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The Fučík Spectrum Structure:
Known Results, Experiments and Open Problems

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Boundary Value Problems and Related Topics

Hejnice, September 16 – 20

Outline

The Fučík Spectrum

$$\Sigma(L) := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\},$$

where

$$u^+ := \max\{u, 0\}, \quad u^- := \max\{-u, 0\}.$$

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- $$Lu = (\lambda + \varepsilon\eta)u + \varepsilon|u|$$
- Let $\lambda \in \sigma_d(L) = \sigma(L)$. For $\varepsilon \neq 0$, we have

$$\left\{ \begin{array}{c} u = Pu + \varepsilon(L-\lambda I)^{-1}\left[(I-P)(|u|+\eta u)\right] + P(|u|+\eta u), \\ \|u\|^2 = 1. \end{array} \right. \tag{1}$$

The Local Existence of $\Sigma(L)$ Close to the Diagonal

Definition

Let us define the functional $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \subset L^2(\Omega) \to \mathbb{R}$

$$\mathrm{dom}(\mathcal{P}) := \ker(L - \lambda I), \qquad \mathcal{P}(z) := \langle |z|, z \rangle .$$

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Theorem (Ben-Naoum, Fabry and Smets)

Let $\lambda \in \sigma_d(L)$. Assume that $u \neq 0$ a.e. in Ω for all $u \in \ker(L - \lambda I) \setminus \{0\}$ and that the functional $\mathcal{P}$, restricted to the unit sphere, has a stationary point z_0. Moreover, let us assume that the following nondegeneracy condition is satisfied at this stationary point z_0

$$y \in \ker(L - \lambda I),\ \langle z_0, y \rangle = 0,\ P(\mathrm{sgn}(z_0)y) = \langle |z_0|, z_0 \rangle\, y \quad \Rightarrow \quad y = 0. \qquad \text{(ND1)}$$

Then, there exists a neighborhood $\mathcal{U}(0) \subset \mathbb{R}$ of 0 and two continuous mappings $\eta : \mathcal{U}(0) \to \mathbb{R}$ and $u : \mathcal{U}(0) \to \mathrm{dom}(L) \subset L^2(\Omega)$ such that

$$\begin{aligned} u(0) &= z_0, \\ \eta(0) &= -\langle |z_0|, z_0 \rangle, \\ Lu(\varepsilon) &= \varepsilon |u(\varepsilon)| + (\lambda + \varepsilon\eta(\varepsilon))u(\varepsilon), \quad \|u(\varepsilon)\| = 1 \qquad \textit{for } \varepsilon \in \mathcal{U}(0). \end{aligned}$$

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For $\dim \ker(L - \lambda I) \geq 2$, (ND1) can be replaced by

$$\exists \mathcal{V}(z_0) \subset S \text{ such that } \begin{cases} \max\{\mathcal{P}(z) : z \in \mathcal{V}(z_0)\} = \mathcal{P}(z_0), & \mathcal{P}(z) < \mathcal{P}(z_0) \quad \forall z \in \partial\mathcal{V}(z_0), \\ \min\{\mathcal{P}(z) : z \in \mathcal{V}(z_0)\} = \mathcal{P}(z_0), & \mathcal{P}(z) > \mathcal{P}(z_0) \quad \forall z \in \partial\mathcal{V}(z_0), \end{cases} \tag{ND1'}$$

where S is a unit sphere in $\ker(L - \lambda I)$.

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the slope of the tangent line

$$\beta'(\lambda) = \frac{\mathcal{P}(z_0) + 1}{\mathcal{P}(z_0) - 1}$$

The Existence of $\Sigma(L)$ Away from the Diagonal

Theorem (Ben-Naoum, Fabry and Smets)

Let $E \subset \mathbb{R}$ *and* u_0, $\alpha_0 \neq \beta_0$ *be such that* $(\alpha_0, \beta_0) \in (E \times E) \cap \Sigma(L)$, $Lu_0 = \alpha_0 u_0^+ - \beta_0 u_0^-$, $\|u_0\| = 1$. *Moreover, let*

- *the equation* $Lu = \lambda u$ *has no solution of a constant sign for* $\lambda \in E$,
- *the nondegeneracy condition (ND2) holds for* $(\alpha, \beta) = (\alpha_0, \beta_0)$,
- $\mathrm{dom}(L) \subset L^p(\Omega)$ *for some* $p > 2$ *and the injection is continuous when* $\mathrm{dom}(L)$ *is equipped with the graph norm.*

The Existence of $\Sigma(L)$ Away from the Diagonal

Nondegeneracy Condition:

For any $u \neq 0$ verifying $Lu = \alpha u^+ - \beta u^-$, we have
$u \neq 0$ a.e. on Ω and $\dim \ker \left[L - (\alpha \chi_{\{u>0\}} + \beta \chi_{\{u<0\}})I\right] = 1.$ (ND2)

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Then, there exist neighborhoods $\mathcal{A}(\alpha_0) \subset \mathbb{R}$, $\mathcal{B}(\beta_0) \subset \mathbb{R}$, $\mathcal{U}(u_0) \subset \operatorname{dom}(L)$ *and continuous mappings* $\beta = \beta(\alpha) : \mathcal{A}(\alpha_0) \to \mathcal{B}(\beta_0)$*, and* $u = u(\alpha) : \mathcal{A}(\alpha_0) \to \mathcal{U}(u_0)$ *such that*

- $\beta(\alpha_0) = \beta_0$ and $u(\alpha_0) = u_0$,
- $Lu(\alpha) = \alpha u^+(\alpha) - \beta(\alpha)u^-(\alpha)$ *with* $\|u(\alpha)\| = 1$ *for* $\alpha \in \mathcal{A}(\alpha_0)$,
- $Lu = \alpha u^+ - \beta u^-$ *with* $\|u\| = 1$ *and* $u \in \mathcal{U}(u_0)$, $\alpha \in \mathcal{A}(\alpha_0)$, $\beta \in \mathcal{B}(\beta_0) \Rightarrow$ $u = u(\alpha)$ *and* $\beta = \beta(\alpha)$.

