On new regions of existence and nonexistence of solutions for a system of p-q-laplacians

Based on a joint work with Ph. Clément, I. Guerra, and R. Manásevich

Marta García-Huidobro

[1]Facultad de Matemáticas, Pontificia Universidad Católica de Chile,

Workshop on Differential Equations, Boundary Value Problems and Related Topics,
September 16-20, 2007 - Hejnice, Czech Republic

Consider the quasilinear elliptic system

$$(S_R)\qquad \begin{array}{l} -\Delta_p u = v^\delta \quad v > 0 \quad \text{in} \quad B, \\ -\Delta_q v = u^\mu \quad u > 0 \quad \text{in} \quad B, \\ u = v = 0 \quad \text{on} \quad \partial B, \end{array}$$

where B is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$. Here $\delta, \mu > 0$ and

$$\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$$

is the $m-$Laplacian operator for $m > 1$.
In view of the invariance of problem (S_R) under rotations, it is natural to look for radially symmetric solutions. If we still denote by u, v the solutions as functions of $r = |x|$, we obtain the system of ode's

$$\begin{array}{l} -(r^{N-1}|u'(r)|^{p-2}u'(r))' = r^{N-1}|v(r)|^\delta, \quad 0 < r < R \\ -(r^{N-1}|v'(r)|^{q-2}v'(r))' = r^{N-1}|u(r)|^\mu, \quad 0 < r < R, \end{array} \qquad (1)$$

with appropriate boundary conditions. We are primarily interested in the existence of (regular) solutions of (1), i.e., $(u, v) \in (C^1[0, R] \cap C^2(0, R])^2$ satisfying (1) and $u'(0) = v'(0) = 0$, $u(R) = v(R) = 0$.

with appropriate boundary conditions. We are primarily interested in the existence of (regular) solutions of (1), i.e., $(u, v) \in (C^1[0, R] \cap C^2(0, R])^2$ satisfying (1) and $u'(0) = v'(0) = 0$, $u(R) = v(R) = 0$.
Clearly, either both u and v are identically 0, or both u and v are strictly positive and decreasing on $[0, R)$.

with appropriate boundary conditions. We are primarily interested in the existence of (regular) solutions of (1), i.e., $(u, v) \in (C^1[0, R] \cap C^2(0, R])^2$ satisfying (1) and $u'(0) = v'(0) = 0$, $u(R) = v(R) = 0$.

Clearly, either both u and v are identically 0, or both u and v are strictly positive and decreasing on $[0, R)$.

Observe that system (S_R) is homogeneous in the sense that if (u, v) is a solution, then $(\lambda u, \nu v)$ is also a solution provided that $\lambda,\ \nu > 0$ and $\lambda^{1-p} = \nu^{\delta}$ and $\nu^{1-q} = \lambda^{\mu}$. So it is natural to call the system *superhomogeneous* when

$$(H_1) \qquad d := \delta\mu - (p-1)(q-1) > 0 \quad \delta > 0,\ \mu > 0.$$

In case that $p = q = 2$, condition (H_1) is usually called *superlinear condition* and it is equivalent to the condition

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} < 1.$$

Previous work on the problem

The case $p = q = 2$: In this case, the system becomes the so called Lane-Emden system,

$$\begin{array}{l} -\Delta u = v^{\delta} \quad v > 0 \quad \text{in} \quad B, \\ -\Delta v = u^{\mu} \quad u > 0 \quad \text{in} \quad B, \\ u = v = 0 \quad \text{on} \quad \partial B, \end{array} \tag{2}$$

where $N \geq 3$.

Previous work on the problem

The case $p = q = 2$: In this case, the system becomes the so called Lane-Emden system,

$$\begin{cases} -\Delta u = v^{\delta} \quad v > 0 \quad \text{in} \quad B, \\ -\Delta v = u^{\mu} \quad u > 0 \quad \text{in} \quad B, \\ u = v = 0 \quad \text{on} \quad \partial B, \end{cases} \tag{2}$$

where $N \geq 3$.
From the point of view of Liouville results one is thus led to study respectively the range of the exponents $\delta, \mu > 0$ for which there are no positive classical solution in $\mathbb{R}^N$, that is, the values of $\delta, \mu > 0$ for which the system

$$\begin{cases} -\Delta u = v^{\delta} \quad v > 0 \quad \text{in} \quad \mathbb{R}^N, \\ -\Delta v = u^{\mu} \quad u > 0 \quad \text{in} \quad \mathbb{R}^N \end{cases} \tag{3}$$

has no solutions. Nonexistence will then provide the existence of the required a priori bounds.

It has been conjectured, see for example the work of Felmer and de Figueiredo [F-F], that the hyperbola – referred as *Sobolev's hyperbola* –

$$\left\{ \delta > 0,\ \mu > 0 \;\middle|\; \frac{1}{\delta + 1} + \frac{1}{\mu + 1} = 1 - \frac{2}{N} \right\}, \tag{4}$$

is the dividing curve between existence and nonexistence of positive solutions in $\mathbb{R}^N$ for (3), the region for nonexistence corresponding to the region below the Sobolev curve.

