On two-point boundary value problem for third-order functional differential equations

Robert Hakl, Brno, Czech Republic

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t) \tag{1}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

- $p, g : C([a, b]; R) \to L([a, b]; R)$ are linear nondecreasing operators $(p, g \in \mathcal{P}_{ab})$
- $q \in L([a, b]; R), \quad c_i \in R \quad (i = 1, 2, 3)$

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t) \tag{1}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

- $p, g : C([a,b]; R) \to L([a,b]; R)$ are linear nondecreasing operators $(p, g \in \mathcal{P}_{ab})$
- $q \in L([a,b]; R), \quad c_i \in R \quad (i = 1, 2, 3)$

Solution:

- $u : [a,b] \to R$
- u, u', u'' are absolutely continuous $(u \in \widetilde{C}^2([a,b]; R))$
- u satisfies (1) almost everywhere on $[a,b]$
- u satisfies (2)

Definition

Let $\ell : C([a,b];R) \to L([a,b];R)$ be a linear operator. Then $\ell \in \mathcal{V}([a,b])$ iff

$$u \in \widetilde{C}^2([a,b];R), \quad \left.\begin{array}{c} u'''(t) \le \ell(u)(t) \quad \text{for} \quad t \in [a,b] \\ u(a) \ge 0, \quad u(b) \ge 0, \quad u'(a) \ge 0 \end{array}\right\} \implies u(t) \ge 0 \quad \text{for} \quad t \in [a,b]$$

Definition

Let $\ell : C([a,b];R) \to L([a,b];R)$ be a linear operator. Then $\ell \in \mathcal{V}([a,b])$ iff

$$u \in \widetilde{C}^2([a,b];R), \quad \left.\begin{array}{c} u'''(t) \le \ell(u)(t) \quad \text{for} \quad t \in [a,b] \\ u(a) \ge 0, \quad u(b) \ge 0, \quad u'(a) \ge 0 \end{array}\right\} \Longrightarrow \quad u(t) \ge 0 \quad \text{for} \quad t \in [a,b]$$

Definition

Let $\ell : C([a,b];R) \to L([a,b];R)$ be a linear operator. Then $\ell \in \mathcal{V}^0([a,b])$ iff

$$u \in \widetilde{C}^2([a,b];R), \quad \left.\begin{array}{c} u'''(t) \le \ell(u)(t) \quad \text{for} \quad t \in [a,b] \\ u(a) = 0, \quad u(b) = 0, \quad u'(a) \ge 0 \end{array}\right\} \Longrightarrow \quad u(t) \ge 0 \quad \text{for} \quad t \in [a,b]$$

Definition

Let $\ell : C([a,b];R) \to L([a,b];R)$ be a linear operator. Then $\ell \in \mathcal{V}([a,b])$ iff

$$u \in \widetilde{C}^2([a,b];R), \quad \left.\begin{array}{c} u'''(t) \le \ell(u)(t) \quad \text{for} \quad t \in [a,b] \\ u(a) \ge 0, \quad u(b) \ge 0, \quad u'(a) \ge 0 \end{array}\right\} \Longrightarrow \quad u(t) \ge 0 \quad \text{for} \quad t \in [a,b]$$

Definition

Let $\ell : C([a,b];R) \to L([a,b];R)$ be a linear operator. Then $\ell \in \mathcal{V}^0([a,b])$ iff

$$u \in \widetilde{C}^2([a,b];R), \quad \left.\begin{array}{c} u'''(t) \le \ell(u)(t) \quad \text{for} \quad t \in [a,b] \\ u(a) = 0, \quad u(b) = 0, \quad u'(a) \ge 0 \end{array}\right\} \Longrightarrow \quad u(t) \ge 0 \quad \text{for} \quad t \in [a,b]$$

Remark. If $\ell \in \mathcal{V}([a,b])$ or $\ell \in \mathcal{V}^0([a,b])$ then

$$u'''(t) = \ell(u)(t); \qquad u(a) = 0, \quad u(b) = 0, \quad u'(a) = 0$$

has only the trivial solution.