Moreover, the function $\beta = \beta(\alpha)$ *is differentiable at* α_0 *and* $\beta'(\alpha_0) = -\dfrac{\|u_0^+\|^2}{\|u_0^-\|^2} < 0.$

Outline

Example 1

$$\begin{cases} u^{\mathrm{IV}}(x) + (m^2 + n^2)u''(x) = \alpha u^+(x) - \beta u^-(x), \quad x \in (0, \pi), \\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0. \end{cases} \tag{2}$$

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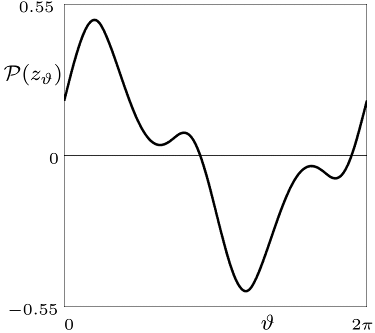
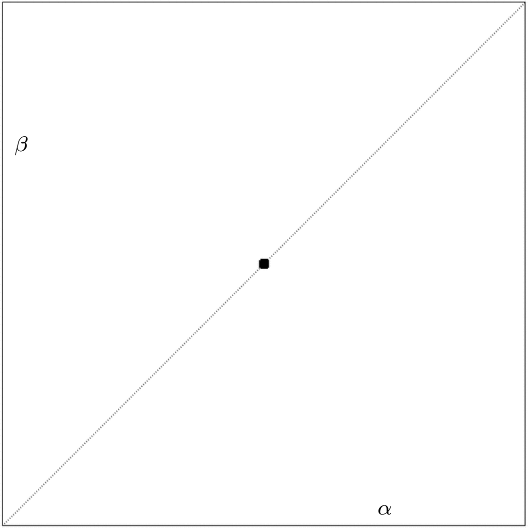
$$\alpha = \beta = -m^2 n^2, \quad m = 5, n = 9,$$

$$z_\vartheta(x) := \varphi_1(x)\cos\vartheta + \varphi_2(x)\sin\vartheta,$$

$$\vartheta \in [0, 2\pi).$$





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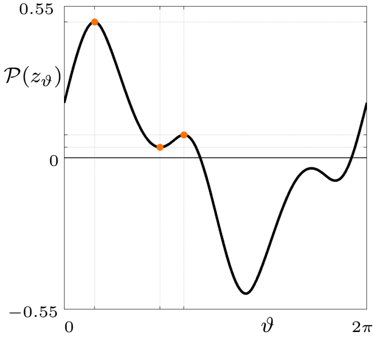

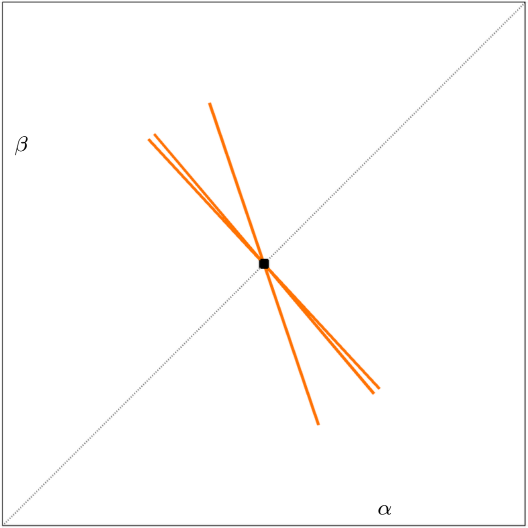

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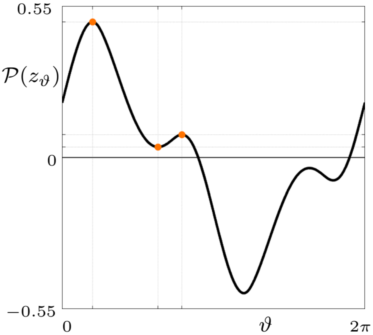

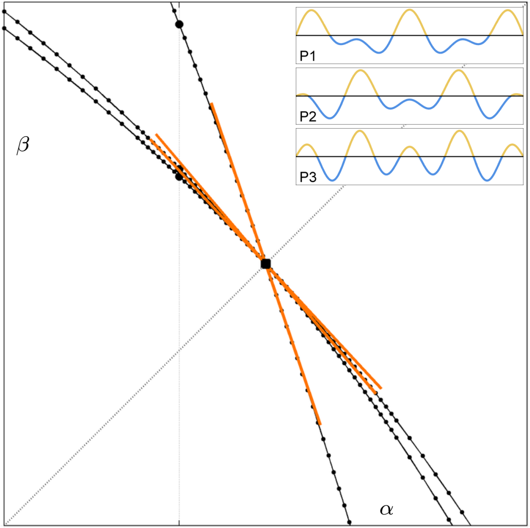

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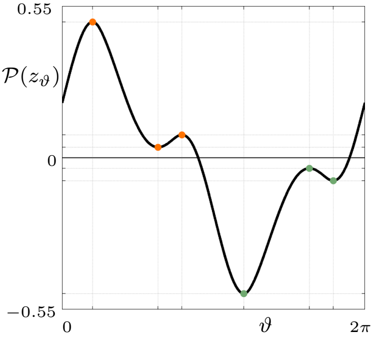