Strong evidence to support this conjecture comes from the work in [Mi93], [PvdV], and [vdV], where it has been shown that under (H_1), when $N > 2$, a necessary and sufficient condition for the existence of radial solutions to (S_R) is

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} > \frac{N-2}{N}. \tag{5}$$

Strong evidence to support this conjecture comes from the work in [Mi93], [PvdV], and [vdV], where it has been shown that under (H_1), when $N > 2$, a necessary and sufficient condition for the existence of radial solutions to (S_R) is

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} > \frac{N-2}{N}. \tag{5}$$

In case that $m = p = q \neq 2$ and $\delta = \mu$ we have that if (u, v) is a solution, then $u = v$ and hence the system reduces to an equation. It follows then from results of [O] that a solution exists in that case (for $m < N$) if and only if

$$\frac{1}{\delta+1} > \frac{N-m}{Nm}. \tag{6}$$

Strong evidence to support this conjecture comes from the work in [Mi93], [PvdV], and [vdV], where it has been shown that under (H_1), when $N > 2$, a necessary and sufficient condition for the existence of radial solutions to (S_R) is

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} > \frac{N-2}{N}. \tag{5}$$

In case that $m = p = q \neq 2$ and $\delta = \mu$ we have that if (u, v) is a solution, then $u = v$ and hence the system reduces to an equation. It follows then from results of [O] that a solution exists in that case (for $m < N$) if and only if

$$\frac{1}{\delta+1} > \frac{N-m}{Nm}. \tag{6}$$

Apart from these cases no necessary and sufficient condition for the existence of solutions is known. Sufficient conditions have been obtained in [CMM93] where a-priori estimates are established by means of a blow up method in the sense of Gidas and Spruck, see [G-S], and a degree argument.

The question for the more general case, i.e. without assuming radial symmetry, and still for $p = q = 2$, is more involved and to the best of our knowledge it has not been completely answered yet. Partial answers for nonexistence are known. In [F-F] Felmer and de Figueiredo proved that if

$$0 < \delta, \mu \le \frac{N+2}{N-2} \quad , \quad (\delta, \mu) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right) \tag{7}$$

then problem (3) has no positive solutions in $\mathbb{R}^N$.

The question for the more general case, i.e. without assuming radial symmetry, and still for $p = q = 2$, is more involved and to the best of our knowledge it has not been completely answered yet. Partial answers for nonexistence are known. In [F-F] Felmer and de Figueiredo proved that if

$$0 < \delta, \mu \leq \frac{N+2}{N-2} \quad , \quad (\delta, \mu) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2} \right) \tag{7}$$

then problem (3) has no positive solutions in $\mathbb{R}^N$. In [S-Z], Serrin and Zou proved that if $N > 3$, $\delta > 0$, $\mu > 0$ satisfy (5), and if the solutions have sufficient decay at infinity then problem (3) has no positive solutions.

The question for the more general case, i.e. without assuming radial symmetry, and still for $p = q = 2$, is more involved and to the best of our knowledge it has not been completely answered yet. Partial answers for nonexistence are known. In [F-F] Felmer and de Figueiredo proved that if

$$0 < \delta, \mu \leq \frac{N+2}{N-2} \quad , \quad (\delta, \mu) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right) \tag{7}$$

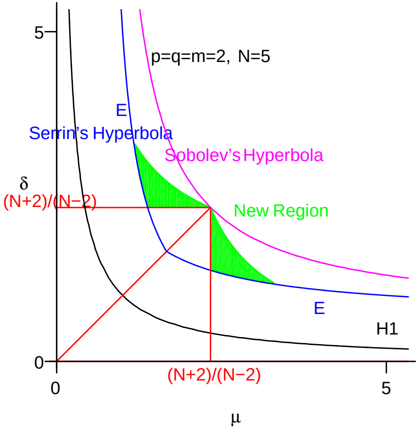
then problem (3) has no positive solutions in $\mathbb{R}^N$. In [S-Z], Serrin and Zou proved that if $N > 3$, $\delta > 0$, $\mu > 0$ satisfy (5), and if the solutions have sufficient decay at infinity then problem (3) has no positive solutions. In a recent paper by Busca and Manásevich, see [B-M], a new region of nonexistence of positive solutions in $\mathbb{R}^N$ for (3) which enlarges that of [F-F] was discovered.

The question for the more general case, i.e. without assuming radial symmetry, and still for $p = q = 2$, is more involved and to the best of our knowledge it has not been completely answered yet. Partial answers for nonexistence are known. In [F-F] Felmer and de Figueiredo proved that if

$$0 < \delta, \mu \leq \frac{N+2}{N-2} \quad , \quad (\delta, \mu) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2} \right) \tag{7}$$

then problem (3) has no positive solutions in $\mathbb{R}^N$. In [S-Z], Serrin and Zou proved that if $N > 3$, $\delta > 0$, $\mu > 0$ satisfy (5), and if the solutions have sufficient decay at infinity then problem (3) has no positive solutions. In a recent paper by Busca and Manásevich, see [B-M], a new region of nonexistence of positive solutions in $\mathbb{R}^N$ for (3) which enlarges that of [F-F] was discovered.
Nevertheless the full conjecture remains open.

The regions of existence / nonexistence



- The main goal of this work was to exhibit a new region of existence or nonexistence of solutions to (S_R). This is done in Theorem 1 for the case $p,\ q \le 2$ and in Theorem 2 for the case $2 \le p,\ q$..
  To our knowledge, when $p \neq q$ or $p = q \neq 2$ and $\delta \neq \mu$, there are no nonexistence results (of Pohozaev type) in the literature.

- The main goal of this work was to exhibit a new region of existence or nonexistence of solutions to (S_R). This is done in Theorem 1 for the case $p,\ q \leq 2$ and in Theorem 2 for the case $2 \leq p,\ q$..
  To our knowledge, when $p \neq q$ or $p = q \neq 2$ and $\delta \neq \mu$, there are no nonexistence results (of Pohozaev type) in the literature.
- An important ingredient in the proof of our existence results is the observation that under condition (H_1), the absence of positive "ground states" implies existence of solutions for (S_R). The result is contained implicitly in [CMM93], [CFMdT], but for the sake of completeness we state it below.