Definition

Let $\ell : C([a,b];R) \to L([a,b];R)$ be a linear operator. Then $\ell \in \mathcal{V}([a,b])$ iff

$$u \in \widetilde{C}^2([a,b];R), \quad \left.\begin{array}{c} u'''(t) \le \ell(u)(t) \quad \text{for} \quad t \in [a,b] \\ u(a) \ge 0, \quad u(b) \ge 0, \quad u'(a) \ge 0 \end{array}\right\} \implies u(t) \ge 0 \quad \text{for} \quad t \in [a,b]$$

Definition

Let $\ell : C([a,b];R) \to L([a,b];R)$ be a linear operator. Then $\ell \in \mathcal{V}^0([a,b])$ iff

$$u \in \widetilde{C}^2([a,b];R), \quad \left.\begin{array}{c} u'''(t) \le \ell(u)(t) \quad \text{for} \quad t \in [a,b] \\ u(a) = 0, \quad u(b) = 0, \quad u'(a) \ge 0 \end{array}\right\} \implies u(t) \ge 0 \quad \text{for} \quad t \in [a,b]$$

Remark. If $\ell \in \mathcal{V}([a,b])$ or $\ell \in \mathcal{V}^0([a,b])$ then

$$u'''(t) = \ell(u)(t); \qquad u(a) = 0, \quad u(b) = 0, \quad u'(a) = 0$$

has only the trivial solution.

Remark. Obviously $\mathcal{V}([a,b]) \subseteq \mathcal{V}^0([a,b])$. Moreover, if $-\ell \in \mathcal{P}_{ab}$ then

$$\ell \in \mathcal{V}([a,b]) \qquad \iff \qquad \ell \in \mathcal{V}^0([a,b]).$$

$$v \in \widetilde{C}_a(\,]a,b[\,;R) \qquad \Longleftrightarrow \qquad \begin{cases} v \in \widetilde{C}^2_{loc}(\,]a,b[\,;R) \cap C([a,b];R) \\ \exists\, v'(a+) \end{cases}$$

$$v \in \widetilde{C}_a(\,]a,b[\,;R) \qquad \Longleftrightarrow \qquad \begin{cases} v \in \widetilde{C}^2_{loc}(\,]a,b[\,;R) \cap C([a,b];R) \\ \exists\, v'(a+) \end{cases}$$

Theorem

Let $p \in \mathcal{P}_{ab}$. Then $-p \in \mathcal{V}([a,b])$ if and only if there exists $\gamma \in \widetilde{C}_a(\,]a,b[\,;R)$ such that

$$\gamma(t) > 0 \qquad \textit{for } \ t \in \,]a,b[\,, \qquad \gamma'(a+) \geq 0,$$

$$\gamma'''(t) \leq -p(\gamma)(t) \qquad \textit{for } \ t \in [a,b],$$

$$\gamma(a) + \gamma(b) + \gamma'(a+) + \operatorname{meas}\left\{t : \gamma'''(t) < -p(\gamma)(t)\right\} > 0.$$

$$v \in \widetilde{C}_a(\,]a,b[\,;R) \quad \Longleftrightarrow \quad \begin{cases} v \in \widetilde{C}^2_{loc}(\,]a,b[\,;R) \cap C([a,b];R) \\ \exists\, v'(a+) \end{cases}$$

Theorem

Let $p \in \mathcal{P}_{ab}$. Then $-p \in \mathcal{V}([a,b])$ if and only if there exists $\gamma \in \widetilde{C}_a(\,]a,b[\,;R)$ such that

$$\gamma(t) > 0 \qquad \textit{for}\ \ t \in\,]a,b[\,, \qquad \gamma'(a+) \geq 0,$$

$$\gamma'''(t) \leq -p(\gamma)(t) \qquad \textit{for}\ \ t \in [a,b],$$

$$\gamma(a) + \gamma(b) + \gamma'(a+) + \operatorname{meas}\left\{t : \gamma'''(t) < -p(\gamma)(t)\right\} > 0.$$

Proof.