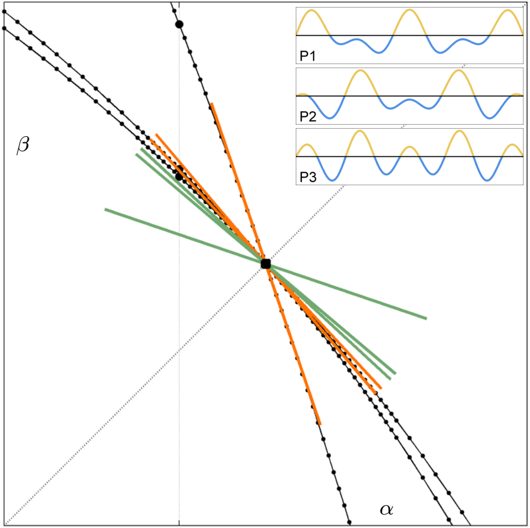

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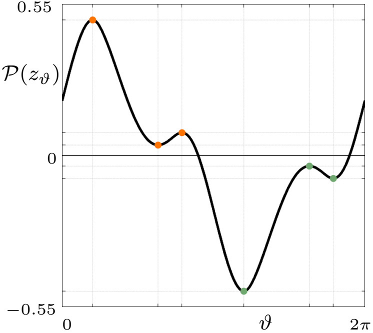
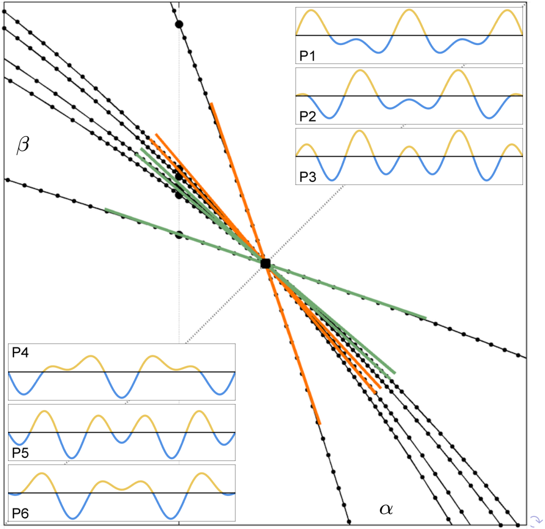
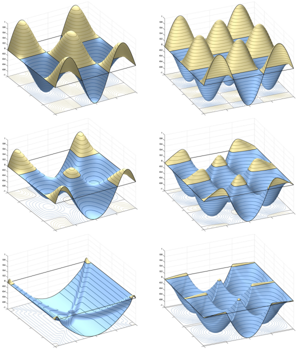
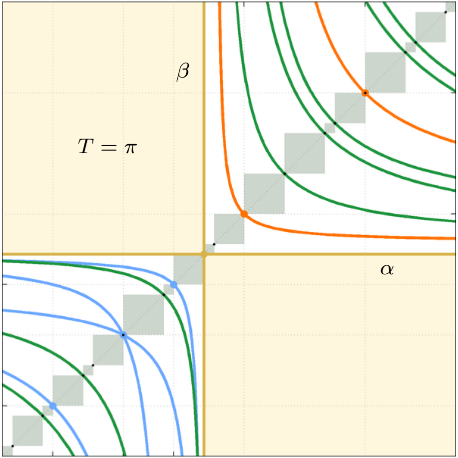




Example 2

$$\begin{cases} -(u_{tt}(x,t) - u_{xx}(x,t)) = \alpha u^{+}(x,t) - \beta u^{-}(x,t), & (x,t) \in (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbb{R}, \\ u(x,t) = u(x,t+T), & (x,t) \in (0,\pi)\times\mathbb{R}, \end{cases} \tag{3}$$





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$$\alpha = \beta = -9,$$

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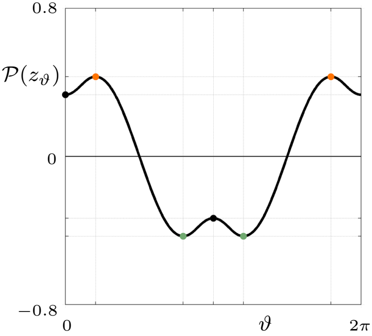