Fundamental Proposition [CMM93, CFMdT]

Let $p,\ q > 1,\ \delta, \mu > 0$ *be such that* (H_1) *holds. If the system*

$$(S_\infty) \qquad \begin{aligned} -\Delta_p u &= |v|^\delta \quad v > 0 \quad in \quad \mathbb{R}^N \\ -\Delta_q v &= |u|^\mu \quad u > 0 \quad in \quad \mathbb{R}^N \end{aligned}$$

has no radially symmetric solutions (u, v) *in* $(C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})^2$, *then system* (S_R) *possesses a nontrivial solution for any* $R > 0$.

Some remarks

- We do not know if the converse of this proposition is true as it is in the case of a single equation or the case of the system with $p = q = 2$, see [CFM92], [Mi93], [PvdV].

Some remarks

- We do not know if the converse of this proposition is true as it is in the case of a single equation or the case of the system with $p = q = 2$, see [CFM92], [Mi93], [PvdV].

- If $p \geq N$, then $-\Delta_p u \geq 0$ and $u \geq 0$ in $\mathbb{R}^N$ imply $u = Const.$, see [N-S85], [N-S86]. Hence from $-\Delta_p u = 0$ it follows that $v = 0$, and from the second equation it follows that $u = 0$. Therefore, if $p \geq N$ or / and $q \geq N$ it follows from our Fundamental Proposition 1 that (S_R) possesses at least one solution (u, v). Hence for our nonexistence results we may assume without loss of generality that $\max\{p, q\} < N$.

- In [CMM93] is has been shown that if (region below Serrin's hyperbola)

$$\max\left\{\alpha - \frac{N-p}{p-1}, \beta - \frac{N-q}{q-1}\right\} \geq 0 \qquad (8)$$

where $\alpha = \frac{1}{d}[p(q-1)+\delta q] \quad \beta = \frac{1}{d}[q(p-1)+\mu p]$ and (H_1), then the assumptions of Proposition 1 are satisfied. Hence in this case, the existence of solutions to (S_R) follows. Observe that in case that $p = q = m$ and $\delta = \mu$, then

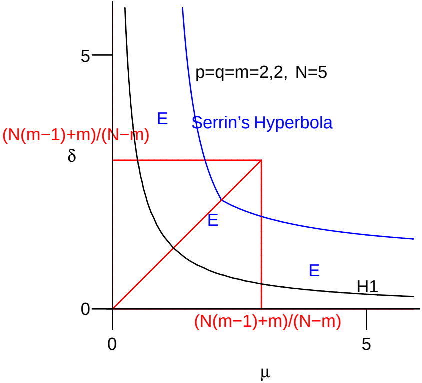
$$\alpha = \beta = \frac{m(\delta + m - 1)}{\delta^2 - (m-1)^2} = \frac{m}{\delta - (m-1)}, \quad \frac{N-p}{p-1} = \frac{N-q}{q-1} = \frac{N-m}{m-1},$$

and thus the condition (8) is equivalent to

$$\frac{m-1}{\delta - (m-1)} > \frac{N-m}{m}, \quad \Leftrightarrow \quad \delta \leq \frac{N(m-1)}{N-m}, \; m > 1$$

which is more restrictive than (6), which says $\delta < \frac{N(m-1)+m}{N-m}$.

Region of Existence [CMM93]



An important information about the system (S_R) is that an equivalent of the Sobolev curve is not known to exist, and thus to obtain non existence or existence results above the Serrin curve, even close to this curve is important.

Main Results

Theorem (1)

Suppose $N \geq 2$ *and that* $\delta,\ \mu > 0$ *satisfy* (H_1). *Let also* $\underline{m} = \min\{p, q\}$.

1. *Let* $\frac{2N}{N+1} < p,\ \ q \leq 2$. *Then problem* (S_R) *possesses a solution* (u, v) *provided that*

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} > \frac{N-\underline{m}}{N(\underline{m}-1)}. \tag{9}$$

2. *If* $N/(N-1) < p,\ \ q \leq 2$ *and*

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} \leq \frac{N(\underline{m}-1)-\underline{m}}{N(\underline{m}-1)}, \tag{10}$$

then system (S_R) *has no solutions (regular or not).*

Remark

Observe that when $p = q = 2$, condition (9) and (10) are optimal, see [CFM92], [Mi93], [PvdV].
When $m = p = q \neq 2$, $m < 2$ and $\delta = \mu$, condition (9) reads

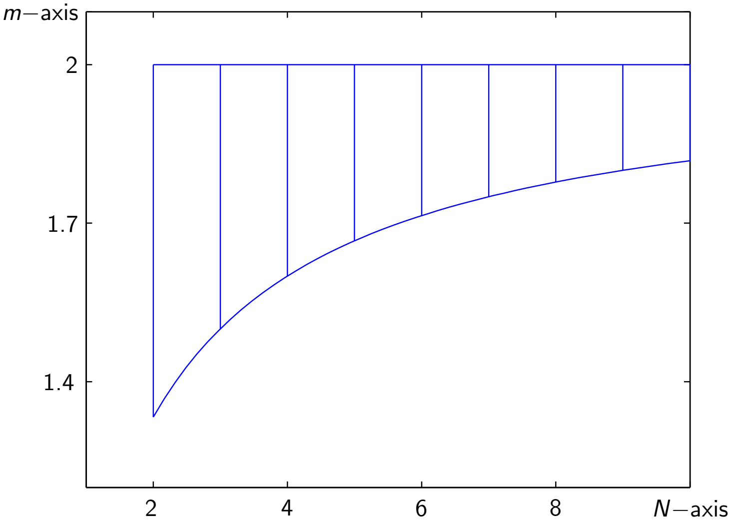
$$\delta + 1 < \frac{2N(m-1)}{N-m} \quad (\text{which for } m < 2 \text{ is } < \frac{Nm}{N-m}),$$

hence condition (9) is not optimal.
Also, when $m < 2$, we note that (9) gives a new region of existence provided that