Sufficiency:

$$u'''(t) \leq -p(u)(t) \qquad \text{for}\ \ t \in [a,b] \qquad u(a) \geq 0, \quad u(b) \geq 0, \quad u'(a) \geq 0$$

$$\lambda \overset{def}{=} \sup\left\{-\frac{u(t)}{\gamma(t)} : t \in\,]a,b[\,\right\} > 0; \qquad w(t) \overset{def}{=} \lambda\gamma(t) + u(t) \qquad \text{for}\ \ t \in [a,b]$$

Studying properties of w, we get a contradiction.

$$v \in \widetilde{C}_a(\,]a,b[\,;R) \qquad \Longleftrightarrow \qquad \begin{cases} v \in \widetilde{C}^2_{loc}(\,]a,b[\,;R) \cap C([a,b];R) \\ \exists\, v'(a+) \end{cases}$$

Theorem

Let $p \in \mathcal{P}_{ab}$. Then $-p \in \mathcal{V}([a,b])$ if and only if there exists $\gamma \in \widetilde{C}_a(\,]a,b[\,;R)$ such that

$$\gamma(t) > 0 \qquad \text{for } \ t \in \,]a,b[\,, \qquad \gamma'(a+) \geq 0,$$

$$\gamma'''(t) \leq -p(\gamma)(t) \qquad \text{for } \ t \in [a,b],$$

$$\gamma(a) + \gamma(b) + \gamma'(a+) + \operatorname{meas}\left\{t : \gamma'''(t) < -p(\gamma)(t)\right\} > 0.$$

Proof.

Necessity:

$-p \in \mathcal{V}([a,b])$

$\gamma'''(t) = -p(\gamma)(t); \qquad \gamma(a) = 1, \quad \gamma(b) = 1, \quad \gamma'(a) = 0$

The function γ has the properties required.

□

$\ell : C([a,b];R) \to L([a,b];R)$ is an a-Volterra operator if for every $c \in \,]a,b]$

$$u(t) = 0 \qquad \text{for} \quad t \in [a,c] \qquad \Longrightarrow \qquad \ell(u)(t) = 0 \qquad \text{for} \quad t \in [a,c]$$

$\ell : C([a,b];R) \to L([a,b];R)$ is an a-Volterra operator if for every $c \in \,]a,b]$

$$u(t) = 0 \quad \text{for} \;\; t \in [a,c] \qquad \Longrightarrow \qquad \ell(u)(t) = 0 \quad \text{for} \;\; t \in [a,c]$$

$$\ell(u)(t) \overset{def}{=} h(t)u(\tau(t)), \quad h \in L([a,b];R), \quad \tau : [a,b] \to [a,b]$$

$$\ell \;\; \text{is an } a\text{-Volterra operator} \qquad \Longleftrightarrow \qquad |h(t)|(\tau(t) - t) \le 0 \quad \text{for} \;\; t \in [a,b]$$

$\ell : C([a,b];R) \to L([a,b];R)$ is an a-Volterra operator if for every $c \in\,]a,b]$

$$u(t) = 0 \qquad \text{for} \;\; t \in [a,c] \qquad \Longrightarrow \qquad \ell(u)(t) = 0 \qquad \text{for} \;\; t \in [a,c]$$

$$\ell(u)(t) \overset{def}{=} h(t)u(\tau(t)), \quad h \in L([a,b];R), \quad \tau : [a,b] \to [a,b]$$

$$\ell \quad \text{is an } a\text{-Volterra operator} \qquad \Longleftrightarrow \qquad |h(t)|(\tau(t) - t) \leq 0 \qquad \text{for} \;\; t \in [a,b]$$