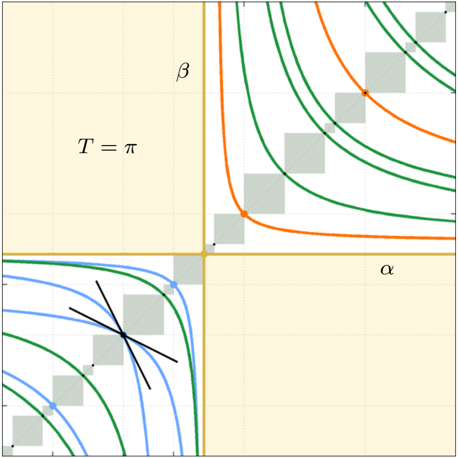

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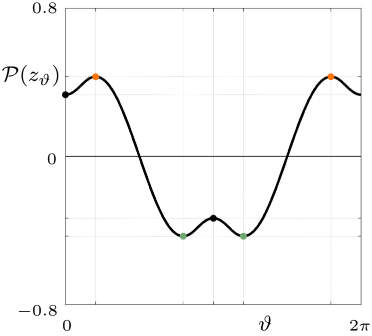
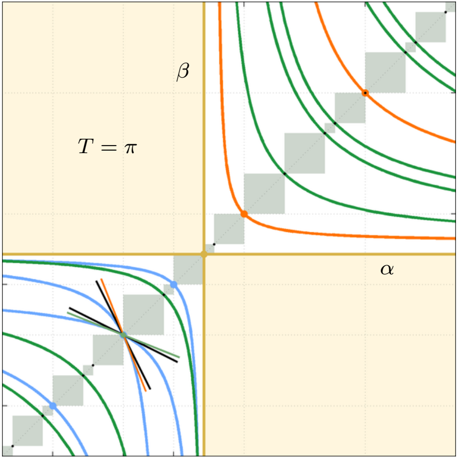
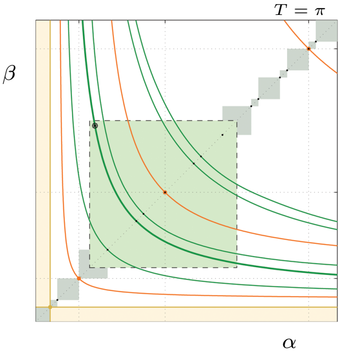




Verification of Explored Fučík Curves

- $E \subset \mathbb{R}$, $\quad E \cap \sigma(L) = \{\lambda_1, \dots, \lambda_p\}$
- $Z := \oplus_{i=1}^{p} \ker(L - \lambda_i I)$
- P – orthogonal projection onto Z



Lemma

Let $(\alpha, \beta) \in E \times E$, $\alpha \neq \beta$.
$(\alpha, \beta) \in \Sigma(L) \quad \Longleftrightarrow \quad \exists\, u \in L^2(\Omega)$ *such that* $Pu \neq 0$ *and that*

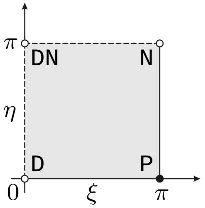
$$\left\{ \begin{array}{ll} \langle |u|, \varphi_i \rangle = 0 & \text{for } \langle u, \varphi_i \rangle = 0, \\ \dfrac{\langle |u|, \varphi_i \rangle}{\langle u, \varphi_i \rangle} = \dfrac{\lambda_i - \frac{\alpha+\beta}{2}}{\frac{\alpha-\beta}{2}} & \text{for } \langle u, \varphi_i \rangle \neq 0, \end{array} \right\}$$

for every eigenfunction $\varphi_i \in \ker(L - \lambda_i I)$, $i = 1, \dots, p$.

Outline

Four-Point BVP

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), \quad x \in (0,\pi), \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), \quad \xi \in (0,\pi), \quad \eta \in (0,\pi). \end{cases} \tag{4}$$



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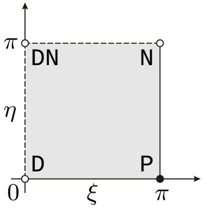
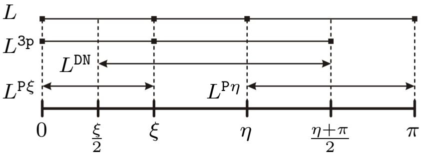




Definition

Let us define the following operators

$$Lu := L^{\mathtt{P}\xi}u := L^{\mathtt{P}\eta}u := L^{\mathtt{DN}}u := L^{\mathtt{3p}}u := -u''$$

for $\xi, \eta \in (0,\pi)$ by

$$D(L) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u(\eta) = u(\pi) \quad \},$$

$$D\left(L^{\mathtt{P}\xi}\right) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u(0) = u(\xi) \quad \},$$

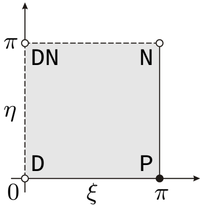
$$D\left(L^{\mathtt{P}\eta}\right) := \{u \in C^2([0,\pi]) : \quad u'(\eta) = u'(\pi), \quad u(\eta) = u(\pi) \quad \},$$

$$D\left(L^{\mathtt{DN}}\right) := \{u \in C^2([0,\pi]) : \quad u\left(\tfrac{\xi}{2}\right) = 0, \quad u'\left(\tfrac{\eta+\pi}{2}\right) = 0 \quad \},$$

$$D\left(L^{\mathtt{3p}}\right) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u'\left(\tfrac{\eta+\pi}{2}\right) = 0, \quad u(0)u(\xi) \le 0\}.$$

Four-Point BVP

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), & x \in (0,\pi), \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi \in (0,\pi), \quad \eta \in (0,\pi). \end{cases} \tag{4}$$



Remark

The spectra of $L^{\mathtt{P}\xi}$, $L^{\mathtt{P}\eta}$ and $L^{\mathtt{DN}}$ are pure point discrete spectra made only of the following real eigenvalues

