system (2.1) the uniqueness of a solution of the Cauchy problem is violated.

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n). \tag{2.1}$$

$$f_i \in \mathcal{K}_{loc}\left(R_+ \times R^n(g)\right), \; f_i(t, 0, \ldots, 0) \equiv 0 \; (i = 1, \ldots, n).$$

For the system (2.1) we have to consider the following boundary value problems

$$x_i(a) = c_i \quad (i = 1, \ldots, m), \qquad x_i(b) = c_i \quad (i = m+1, \ldots, n) \tag{2.2}$$

and

$$x_i(a) = c_i \quad (i = 1, \ldots, m), \qquad \limsup_{t \to +\infty} |x_i(t)| < +\infty \quad (i = m+1, \ldots, n), \tag{2.3}$$

where $m \in \{1, \ldots, n-1\}$.

Definition 2.1. A trivial solution of the system (2.1) is said to be $(m, n-m)$-stable on R_+ if for any $e \in \,]0, g[$ there exists $d \in \,]0, e[$ such that:

(i) for arbitrary $a \in R_+$, $b \in \,]a, +\infty[$, and $c_i \in [-d, d]$ $(i = 1, \ldots, n)$, the problem (2.1), (2.2) has at least one solution, and every solution of that problem on $[a, b]$ satisfies the inequality

$$\sum_{i=1}^{n} |x_i(t)| < e\,; \tag{2.4}$$

(ii) for arbitrary $a \in R_+$, and $c_i \in [-d, d]$ $(i = 1, \ldots, m)$, the problem (2.1), (2.3) has at least one solution, and every solution of that problem satisfies on $[a, +\infty[$ the inequality (2.4).

Definition 2.2. A trivial solution of the system (2.1) is said to be asymptotically $(m, n-m)$-stable on R_+ if it is $(m, n-m)$-stable on R_+ and there exists $d_0 \in \,]0, g[$ such that for arbitrary $a \in R_+$, and $c_i \in [-d_0, d_0]$ $(i = 1, \ldots, m)$ every solution of the problem (2.1), (2.3) vanishes at $+\infty$.

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n). \tag{2.1}$$

$$f_i \in \mathcal{K}_{loc}\big(R_+ \times R^n(g)\big), \;\; f_i(t, 0, \dots, 0) \equiv 0 \;\; (i = 1, \dots, n).$$

Theorem 2.1. *Let on* $R_+ \times R^n(g)$ *the inequalities*

$$s_i f_i(t, x_1, \dots, x_n)\operatorname{sgn}(x_i) \le g_i(t, x_1, \dots, x_n)\sum_{k=1}^{n} h_{ik}|x_k| \quad (i = 1, \dots, n) \tag{2.5}$$

be fulfilled, where $s_1 = \dots = s_m = 1$, $s_{m+1} = \dots = s_n = -1$, $(h_{ik})_{i,k=1}^{n} \in A_s$, *and* $g_i \in \mathcal{K}_{loc}\big(R_+ \times R^n(g)\big)$ $(i = 1, \dots, n)$ *are nonnegative functions. If, moreover, the conditions*

$$\int_0^{+\infty} p_i(s)ds = +\infty \quad (i = m+1, \dots, n), \tag{2.6}$$

where

$$p_i(t) = \inf\left\{g_i(t, x_1, \dots, x_n) : (x_k)_{k=1}^{n} \in R^n\right\}, \tag{2.7}$$

are fulfilled, then a trivial solution of the system (2.1) *is* $(m, n-m)$*-stable on* R_+. *If, however, instead of* (2.6) *we have*

$$\int_0^{+\infty} p_i(s)ds = +\infty \quad (i = 1, \dots, n), \tag{2.8}$$

then a trivial solution of the system (2.1) *is asymptotically* $(m, n-m)$*-stable on* R_+.

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n). \tag{2.1}$$

$$\int_0^{+\infty} p_i(s)\,ds = +\infty \quad (i = m+1, \dots, n). \tag{2.6}$$

$$\int_0^{+\infty} p_i(s)\,ds = +\infty \quad (i = 1, \dots, n). \tag{2.8}$$

As an example, consider the case when

$$f_i(t, x_1, \dots, x_n) = -p_i(t)x_i \quad (i = 1, \dots, m),$$
$$f_i(t, x_1, \dots, x_n) = p_i(t)x_i \quad (i = m+1, \dots, n),$$

and $p_i \in L_{loc}(R_+)$ $(i = 1, \dots, n)$ are nonnegative functions. Then for the system (2.1) to be $(m, n-m)$-stable (asymptotically $(m, n-m)$-stable), it is necessary and sufficient that the conditions (2.6) (the conditions (2.8)) be fulfilled. Consequently, conditions (2.6), (conditions (2.8)) in Theorem 2.1 are optimal, and they cannot be weakened.