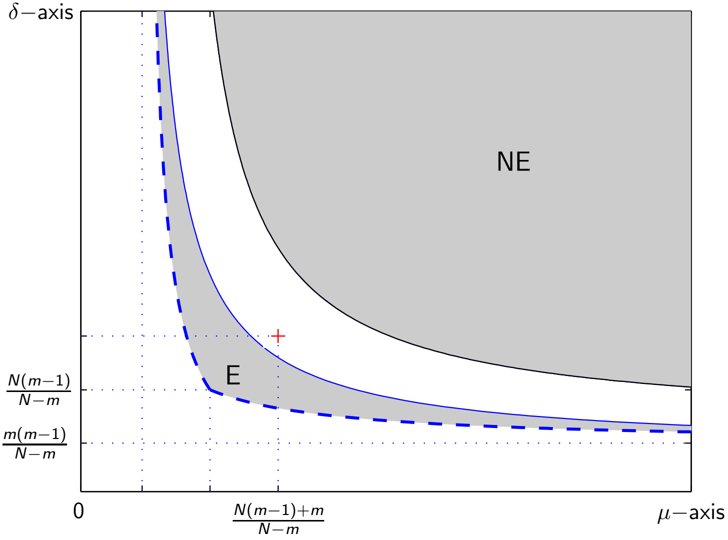
$$\frac{N\,(m-1)}{N-m} < \frac{(2\,N+1)\,m - 3\,N}{N-m},$$

which holds if $m > \frac{2N}{N+1}$. Since $2N/(N+1) < 2$, there is always room for some $m < 2$, as is shown in Figure 1.

The values of $m < 2$ for which we obtain a new region of existence



$m = 1.9$ and $N = 4$



Main Results

Theorem (2)

Suppose that $N > 2$ and $\delta, \mu > 0$ satisfy (H_1). Let also $\overline{m} = \max\{p, q\}$

1. *Let $2 \leq p,\ q < N$. Then problem (S_R) possesses a solution (u, v) provided that*

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} > \frac{N(\overline{m} - 1) - \overline{m}}{N(\overline{m} - 1)}. \tag{11}$$

2. *If $2 \leq p,\ q < N$, and*

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} \leq \frac{N - \overline{m}}{N(\overline{m} - 1)}, \tag{12}$$

then system (S_R) has no solutions (regular or not).

Remark

When $m > 2$, we note that (11) gives a new region of existence provided that

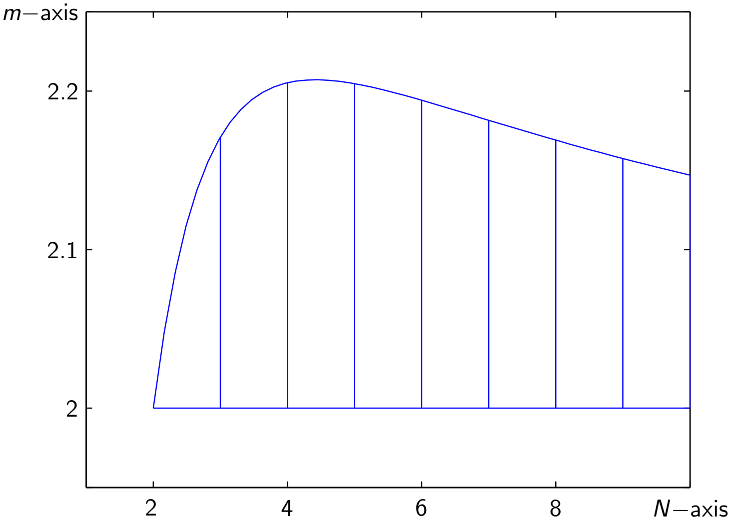
$$\frac{N(m-1)}{N-m} < \frac{N(m-1)+m}{N(m-1)-m},$$

which holds if

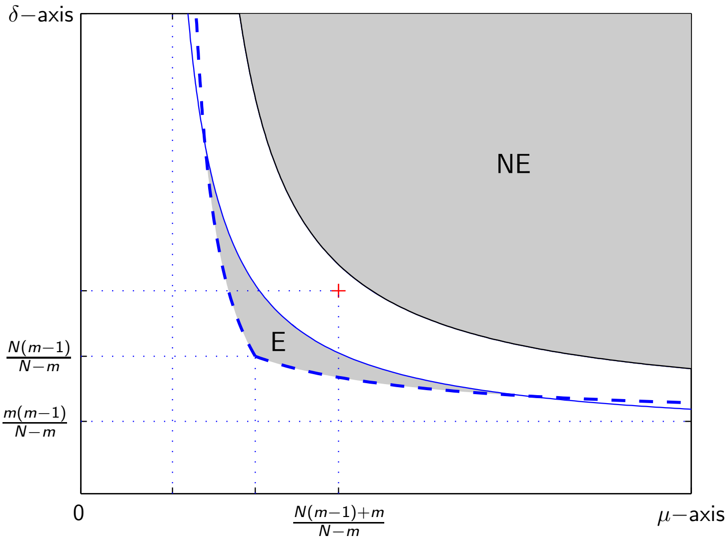
$$m < \frac{\left(3\,N+1+\sqrt{(N-1)(N+7)}\right)\,N}{2N^2+2}.$$

Since the right hand side of this inequality is greater than 2 for $N > 2$, there is always room for some $m > 2$, as is shown in the next figure.

The values of $m > 2$ for which we obtain a new region of existence.