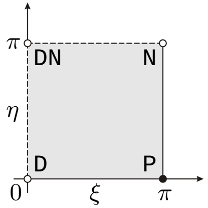
$$\lambda_k^{\mathtt{P}\xi} := \left(\frac{2k\pi}{\xi}\right)^2, \ \lambda_m^{\mathtt{P}\eta} := \left(\frac{2m\pi}{\pi-\eta}\right)^2 \quad \textit{and} \quad \lambda_l^{\mathtt{DN}} := \left(\frac{(2l+1)\pi}{\pi+\eta-\xi}\right)^2, \ k,l,m \in \mathbb{N}_0,$$

and

$$\sigma(L) = \sigma(L^{\mathtt{P}\xi}) \cup \sigma(L^{\mathtt{P}\eta}) \cup \sigma(L^{\mathtt{DN}}).$$

Four-Point BVP

$$\begin{cases} -u''(x) = \alpha u^{+}(x) - \beta u^{-}(x), & x \in (0,\pi), \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi \in (0,\pi), \quad \eta \in (0,\pi). \end{cases} \tag{4}$$



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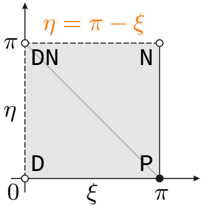
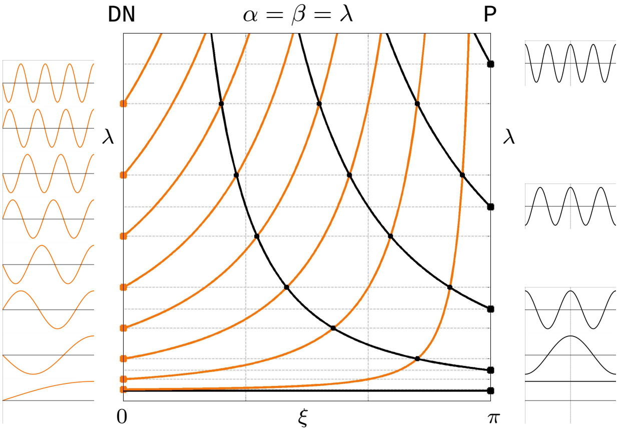
$$\sigma(L) = \sigma(L^{\mathrm{P}\xi}) \cup \sigma(L^{\mathrm{P}\eta}) \cup \sigma(L^{\mathrm{DN}}).$$

- $\sigma(L^{3\mathrm{p}}) = \sigma(L^{\mathrm{DN}})$,
- $\sigma(L^{\mathrm{P}\xi}) = \sigma(L^{\mathrm{P}\eta})$ for $\eta = \pi - \xi$.

Four-Point BVP

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), & x \in (0,\pi), \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi \in (0,\pi), \quad \eta \in (0,\pi). \end{cases} \tag{4}$$





Four-Point BVP

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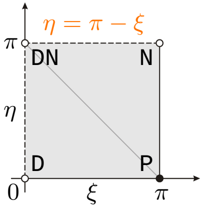
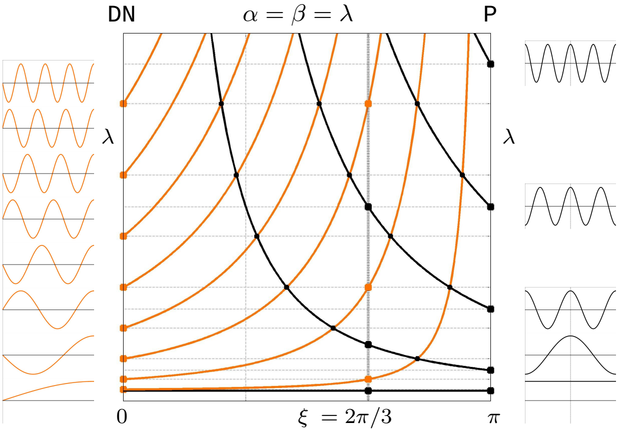
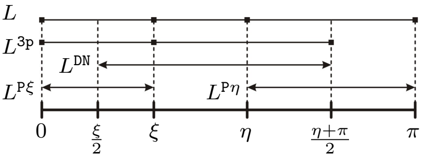


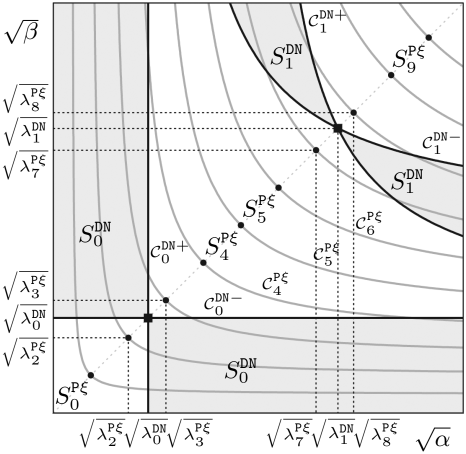


Outline

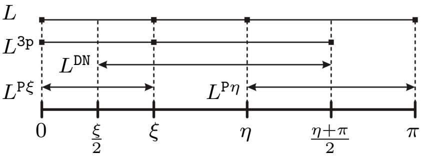
Construction of the Fučík Spectrum



- $\sigma(L) = \sigma(L^{\mathrm{P}\xi}) \cup \sigma(L^{\mathrm{P}\eta}) \cup \sigma(L^{\mathrm{DN}})$,