Consider the linear differential system

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} p_{ik}(t)x_k \quad (i=1,\dots,n) \tag{2.9}$$

with coefficients $p_{ik} \in L_{loc}(R_+)$ $(i,k=1,\dots,n)$. We call this system $(m,n-m)$-stable (asymptotically $(m,n-m)$-stable) on R_+ if its trivial solution is $(m,n-m)$-stable (asymptotically $(m,n-m)$-stable) on R_+.

From Theorem 2.1 it follows

Corollary 2.1. *Let on R_+ the conditions*

$$p_{ii}(t) = s_i h_{ii} p_i(t), \quad |p_{ik}(t)| \le h_{ik} p_i(t) \quad (i,k=1,\dots,n; \quad i \ne k) \tag{2.10}$$

be satisfied, where $s_1 = \dots = s_m = 1$, $s_{m+1} = \dots = s_n = -1$, and $(h_{ik})_{i,k=1}^{n} \in A_s$. If, moreover, the equalities

$$\int_0^{+\infty} p_i(s)ds = +\infty \quad (i=m+1,\dots,n) \tag{2.6}$$

(*the equalities*

$$\int_0^{+\infty} p_i(s)ds = +\infty \quad (i=1,\dots,n) \tag{2.8}$$

) *are fulfilled, then the system* (2.9) *is* $(m,n-m)$*-stable* (*asymptotically* $(m,n-m)$*-stable*) *on* R_+.

It can be easily seen that under the conditions of Corollary 2.1 the bounded solutions of the system (2.9) form an m-dimensional linear space. Moreover, if the equalities (2.8) are fulfilled, then all solutions from the above-mentioned space are vanishing at $+\infty$.

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n). \tag{2.1}$$

Let us now consider the case, where for some $g > 0$,

$$f_i \in \mathcal{K}_{loc}\big(R \times R^n(g)\big), \;\; f_i(t, 0, \ldots, 0) \equiv 0 \;\; (i = 1, \ldots, n).$$

In this case, for the system (2.1) along with the problems (2.2) and (2.3) we have to consider also the problem

$$x_i(a) = c_{i-m} \quad (i = m+1, \ldots, n), \quad \limsup_{t \to -\infty} |x_i(t)| < +\infty \quad (i = 1, \ldots, m). \tag{2.11}$$

Definition 2.3. A trivial solution of the system (2.1) is said to be $(m, n-m)$-stable on R if for any $e \in \left]0, g\right[$ there exists $d \in \left]0, e\right[$ such that:

(i) for arbitrary $a \in R$, $b \in \left]a, +\infty\right[$, $c_i \in [-d, d]$ $(i = 1, \ldots, n)$, the problem (2.1), (2.2) has at least one solution, and every solution of that problem satisfies on $[a, b]$ the inequality (2.4);

(ii) for arbitrary $a \in R$ and $c_i \in [-d, d]$ $(i = 1, \ldots, m)$ (for arbitrary $a \in R$ and $c_i \in [-d, d]$ $(i = 1, \ldots, n-m)$), the problem (2.1), (2.3) (the problem (2.1), (2.11)) has at least one solution, and every solution of that problem satisfies on $\left[a, +\infty\right[$ (on $\left]-\infty, a\right]$) the inequality (2.4).

Definition 2.4. A trivial solution of the system (2.1) is said to be asymptotically $(m, n-m)$-stable on R if it is $(m, n-m)$-stable on R, and there exists $d_0 \in \left]0, g\right[$ such that for arbitrary $a \in R$, $c_i \in [-d_0, d_0]$ $(i = 1, \ldots, m)$ (for arbitrary $a \in R$ and $c_i \in [-d_0, d_0]$ $(i = 1, \ldots, n-m)$) every solution of the problem (2.1), (2.3) (of the problem (2.1), (2.11)) satisfies the equalities

$$\lim_{t \to +\infty} x_i(t) = 0 \quad (i = 1, \ldots, n), \qquad \left(\lim_{t \to -\infty} x_i(t) = 0 \quad (i = 1, \ldots, n) \right).$$