Theorem

Let $g \in \mathcal{P}_{ab}$ be an a-Volterra operator. Then $g \in \mathcal{V}([a,b])$ if and only if there exists $\beta \in \widetilde{C}_a(\,]a,b[\,;R)$ such that

$$\beta(t) > 0 \qquad \textit{for} \;\; t \in [a,b[\,, \qquad \beta'(t) \leq 0 \qquad \textit{for} \;\; t \in \,]a,b[\,,$$

$$\beta'''(t) \geq g(\beta)(t) \qquad \textit{for} \;\; t \in [a,b].$$

$\ell : C([a,b];R) \to L([a,b];R)$ is an a-Volterra operator if for every $c \in]a,b]$

$$u(t) = 0 \quad \text{for } t \in [a,c] \quad \Longrightarrow \quad \ell(u)(t) = 0 \quad \text{for } t \in [a,c]$$

$$\ell(u)(t) \overset{def}{=} h(t)u(\tau(t)), \quad h \in L([a,b];R), \quad \tau : [a,b] \to [a,b]$$

$$\ell \text{ is an } a\text{-Volterra operator} \quad \Longleftrightarrow \quad |h(t)|(\tau(t) - t) \leq 0 \quad \text{for } t \in [a,b]$$

Theorem

Let $g \in \mathcal{P}_{ab}$ be an a-Volterra operator. Then $g \in \mathcal{V}([a,b])$ if and only if there exists $\beta \in \widetilde{C}_a(\,]a,b[\,;R)$ such that

$$\beta(t) > 0 \quad \textit{for } t \in [a,b[\,, \qquad \beta'(t) \leq 0 \quad \textit{for } t \in \,]a,b[\,,$$

$$\beta'''(t) \geq g(\beta)(t) \quad \textit{for } t \in [a,b].$$

Theorem

Let $g \in \mathcal{P}_{ab}$ be an a-Volterra operator. Then $g \in \mathcal{V}^0([a,b])$ if and only if there exists $\beta \in \widetilde{C}_a(\,]a,b[\,;R)$ such that

$$\beta(t) > 0 \quad \textit{for } t \in \,]a,b[\,, \qquad \beta'(b-) \leq 0, \qquad \beta'''(t) \geq g(\beta)(t) \quad \textit{for } t \in [a,b].$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p+g \in \mathcal{V}([a,b]).$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p + g \in \mathcal{V}([a,b]).$$

Proof.

$$u'''(t) \leq -p(u) + g(u)(t) \quad \text{for} \quad t \in [a,b] \qquad u(a) \geq 0, \quad u(b) \geq 0, \quad u'(a) \geq 0$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. Then

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p + g \in \mathcal{V}([a,b]).$$

Proof.

$$u'''(t) \leq -p(u) + g(u)(t) \quad \text{for} \quad t \in [a,b] \qquad u(a) \geq 0, \quad u(b) \geq 0, \quad u'(a) \geq 0$$

$$\alpha'''(t) = g(\alpha)(t) + p([u]_-)(t); \qquad \alpha(a) = 0, \quad \alpha(b) = 0, \quad \alpha'(a) = 0$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p + g \in \mathcal{V}([a,b]).$$

Proof.

$$u'''(t) \le -p(u) + g(u)(t) \quad \text{for} \quad t \in [a,b] \qquad u(a) \ge 0, \quad u(b) \ge 0, \quad u'(a) \ge 0$$

$$\alpha'''(t) = g(\alpha)(t) + p([u]_-)(t); \qquad \alpha(a) = 0, \quad \alpha(b) = 0, \quad \alpha'(a) = 0$$

$$g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad \alpha(t) \le 0 \quad \text{for} \quad t \in [a,b]$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p+g \in \mathcal{V}([a,b]).$$

Proof.

$$u'''(t) \le -p(u) + g(u)(t) \quad \text{for} \quad t \in [a,b] \qquad u(a) \ge 0, \quad u(b) \ge 0, \quad u'(a) \ge 0$$

$$\alpha'''(t) = g(\alpha)(t) + p([u]_-)(t); \qquad \alpha(a) = 0, \quad \alpha(b) = 0, \quad \alpha'(a) = 0$$

$$g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad \alpha(t) \le 0 \quad \text{for} \quad t \in [a,b]$$

$$\Longrightarrow \qquad \alpha(t) \le u(t) \quad \text{for} \quad t \in [a,b]$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p + g \in \mathcal{V}([a,b]).$$

Proof.