Construction of the Fučík Spectrum



- $\sigma(L) = \sigma(L^{\mathrm{P}\xi}) \cup \sigma(L^{\mathrm{P}\eta}) \cup \sigma(L^{\mathrm{DN}})$,
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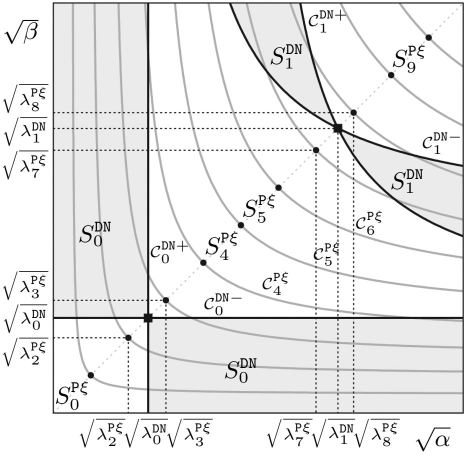
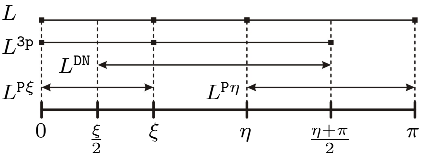
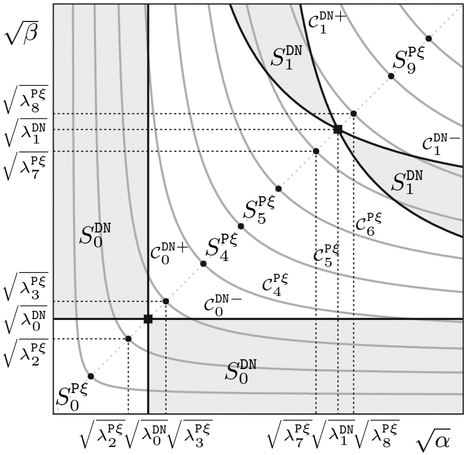


Construction of the Fučík Spectrum





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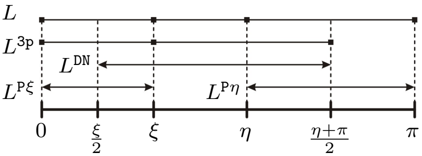
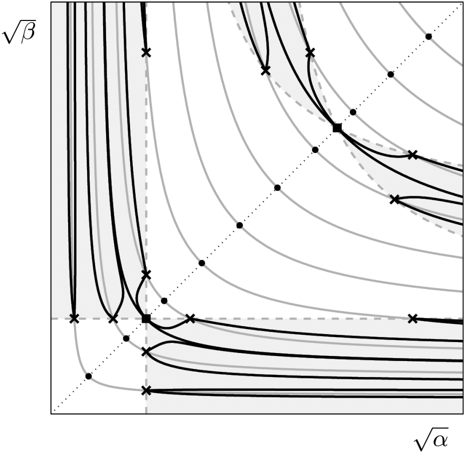
Construction of the Fučík Spectrum





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Construction of the Fučík Spectrum

Theorem

The Fučík spectrum of the four-point problem (4) is given by

$$\Sigma(L) = \Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta}) \cup \Sigma(L^{\mathtt{3p}}).$$

Construction of the Fučík Spectrum

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The Fučík spectrum of the four-point problem (4) is given by

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Proof

- $\Sigma(L) \subset \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta}) \cup \Sigma(L^{\mathrm{3p}})$
- $\Sigma(L) \supset \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta}) \cup \Sigma(L^{\mathrm{3p}})$

Construction of the Fučík Spectrum

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The Fučík spectrum of the four-point problem (4) is given by

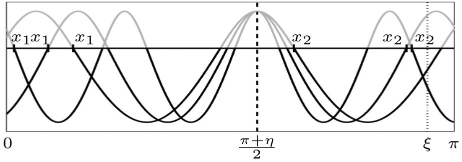
$$\Sigma(L) = \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{3p}).$$

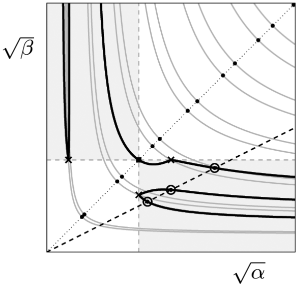
Proof

- $\Sigma(L) \subset \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{3p})$
- $\Sigma(L) \supset \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{3p})$
- Let $\alpha, \beta > 0$ and $u \in C^2([0,\pi])$ be the nontrivial solution of $-u'' = \alpha u^+ - \beta u^-$ on $(0,\pi)$. Then the following statements hold ($x_1, x_2 \in [0,\pi]$):
  - If $u(x_1) = u(x_2)$ then $u'(x_1) = u'(x_2)$ or $u'(x_1) = -u'(x_2)$. (5)
  - $u(x_1) = u(x_2)$ and $u'(x_1) = -u'(x_2)$ if and only if $u'(\frac{x_1+x_2}{2}) = 0$. (6)
  - If $u'(x_1) = u'(x_2)$ and $u(x_1)u(x_2) > 0$ then $u(x_1) = u(x_2)$. (7)

∎

Construction of $\Sigma(L^{3\mathrm{p}})$





Proposition

The Fučík spectrum of $L^{3\mathrm{p}}$ on $\mathbb{R}^+ \times \mathbb{R}^+$ is given by

$$\Sigma\left(L^{3\mathrm{p}}\right) = \bigcup_{l \in \mathbb{N}_0} \mathcal{C}_l^{3\mathrm{p}}, \qquad \textit{where } \mathcal{C}_l^{3\mathrm{p}} := \bigcup_{k \in \mathbb{N}_0} \left(\mathcal{C}_{k,l}^{3\mathrm{p}+} \cup \mathcal{C}_{k,l}^{3\mathrm{p}-}\right), \quad l \in \mathbb{N}_0,$$