$m = 2.1$ and $N = 4$



Theorem (3)

Suppose that $N > 2$ *and* $\delta, \mu > 0$ *satisfy* (H_1) *and let* $1 < p \leq 2 \leq q < N$

1. *Let* $2N/(N+1) < p$. *Then problem* (S_R) *possesses a solution* (u, v) *provided that*

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} > \frac{N-k_1}{N}, \tag{13}$$

   where $k_1 = \min\{p + \frac{N-p}{p-1}(p-2),\ \frac{q}{q-1}\}$

2. *If* $N/(N-1) < p$, *and*

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} \leq \frac{N-k_2}{N}, \tag{14}$$

   where $k_2 = \max\{q + \frac{N-q}{q-1}(q-2),\ \frac{p}{p-1}\}$ *then system* (S_R) *has no solutions (regular or not).*

Our main theorems are based on the two following lemmas, which give appropriate generalizations of the Pohozaev identity used to deal with the case $p = q = 2$, see [Mi93].

Lemma 1: A Pohozaev type identity

Let $(u,v) \in (C^1[0,\infty) \cap C^2(0,\infty))^2$ be a solution of the system

$$(S_\infty) \qquad \begin{aligned} &-(r^{N-1}|u'|^{p-2}u')' = r^{N-1}v^{\delta} \\ &-(r^{N-1}|v'|^{q-2}v')' = r^{N-1}u^{\mu} \\ &u(r) > 0, \quad v(r) > 0, \quad r \in [0,\infty), \end{aligned}$$

with $\delta, \ \mu > 0$, and assume that either

$$2N/(N+1) \le p \le q \le 2 \quad \textit{or} \quad 2 \le p \le q < N.$$

Let us define

$$E_1(r) = r^{N+k_1-2}|u'|^{p-1}|v'|^{q-1} - \frac{N}{\delta+1} r^{N-1}|u'|^{p-1}\int_r^\infty s^{k_1-2}|v'|^{q-1}\,ds$$
$$- \frac{N}{\mu+1} r^{N-1}|v'|^{q-1}\int_r^\infty s^{k_1-2}|u'|^{p-1}\,ds$$
$$+ r^N\int_r^\infty s^{k_1-2}|v'|^{q-1}v^{\delta}\,ds + r^N\int_r^\infty s^{k_1-2}|u'|^{p-1}u^{\mu}\,ds, \ r \in (0,\infty),$$

where

$$k_1 = \begin{cases} p + \frac{N-p}{p-1}(p-2) & \text{if } 2N/(N+1) \le p \le q \le 2, \\ \frac{q}{q-1} & \text{if } 2 \le p \le q. \end{cases} \tag{15}$$

Then for $r \in (0, \infty)$ we have

$$E_1'(r) = \Big(k_1 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\Big) r^{N+k_1-3}|u'|^{p-1}|v'|^{q-1}$$
$$-\frac{N}{\delta+1} r^{N-1} v^{\delta} \int_r^{\infty} s^{k_1-2}|v'|^{q-1}\,ds + N r^{N-1} \int_r^{\infty} s^{k_1-2}|v'|^{q-1} v^{\delta}\,ds$$
$$-\frac{N}{\mu+1} r^{N-1} u^{\mu} \int_r^{\infty} s^{k_1-2}|u'|^{p-1}\,ds + N r^{N-1} \int_r^{\infty} s^{k_1-2}|u'|^{p-1} u^{\mu}\,ds. \tag{16}$$

Lemma 2: A Pohozaev type identity

Let $(u,v) \in (C^1[0,R] \cap C^2(0,R])^2$ be a solution of the system (S_R) with $\delta,\ \mu > 0$, and assume that either

$$N/(N-1) < p \le q \le 2 \quad \text{or} \quad 2 \le p \le q < N.$$

Let us define

$$E_2(r) = r^{N+k_2-2}|u'|^{p-1}|v'|^{q-1} - \frac{N}{\delta+1} r^{N-1}|u'|^{p-1} \int_r^R s^{k_2-2}|v'|^{q-1}\,ds$$
$$-\frac{N}{\mu+1} r^{N-1}|v'|^{q-1} \int_r^R s^{k_2-2}|u'|^{p-1}\,ds$$
$$+r^N \int_r^R s^{k_2-2}|v'|^{q-1}v^{\delta}\,ds + r^N \int_r^R s^{k_2-2}|u'|^{p-1}u^{\mu}\,ds,\ r \in (0,R],$$

where

$$k_2 = \begin{cases} q + \frac{N-q}{q-1}(q-2) & \text{if } 2 \le p \le q < N, \\ \frac{p}{p-1} & \text{if } N/(N-1) < p \le q \le 2. \end{cases} \qquad (17)$$

Then for $r \in (0, R)$ we have

$$E_2'(r) = \left(k_2 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\right) r^{N+k_2-3}|u'|^{p-1}|v'|^{q-1}$$
$$-\frac{N}{\delta+1} r^{N-1} v^{\delta} \int_r^R s^{k_2-2}|v'|^{q-1}\, ds + N r^{N-1} \int_r^R s^{k_2-2}|v'|^{q-1} v^{\delta}\, ds$$
$$-\frac{N}{\mu+1} r^{N-1} u^{\mu} \int_r^R s^{k_2-2}|u'|^{p-1}\, ds + N r^{N-1} \int_r^R s^{k_2-2}|u'|^{p-1} u^{\mu}\, ds. \qquad (18)$$

We observe that from (15) and (17),

$$p = q = 2 \Rightarrow k_1 = k_2 = 2,$$

and therefore

$$E_1(r) = E_2(r) = E(r),$$

where

$$E(r) = r^N|u'||v'| - \frac{N}{\delta+1} r^{N-1}|u'|v - \frac{N}{\mu+1} r^{N-1}|v'|u + r^N \frac{v^{\delta+1}}{\delta+1} + r^N \frac{u^{\mu+1}}{\mu+1},$$

and

$$E'(r) = \left(2 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\right) r^{N-1}|u'||v'|.$$