$$u'''(t) \leq -p(u) + g(u)(t) \quad \text{for} \quad t \in [a,b] \qquad u(a) \geq 0, \quad u(b) \geq 0, \quad u'(a) \geq 0$$

$$\alpha'''(t) = g(\alpha)(t) + p([u]_-)(t); \qquad \alpha(a) = 0, \quad \alpha(b) = 0, \quad \alpha'(a) = 0$$

$$g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad \alpha(t) \leq 0 \quad \text{for} \quad t \in [a,b]$$

$$\Longrightarrow \qquad \alpha(t) \leq u(t) \quad \text{for} \quad t \in [a,b]$$

$$\Longrightarrow \qquad \alpha(t) \leq -[u]_-(t) \quad \text{for} \quad t \in [a,b]$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p + g \in \mathcal{V}([a,b]).$$

Proof.

$$u'''(t) \le -p(u) + g(u)(t) \quad \text{for} \quad t \in [a,b] \qquad u(a) \ge 0, \quad u(b) \ge 0, \quad u'(a) \ge 0$$

$$\alpha'''(t) = g(\alpha)(t) + p([u]_-)(t); \qquad \alpha(a) = 0, \quad \alpha(b) = 0, \quad \alpha'(a) = 0$$

$$g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad \alpha(t) \le 0 \quad \text{for} \quad t \in [a,b]$$

$$\Longrightarrow \qquad \alpha(t) \le u(t) \quad \text{for} \quad t \in [a,b]$$

$$\Longrightarrow \qquad \alpha(t) \le -[u]_-(t) \quad \text{for} \quad t \in [a,b]$$

$$\alpha'''(t) \le g(\alpha)(t) - p(\alpha)(t) \le -p(\alpha)(t); \qquad \alpha(a) = 0, \quad \alpha'(a) = 0, \quad \alpha(b) = 0$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p + g \in \mathcal{V}([a,b]).$$

Proof.

$$u'''(t) \leq -p(u) + g(u)(t) \quad \text{for} \quad t \in [a,b] \qquad u(a) \geq 0, \quad u(b) \geq 0, \quad u'(a) \geq 0$$

$$\alpha'''(t) = g(\alpha)(t) + p([u]_-)(t); \qquad \alpha(a) = 0, \quad \alpha(b) = 0, \quad \alpha'(a) = 0$$

$$g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad \alpha(t) \leq 0 \quad \text{for} \quad t \in [a,b]$$

$$\Longrightarrow \qquad \alpha(t) \leq u(t) \quad \text{for} \quad t \in [a,b]$$

$$\Longrightarrow \qquad \alpha(t) \leq -[u]_-(t) \quad \text{for} \quad t \in [a,b]$$

$$\alpha'''(t) \leq g(\alpha)(t) - p(\alpha)(t) \leq -p(\alpha)(t); \qquad \alpha(a) = 0, \quad \alpha'(a) = 0, \quad \alpha(b) = 0$$

$$-p \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad \alpha(t) \geq 0 \quad \text{for} \quad t \in [a,b] \qquad \Longrightarrow \qquad u(t) \geq 0 \quad \text{for} \quad t \in [a,b]$$

□

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Then*

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p+g \in \mathcal{V}([a,b]).$$

Remark. If $p, g \in \mathcal{P}_{ab}$ and $-p \in \mathcal{V}([a,b])$, $g \in \mathcal{V}([a,b])$, then

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3$$

has a unique solution u. Moreover,

$$q(t) \leq 0 \quad \text{for} \quad t \in [a,b], \qquad c_i \geq 0 \quad (i = 1,2,3) \qquad \Longrightarrow \qquad u(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. Then

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p+g \in \mathcal{V}([a,b]).$$

Remark. If $p, g \in \mathcal{P}_{ab}$ and $-p \in \mathcal{V}([a,b])$, $g \in \mathcal{V}([a,b])$, then

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3$$

has a unique solution u. Moreover,

$$q(t) \le 0 \quad \text{for} \quad t \in [a,b], \qquad c_i \ge 0 \quad (i = 1,2,3) \qquad \Longrightarrow \qquad u(t) \ge 0 \quad \text{for} \quad t \in [a,b].$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. Then

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}^0([a,b]) \qquad \Longrightarrow \qquad -p+g \in \mathcal{V}^0([a,b]).$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. Then