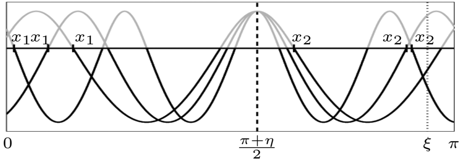
and where we define for $k, l \in \mathbb{N}_0$

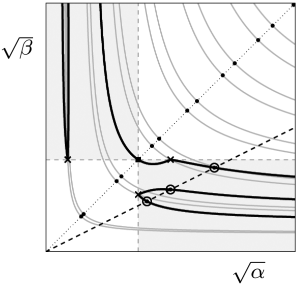
$$\mathcal{C}_{k,l}^{3\mathrm{p}-} := \left\{(\alpha,\beta) \in S_k^{\mathrm{P}\xi} \cap S_l^{\mathrm{DN}} : (\beta,\alpha) \in \mathcal{C}_{k,l}^{3\mathrm{p}+}\right\},$$

$$\mathcal{C}_{k,l}^{3\mathrm{p}+} := \left\{(\alpha,\beta) \in S_k^{\mathrm{P}\xi} \cap S_l^{\mathrm{DN}} : \left\{ \begin{array}{ll} \frac{\pi+\eta}{2\pi} = F_{k,l}(\alpha,\beta) & \textit{for } k+l \textit{ even} \\ \frac{\pi+\eta}{2\pi} = F_{k,l}(\beta,\alpha) & \textit{for } k+l \textit{ odd} \end{array} \right\}\right\},$$

$$F_{k,l}(\alpha,\beta) := \frac{\xi}{\pi}\frac{\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}} + \frac{l-k}{2\sqrt{\alpha}} + \frac{l+k+1}{2\sqrt{\beta}}.$$

Construction of $\Sigma(L^{3\mathrm{p}})$





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$$\mathcal{C}_{k,l}^{3\mathrm{p}+} := \Big\{(\alpha,\beta) \in$$

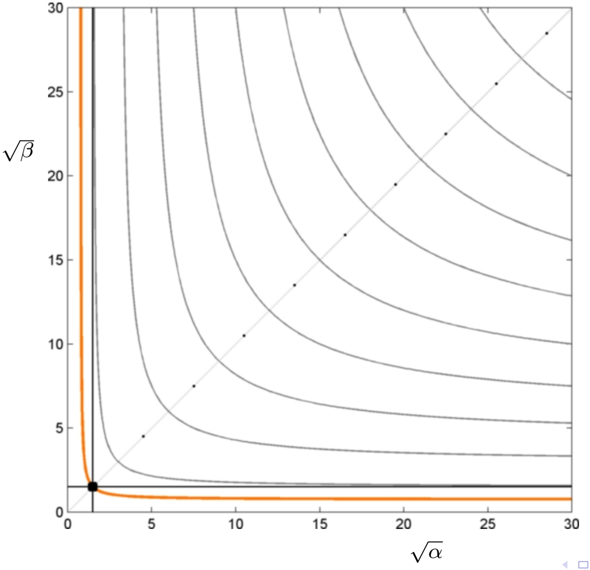
$$F_{k,l}(\alpha,\beta) := \frac{\xi}{\pi}\frac{\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{}}$$

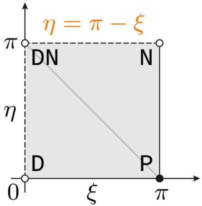
$$\mathcal{C}_l^{3\mathrm{p}} = \bigcup_{k=k_l^{\mathtt{min}}}^{k_l^{\mathtt{max}}-1} \left(\mathcal{C}_{k,l}^{3\mathrm{p}+} \cup \mathcal{C}_{k,l}^{3\mathrm{p}-}\right), \quad l \in \mathbb{N}_0,$$

where

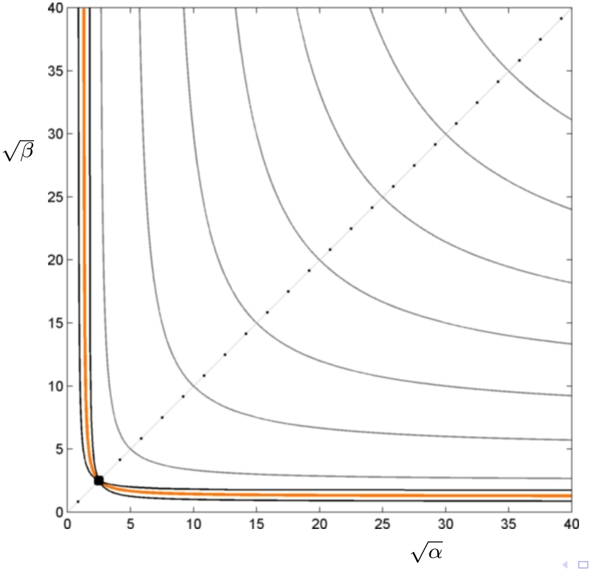
$$k_l^{\mathtt{min}} := \left\lfloor \frac{l\xi}{\pi+\eta-\xi} \right\rfloor \quad \text{and} \quad k_l^{\mathtt{max}} := \left\lceil \frac{(l+1)\xi}{\pi+\eta-\xi} \right\rceil.$$

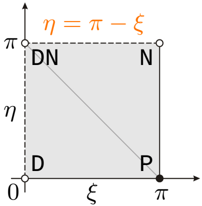
Qualitative Properties of $\Sigma(L^{3\mathrm{p}})$