Recall

$$k_1 = \begin{cases} p + \frac{N-p}{p-1}(p-2) & \text{if } 2N/(N+1) \le p \le q \le 2, \\ \frac{q}{q-1} & \text{if } 2 \le p \le q, \end{cases}$$

$$k_2 = \begin{cases} q + \frac{N-q}{q-1}(q-2) & \text{if } 2 \le p \le q < N, \\ \frac{p}{p-1} & \text{if } N/(N-1) < p \le q \le 2, \end{cases}$$

and let us set, for $i = 1, 2$, $m > 1$, $\gamma > 0$, and $w \in C^1(r, R_i)$, $0 < r < R_i$,

$$G_i(m, \gamma, w)(r) = N \int_r^{R_i} s^{k_i-2} |w'|^{m-1} |w|^{\gamma} \, ds - \frac{N}{\gamma+1} |w|^{\gamma} \int_r^{R_i} s^{k_i-2} |w'|^{m-1} \, ds, \quad (19)$$

where $R_1 = \infty$ and $R_2 = R$.

We note that according to the previous lemmas,

$$E_i'(r) \;=\; \left(k_i - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\right) r^{N+k_i-3}|u'|^{p-1}|v'|^{q-1}$$
$$+r^{N-1}(G_i(p,\mu,u)(r) + G_i(q,\delta,v)(r)).$$

In order to prove the existence parts of Theorems 1,2, we will use the Fundamental Proposition quoted at the beginning by establishing that

$$G_1(p,\mu,u)(r) \geq 0 \qquad G_1(q,\delta,v)(r) \geq 0$$

for any (u, v) solution to (S_∞), $\delta, \mu > 0$ and $2N/(N+1) < p,\ q < N$, and in order to prove the nonexistence parts of Theorems 1, 2, we will prove that

$$G_2(p,\mu,u)(r) \leq 0 \qquad G_2(q,\delta,v)(r) \leq 0$$

for any (u, v) solution to (S_R), $\delta, \mu > 0$ and $N/(N-1) < p,\ q < N$.

In view of the Fundamental Proposition 1, in order to prove the existence part of our theorems we only need to prove that under assumption (9) or (11) system (S_∞) does not possess any radial solution. We will argue by contradiction by assuming that there exists a radially symmetric solution (u, v) to (S_∞).

The idea is to have E_1 strictly increasing with $\lim_{r\to 0^+} E_1(r) = 0$ and $\lim_{r\to\infty} E_1(r) = 0$ which will give a contradiction.

We start by proving that $\lim_{r\to 0} E_1(r) = 0$. Since u and v are regular, a simple application of L'Hôpital's rule gives

$$\lim_{r\to 0} |u'(r)|^{p-1}/r = v(0)^\delta/N \quad \text{and} \quad \lim_{r\to 0} |v'(r)|^{q-1}/r = u(0)^\mu/N.$$

Therefore we need $N + k_1 > 0$, or equivalently, $p > 3N/(2N+1)$. But $p > 2N/(N+1) > 3N/(2N+1)$ if $N > 1$, hence $\lim_{r\to 0} E_1(r) = 0$.

We now verify that $\lim_{r\to\infty} E_1(r) = 0$. We have the bounds near infinity given by

$$u(r) \leq Cr^{-\alpha}, \quad |u'(r)| \leq Cr^{-\alpha-1}, \quad v(r) \leq Cr^{-\beta}, \quad |v'(r)| \leq Cr^{-\beta-1}$$

for some $C > 0$ and r large, see [CMM93, Lemma 2.1] or [CFMdT, Proposition V.1]. Next, by observing that by the definition of $\alpha,\ \beta$ we have

$$1 - \delta\beta = -(\alpha+1)(p-1) \quad \text{and} \quad 1 - \mu\alpha = -(\beta+1)(q-1),$$

in order that $\lim_{r\to\infty} E_1(r) = 0$ it is sufficient to show that

$$N + k_1 - 2 - (\alpha+1)(p-1) - (\beta+1)(q-1) < 0, \tag{20}$$

which follows after some considerations by using (9).

We prove next that under the assumptions of the theorem we have $E_1'(r) > 0$ for all $r > 0$. Since by the choice of k_1

$$k_1 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1} = -\frac{(N-p)}{p-1} + \frac{N}{\delta+1} + \frac{N}{\mu+1},$$

we have by assumption (9) that this term is indeed positive. From (19) we have that

$$G_1(p,\mu,u)(r) = N\int_r^\infty s^{k_1-2}|u'|^{p-1}u^\mu\,ds - \frac{N}{\mu+1}u^\mu\int_r^\infty s^{k_1-2}|u'|^{p-1}\,ds, \tag{21}$$

where $k_1 = p + \frac{N-p}{p-1}(p-2)$. With this notation, for $r \in (0,\infty)$, we have that $E_1'(r)$ can be written as

$$\begin{aligned} E_1'(r) &= \overbrace{\left(k_1 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\right)}^{>0} r^{N+k_1-3}|u'|^{p-1}|v'|^{q-1} \\ &\quad + r^{N-1}G_1(p,\mu,u)(r) + r^{N-1}G_1(q,\delta,v)(r). \end{aligned} \tag{22}$$

hence we need to prove that both $G_1(p,\mu,u)(r)$ and $G_1(q,\delta,v)(r)$ are positive.