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n). \tag{2.1}$$

$$f_i \in \mathcal{K}_{loc}\big(R \times R^n(g)\big), \ \ f_i(t, 0, \dots, 0) \equiv 0 \ \ (i = 1, \dots, n).$$

$$s_i f_i(t, x_1, \dots, x_n)\operatorname{sgn}(x_i) \le g_i(t, x_1, \dots, x_n)\sum_{k=1}^{n} h_{ik}|x_k| \quad (i = 1, \dots, n). \tag{2.5}$$

$$p_i(t) = \inf\left\{g_i(t, x_1, \dots, x_n) : (x_k)_{k=1}^n \in R^n\right\} \ \ (i = 1, \dots, n). \tag{2.7}$$

Theorem 2.2. *Let on* $R \times R^n(g)$ *the inequalities* (2.5) *be satisfied, where* $s_1 = \dots = s_m = 1$, $s_{m+1} = \dots = s_n = -1$, $(h_{ik})_{i,k=1}^n \in$, A_s *and* $g_i \in \mathcal{K}_{loc}\big(R \times R^n(g)\big)$ $(i = 1, \dots, n)$ *are nonnegative functions. If, moreover, the conditions*

$$\int_{-\infty}^{0} p_i(s)ds = +\infty \ \ (i = 1, \dots, m), \ \ \int_{0}^{+\infty} p_i(s)ds = +\infty \ (i = m+1, \dots, n), \tag{2.12}$$

are fulfilled, where each p_i *is the function given by the equality* (2.7), *then a trivial solution of the system* (2.1) *is* $(m, n-m)$-*stable on R. If, however, instead of* (2.12) *we have*

$$\int_{-\infty}^{0} p_i(s)ds = \int_{0}^{+\infty} p_i(s)ds = +\infty \quad (i = 1, \dots, n), \tag{2.13}$$

then a trivial solution of the system (2.1) *is asymptotically* $(m, n-m)$-*stable on R.*

If the right-hand sides of the system (2.1) are of the form $f_i(t, x_1, \dots, x_n) = -p_i(t)x_i$ $(i = 1, \dots, m)$, $f_i(t, x_1, \dots, x_n) = p_i(t)x_i$ $(i = m+1, \dots, n)$ where $p_i \in L_{loc}(R)$ $(i = 1, \dots, n)$ are nonnegative functions, then for a trivial solution of the system (2.1) to be $(m, n-m)$-stable (asymptotically $(m, n-m)$-stable) on R, it is necessary and sufficient that the conditions (2.12) (the conditions (2.13)) be fulfilled. Consequently, the conditions of Theorem 2.2 are in a certain sense unimprovable.

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} p_{ik}(t) x_k \quad (i = 1, \ldots, n). \tag{2.9}$$

$$\int_{-\infty}^{0} p_i(s) ds = +\infty \quad (i = 1, \ldots, m), \quad \int_{0}^{+\infty} p_i(s) ds = +\infty \ (i = m+1, \ldots, n). \tag{2.12}$$

$$\int_{-\infty}^{0} p_i(s) ds = \int_{0}^{+\infty} p_i(s) ds = +\infty \quad (i = 1, \ldots, n). \tag{2.13}$$

At the end, consider the linear differential system (2.9) with coefficients $p_{ik} \in L_{loc}(R)$ $(i, k = 1, \ldots, n)$.

From Theorem 2.2 it follows

Corollary 2.2. *Let on R the conditions*

$$p_{ii}(t) = s_i h_{ii} p_i(t), \quad |p_{ik}(t)| \le h_{ik} p_i(t) \quad (i, k = 1, \ldots, n; \quad i \ne k) \tag{2.10}$$

be satisfied, where $s_1 = \cdots = s_m = 1$, $s_{m+1} = \cdots = s_n = -1$ *and* $(h_{ik})_{i,k=1}^{n} \in A_s$. *If, moreover, the equalities* (2.12) (*the equalities* (2.13)) *are fulfilled, then the system* (2.9) *is* $(m, n-m)$*-stable* (*asymptotically* $(m, n-m)$*-stable*) *on R*.

It is clear that under the conditions of Corollary 2.2 the system (2.9) has no nontrivial bounded solution on R, and a set of bounded on R_+ (bounded on R_-) solutions of that system forms an m-dimensional (an $(n-m)$-dimensional) linear space.