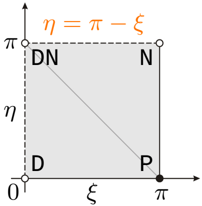


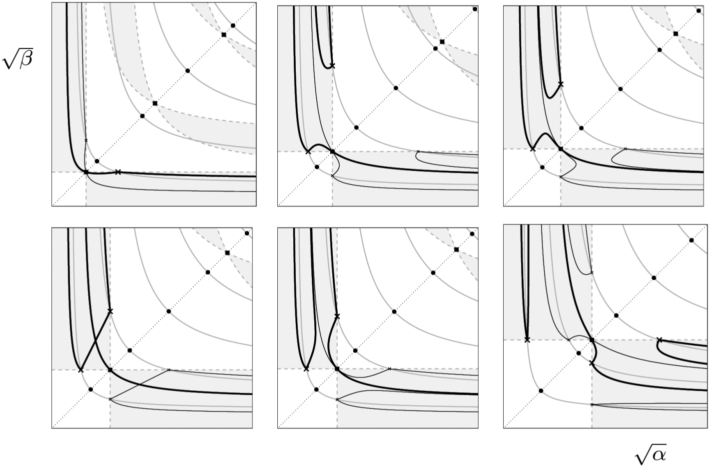
Qualitative Properties of $\Sigma(L^{3\mathrm{p}})$



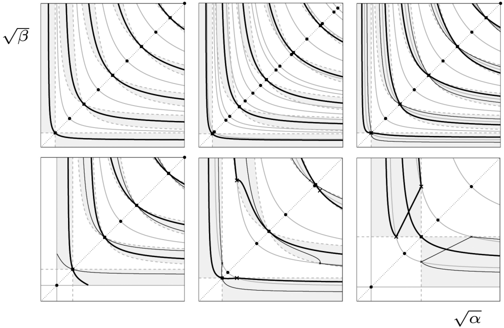


Qualitative Properties of $\Sigma(L^{3\mathrm{p}})$





Conclusion of the Construction



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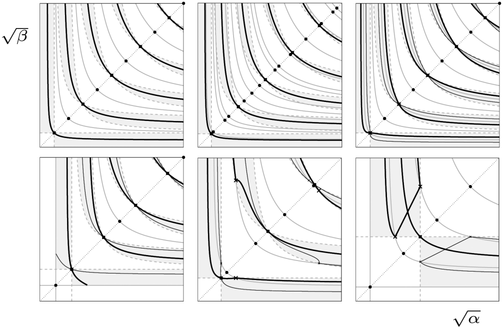


Remark

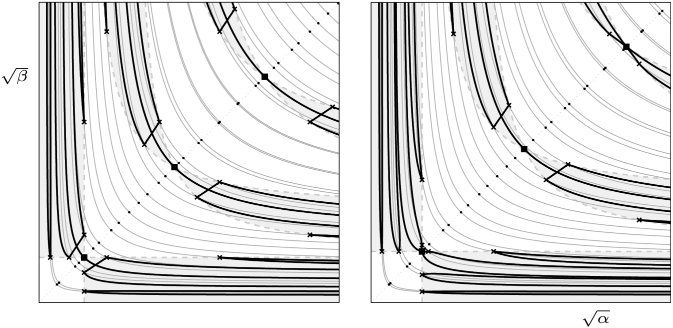
- Since $\sigma(L^{3p}) = \sigma(L^{DN})$ but $\Sigma(L^{3p}) \neq \Sigma(L^{DN})$, we conclude that

$$\begin{array}{rcl} \sigma(L) & = & \sigma(L^{P\xi}) \cup \sigma(L^{P\eta}) \cup \sigma(L^{3p}) \quad = \quad \sigma(L^{P\xi}) \cup \sigma(L^{P\eta}) \cup \sigma(L^{DN}), \\ \Sigma(L) & = & \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{3p}) \quad \neq \quad \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{DN}). \end{array}$$

- The Fučík spectrum $\Sigma(L^{DN})$ determines the intersection of $\Sigma(L^{P\xi})$ and $\Sigma(L^{3p})$:

$$\Sigma(L^{3p}) \cap \Sigma(L^{P\xi}) = \Sigma(L^{DN}) \cap \Sigma(L^{P\xi}) \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^+.$$

Conclusion of the Construction



Remark

- Since $\sigma(L^{3\mathrm{p}}) = \sigma(L^{\mathrm{DN}})$ but $\Sigma(L^{3\mathrm{p}}) \neq \Sigma(L^{\mathrm{DN}})$, we conclude that

$$\begin{array}{rcccl} \sigma(L) & = & \sigma(L^{\mathrm{P}\xi}) \cup \sigma(L^{\mathrm{P}\eta}) \cup \sigma(L^{3\mathrm{p}}) & = & \sigma(L^{\mathrm{P}\xi}) \cup \sigma(L^{\mathrm{P}\eta}) \cup \sigma(L^{\mathrm{DN}}), \\ \Sigma(L) & = & \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta}) \cup \Sigma(L^{3\mathrm{p}}) & \neq & \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta}) \cup \Sigma(L^{\mathrm{DN}}). \end{array}$$

- The Fučík spectrum $\Sigma(L^{\mathrm{DN}})$ determines the intersection of $\Sigma(L^{\mathrm{P}\xi})$ and $\Sigma(L^{3\mathrm{p}})$:

$$\Sigma(L^{3\mathrm{p}}) \cap \Sigma(L^{\mathrm{P}\xi}) = \Sigma(L^{\mathrm{DN}}) \cap \Sigma(L^{\mathrm{P}\xi}) \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^+.$$

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