This is done by proving that $G_1(q,\delta,v)'(r) \leq 0$ and using that $G_1(q,\delta,v)(\infty) = 0$. By differentiating both sides in (21) with respect to r we obtain

$$\begin{aligned} G_1'(p,\mu,u)(r) &= -\frac{N\mu}{\mu+1} r^{k_1-2}|u'|^{p-1}u^{\mu} \\ &\quad + \frac{N\mu}{\mu+1} u^{\mu-1}|u'| \int_r^{\infty} s^{k_1-2}|u'|^{p-1}\,ds. \qquad (23) \\ &= \frac{N\mu|u|u^{\mu-1}}{\mu+1}\Big(-r^{k_1-2}|u'|^{p-2}u + \int_r^{\infty} s^{k_1-2}|u'|^{p-1}\,ds\Big). \end{aligned}$$

Using now that $(r^{N-1}|u'|^{p-1})' \geq 0$, we have, using that $p \leq 2$, we find that $\left(s^{(N-1)/(p-1)}|u'|(s)\right)^{p-2} \leq \left(r^{(N-1)/(p-1)}|u'|(r)\right)^{p-2}$ for $s \geq r$. Therefore,

$$\begin{aligned} \int_r^{\infty} s^{k_1-2}|u'|^{p-1}\,ds &= \int_r^{\infty} \left(s^{(N-1)/(p-1)}|u'|(s)\right)^{p-2}|u'(s)|\,ds \\ &\leq r^{k_1-2}u(r)|u'|^{p-2}. \qquad (24) \end{aligned}$$

Replacing (24) into (23), we obtain

$$G_1'(p,\mu,u)(r) \leq \Big(-N + \frac{N}{\mu+1}(\mu+1)\Big) r^{k-2}|u'|^{p-1}u^{\mu} = 0,$$

hence $G_1'(p,\mu,u)(r) \le 0$ for all $r>0$, and since $G_1(p,\mu,u)(\infty)=0$, we have
$G_1(p,\mu,u)(r) \ge 0$ for all $r>0$.

Finally we show that $G_1(q,\delta,v)(r) \ge 0$ for all $r>0$. Indeed, we define $\bar{k} = q + \frac{N-q}{q-1}(q-2)$ and note that $k_1 \le \bar{k}$ when $q \ge p$. We proceed as above using the following inequality

$$\int_r^\infty s^{k_1-2}|v'|^{q-1}\,ds = \int_r^\infty s^{k_1-\bar{k}}\left(s^{(N-1)/(q-1)}|v'(s)|\right)^{q-2}|v'(s)|\,ds$$

$$\le r^{k_1-\bar{k}}\int_r^\infty \left(s^{(N-1)/(q-1)}|v'(s)|\right)^{q-2}|v'(s)|\,ds \le r^{k_1-2}|v'|^{q-2}v(r).$$

Therefore $E_1'(r)>0$ for all $r>0$ in contradiction with $E_1(0^+)=E_1(\infty)=0$. Thus under the assumptions of the Theorem there cannot exist radially symmetric solutions to (S_∞) and we can use Proposition 1 to obtain the existence of at least one solution to (S_R) for any positive R.

Proof of the nonexistence results

We will argue by contradiction assuming that there exists a solution (u, v) to (S_R). Now we will use Lemma 2.
The idea is to have E_2 decreasing with $E_2(0^+) = 0$ and $E_2(R) > 0$ yielding a contradiction.
We assume $q = \overline{m}$, $p = \underline{m}$, and since the case $p = q = 2$ was proven in [Mi93], we may assume without loss of generality that $p < 2$ for part (2) in Theorem 1 and $q > 2$ for part (2) in Theorem 2.
By direct computation we have that

$$E_2(R) = R^{N+k_2-2}|u'(R)|^{p-1}|v'(R)|^{q-1} > 0.$$

We will show next that $E_2(0^+) = 0$. By [CMM93, Lemma 2.1] or [CFMdT, Proposition V.1], we have that any solution (u, v) to (S_R) satisfies

$$u(r) \leq Kr^{-\alpha}, \quad |u'(r)| \leq Kr^{-\alpha-1}, \quad v(r) \leq Kr^{-\beta}, \quad |v'(r)| \leq Kr^{-\beta-1}$$

for some $K > 0$ and $0 < r \ll 1$. Hence in order to show that $E_2(0^+) = 0$ we need

$$N + k_2 - 2 - (\alpha + 1)(p - 1) - (\beta + 1)(q - 1) > 0. \tag{25}$$

As for (20), this last inequality reduces to

$$L > \frac{(q-1)(L+p)}{\delta + q - 1} + \frac{(p-1)(L+q)}{\mu + p - 1}.$$

where $L := N + k_2 - p - q$, and can be established after some hard work by using (12) or (10).

Finally we prove that $E_2'(r) \leq 0$ for all $r \in (0, R)$. To this end we recall from (19) that

$$G_2(q,\delta,v)(r) = N\int_r^R s^{k_2-2}|v'|^{q-1}v^\delta\, ds - \frac{N}{\delta+1}v^\delta\int_r^R s^{k_2-2}|v'|^{q-1}\, ds, \tag{26}$$

so that

$$E_2'(r) = r^{N-1}\left(k_2 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\right)r^{k_2-2}|u'|^{p-1}|v'|^{q-1} + r^{N-1}(G_2(q,\delta,v)(r) + G_2(p,\mu,u)(r)). \tag{27}$$

We claim that $E_2'(r) \leq 0$ for $r \in (0, R)$. Indeed, we observe first that from (17) we have that

$$N - k_2 = \begin{cases} \frac{N(p-1)-p}{p-1} & \text{for } N/(N-1) < p \leq q \leq 2 \\ \frac{N-q}{q-1} & \text{for } 2 \leq p \leq q < N \end{cases},$$

implying that the first term in (27) is negative by assumptions (10) (or (12)).