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}([a,b]) \qquad \Longrightarrow \qquad -p+g \in \mathcal{V}([a,b]).$$

Remark. If $p, g \in \mathcal{P}_{ab}$ and $-p \in \mathcal{V}([a,b])$, $g \in \mathcal{V}([a,b])$, then

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3$$

has a unique solution u. Moreover,

$$q(t) \le 0 \quad \text{for} \quad t \in [a,b], \qquad c_i \ge 0 \quad (i = 1,2,3) \qquad \Longrightarrow \qquad u(t) \ge 0 \quad \text{for} \quad t \in [a,b].$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. Then

$$-p \in \mathcal{V}([a,b]), \qquad g \in \mathcal{V}^0([a,b]) \qquad \Longrightarrow \qquad -p+g \in \mathcal{V}^0([a,b]).$$

Remark. If $p, g \in \mathcal{P}_{ab}$ and $-p \in \mathcal{V}([a,b])$, $g \in \mathcal{V}^0([a,b])$, then

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3$$

has a unique solution u. Moreover,

$$q(t) \le 0 \quad \text{for} \quad t \in [a,b], \qquad c_1 = 0, \quad c_2 = 0, \quad c_3 \ge 0 \qquad \Longrightarrow \qquad u(t) \ge 0 \quad \text{for} \quad t \in [a,b].$$

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Let, moreover,*

$$-p \in \mathcal{V}([a,b]), \qquad \frac{1}{2} g \in \mathcal{V}^0([a,b]).$$

Then

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3$$

is uniquely solvable.

Theorem

Let $p, g \in \mathcal{P}_{ab}$. *Let, moreover,*

$$-p \in \mathcal{V}([a,b]), \qquad \frac{1}{2} g \in \mathcal{V}^0([a,b]).$$

Then

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3$$

is uniquely solvable.

Theorem

Let $p, g \in \mathcal{P}_{ab}$ *be a-Volterra operators and* $-p \in \mathcal{V}([a,b])$. *Then*

$$u'''(t) = -p(u)(t) + g(u)(t) + q(t); \qquad u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3$$

is uniquely solvable.

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \tag{3}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

$p, g \in L([a, b]; R_+), \quad \tau, \mu : [a, b] \to [a, b]$

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \tag{3}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \to [a,b]$

Corollary

Let $\mu(t) \leq t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s)ds \leq \frac{16}{(b-a)^2}\,, \qquad \int_a^b g(s)ds \leq \frac{8}{(b-a)^2}\,.$$

Then (3), (2) *has a unique solution* u. *Moreover,*

$$q(t) \leq 0 \quad \text{for} \quad t \in [a,b], \qquad c_i \geq 0 \quad (i = 1,2,3) \qquad \Longrightarrow \qquad u(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \tag{3}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \to [a,b]$

Corollary

Let $\mu(t) \le t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s)ds \le \frac{16}{(b-a)^2}, \qquad \int_a^b g(s)ds \le \frac{16}{(b-a)^2}.$$

Then (3), (2) *has a unique solution* u. *Moreover,*

$$q(t) \le 0 \quad for \quad t \in [a,b], \qquad c_1 = 0, \quad c_2 = 0, \quad c_3 \ge 0 \qquad \Longrightarrow \qquad u(t) \ge 0 \quad for \quad t \in [a,b].$$

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \qquad (3)$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \qquad (2)$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \to [a,b]$

Corollary

Let $\mu(t) \le t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s)ds \le \frac{16}{(b-a)^2}, \qquad \int_a^b g(s)ds \le \frac{32}{(b-a)^2}.$$

Then (3), (2) *has a unique solution* u.

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \tag{3}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \to [a,b]$

Corollary

Let $\tau(t) \leq t$, $\mu(t) \leq t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s)ds \leq \frac{16}{(b-a)^2} \; .$$

Then (3), (2) *has a unique solution* u.