Also, differentiating in (26) with respect to r, we obtain

$$G_2'(q,\delta,v)(r) = \left(-N + \frac{N}{\delta+1}\right) r^{k_2-2}|v'|^{q-1}v^{\delta} + \delta\frac{N}{\delta+1}v^{\delta-1}|v'|\int_r^R s^{k_2-2}|v'|^{q-1}\,ds. \quad (28)$$

We will prove that $G_2'(q,\delta,v)(r) \geq 0$, and thus, using that $G_2(q,\delta,v)(R) = 0$, we will obtain that $G_2(q,\delta,v)(r) \leq 0$ for all $r \in (0,R)$.

To this end, we will see that $|u'|^{p-1}/r$ is decreasing for all $r > 0$: indeed, since

$$\frac{|u'|^{p-1}}{r} = \frac{1}{r^N}\int_0^r s^{N-1}v^{\delta}(s)ds,$$

we have that

$$\frac{d}{dr}\frac{(|u'|^{p-1})}{r} = \frac{1}{r^N}r^{N-1}v^{\delta}(r) - N\frac{r^{N-1}}{r^{2N}}\int_0^r s^{N-1}v^{\delta}(s)ds,$$

and thus, using that v is decreasing in $(0,\infty)$ we find that

$$\frac{d}{dr}\frac{(|u'|^{p-1})}{r} \le \frac{1}{r^N}r^{N-1}v^{\delta}(r) - N\frac{r^{N-1}}{r^{2N}}\frac{r^N}{N}v^{\delta}(r) = 0. \qquad (29)$$

Nonexistence part for $N/(N-1) < p \le q \le 2$

We set $\bar{k} = q/(q-1) \le k_2 = p/(p-1)$ to obtain that

$s^{k_2-\bar{k}}$ increases, and since $q < 2$, $\left(\frac{|v'|(s)}{s^{1/(q-1)}}\right)^{q-2}$ also increases implying

$$\begin{aligned}\int_r^R s^{k_2-2}|v'|^{q-1}\,ds &= \int_r^R s^{k_2-\bar{k}}\left(\frac{|v'|(s)}{s^{1/(q-1)}}\right)^{q-2}|v'(s)|\,ds \\ &\ge r^{k_2-2}|v'|^{q-2}v(r). \qquad (30)\end{aligned}$$

Hence in this case we find that

$$G_2'(q,\delta,v)(r) \ge \left(-N + \frac{N}{\delta+1}(\delta+1)\right) r^{k_2-2}|v'|^{q-1}v^{\delta} = 0.$$

Similarly, setting $\bar{k} = k_2$, we find that

$$G_2'(p,\mu,u)(r) \ge \left(-N + \frac{N}{\mu+1}(\mu+1)\right) r^{k_2-2}|v'|^{q-1}v^{\delta} = 0.$$

Since $G_2(q, \delta, v)(R) = G_2(p, \mu, u)(R) = 0$, we conclude

$$G_2(q, \delta, v)(r) \leq 0 \text{ and } G_2(p, \mu, u)(r) \leq 0 \text{ for all } r \in (0, R).$$

Now using that (δ, μ) satisfies (10), $G_2(q, \delta, v)(r) \leq 0$ and $G_2(p, \mu, u)(r) \leq 0$ for all $r \in (0, R]$, by (27) we obtain $E_2'(r) < 0$ for all $r \in (0, R]$, which is a contradiction.

THANK YOU!!!

C. Azizieh, Ph. Clément, and E. Mitidieri, Existence and a priori estimates for positive solutions of p-Laplace systems. *J. Diff. Equations* **184** (2002), no. 2, 422–442

J. Busca and R. Mansevich A Liouville-type theorem for Lane-Emden systems. *Indiana Univ. Math. J.***51** (2002), no. 1, 37–51.

Ph. Clément, D.G. de Figueiredo, and E. Mitidieri, Positive solutions of semilinear elleiptic systems, *Comm in P.D.E.*,**17** (1992) 923-940.

Ph. Clément, R. Manásevich, and E. Mitidieri, Positive solutions for a quasilinear system via blow up, *Comm in P.D.E.*,**18** (1993) 2071-2106.

Ph. Clément, J. Fleckinger, E. Mitidieri, and F. de Thélin, Existence of Positive solutions for a Nonvariational Quasilinear Elliptic System,*J. Diff. Equations* **166** (2000) 455-477.

P. Felmer, D.G. de Figueiredo, A Liouville-type Theorem for systems. *Ann. Sc. Norm. Sup. Pisa* **XXI** (1994), 387-397.

B. Gidas and J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, *Comm. Part. Diff. Equations* **6** (1981), no. 8, 883-901

E. Mitidieri, A Rellich type identity and applications, *Comm. Partial Diff. Equations,* **18**(1-2) 1993, 125-151.

W.-M. Ni and J. Serrin, Non-existence theorems for quasilinear partial differential equations, *Rend. Circ. Mat. Palermo* (suppl.) **8** (1985), 171-185.

W.-M. Ni and J. Serrin, Existence and non-existence for ground states for quasilinear partial differential equations, *Att. Conveg. Lincei* **77** (1985), 231-257.

- W.-M. Ni and J. Serrin, Non-existence theorems for singular solutions of quasilinear partial differential equations, *Comm. Pure Appl. Math.* **39** (1986), 379-399.
- M. Ôtani, Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations, *J. Funct. Anal.*, **76** no. 1, (1988), 140–159.
- L.A. Peletier and R.C.A.M van der Vorst, Existence and nonexistence of positive solutions of non-linear elliptic systems and the biharmonic equation *Diff. Int. Eq.*, **5** no. 4 1992, 747-767.
- J. Serrin, H. Zou, Non-existence of positive solutions of Lane-Emden systems. *Diff. Int. Eq.* **9** (1996), 635-653.
- R.C.A.M van der Vorst, Variational identities and applications to differential systems, *Arch. Rational. Mech. Anal.*, **116**, (1991), 375-398.