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \tag{3}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \to [a,b]$

Corollary

Let $\mu(t) \le t$ for $t \in [a,b]$ and

$$\left(\frac{b-\tau(t)}{b-t}\right)^{1-\frac{\sqrt{3}}{3}} \left(\frac{\tau(t)-a}{t-a}\right)^{1+\frac{\sqrt{3}}{3}} p(t) \le \frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for } t \in]a,b[$$

and the inequality is strict on a set of positive measure. Let, moreover,

$$\left(\frac{b-\mu(t)}{b-t}\right)^{1+\frac{\sqrt{3}}{3}} \left(\frac{\mu(t)-a+\omega}{t-a+\omega}\right)^{1-\frac{\sqrt{3}}{3}} g(t) \le \frac{2\sqrt{3}(b-a+\omega)^3}{9(b-t)^3(t-a+\omega)^3} \qquad \text{for } t \in]a,b[$$

with $\omega = \frac{3-\sqrt{3}}{3+\sqrt{3}}(b-a)$. Then the problem (3), (2) has a unique solution u. Moreover,

$$q(t) \le 0 \quad \text{for } t \in [a,b], \qquad c_i \ge 0 \quad (i=1,2,3) \qquad \Longrightarrow \qquad u(t) \ge 0 \quad \text{for } t \in [a,b].$$

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \tag{3}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \longrightarrow [a,b]$

Corollary

Let $\mu(t) \leq t$ *for* $t \in [a,b]$ *and*

$$\left(\frac{b-\tau(t)}{b-t}\right)^{1-\frac{\sqrt{3}}{3}} \left(\frac{\tau(t)-a}{t-a}\right)^{1+\frac{\sqrt{3}}{3}} p(t) \leq \frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for } \ t \in \,]a,b[$$

and the inequality is strict on a set of positive measure. Let, moreover,

$$\left(\frac{b-\mu(t)}{b-t}\right)^{1+\frac{\sqrt{3}}{3}} \left(\frac{\mu(t)-a}{t-a}\right)^{1-\frac{\sqrt{3}}{3}} g(t) \leq \frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for } \ t \in \,]a,b[\,.$$

Then the problem (3), (2) *has a unique solution* u. *Moreover,*

$$q(t) \leq 0 \quad \text{for} \ \ t \in [a,b], \qquad c_1 = 0, \quad c_2 = 0, \quad c_3 \geq 0 \qquad \Longrightarrow \qquad u(t) \geq 0 \quad \text{for} \ \ t \in [a,b].$$

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \quad (3)$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \quad (2)$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \to [a,b]$

Corollary

Let $\mu(t) \le t$ *for* $t \in [a,b]$ *and*

$$\left(\frac{b-\tau(t)}{b-t}\right)^{1-\frac{\sqrt{3}}{3}} \left(\frac{\tau(t)-a}{t-a}\right)^{1+\frac{\sqrt{3}}{3}} p(t) \le \frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for } t \in \,]a,b[$$

and the inequality is strict on a set of positive measure. Let, moreover,

$$\left(\frac{b-\mu(t)}{b-t}\right)^{1+\frac{\sqrt{3}}{3}} \left(\frac{\mu(t)-a}{t-a}\right)^{1-\frac{\sqrt{3}}{3}} g(t) \le \frac{4\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for } t \in \,]a,b[\,.$$

Then the problem (3), (2) *has a unique solution* u.

$$u'''(t) = -p(t)u(\tau(t)) + g(t)u(\mu(t)) + q(t) \tag{3}$$

$$u(a) = c_1, \qquad u(b) = c_2, \qquad u'(a) = c_3 \tag{2}$$

$p, g \in L([a,b]; R_+), \quad \tau, \mu : [a,b] \to [a,b]$

Corollary

Let $\tau(t) \le t$, $\mu(t) \le t$ *for* $t \in [a,b]$ *and*

$$\left(\frac{b-\tau(t)}{b-t}\right)^{1-\frac{\sqrt{3}}{3}} \left(\frac{\tau(t)-a}{t-a}\right)^{1+\frac{\sqrt{3}}{3}} p(t) \le \frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for } \ t \in \,]a,b[$$

and the inequality is strict on a set of positive measure. Then the problem (3), (2) *has a unique solution* u.