SOME NONLINEAR EQUATIONS WITH SOBOLEV TYPE SINGULARITY

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$$u(t_0) = u_0 \tag{1}$$

$$L\dot{u}(t) = Mu(t) + N(t, u(t)), \quad t \in (t_0, T), \tag{2}$$

$\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, $\ker L \neq \{0\}$, $M \in Cl(\mathfrak{U}; \mathfrak{F})$,
$N : U \to \mathfrak{F}$, $U \subset \mathfrak{U}$

1. SVIRIDYUK G.A., SUKACHEVA T.G. The Cauchy problem for a class of semilinear equations of Sobolev type. **Siberian Math. J.**, 1991, 31, no. 5, 794-802.

2. SVIRIDYUK G.A., YAKUPOV M.M. The phase space of the initial-boundary value problem for the Oskolkov system. **Diff. equations**, 1996, 32, no 11, 1535-1540.

3. SVIRIDYUK G.A., FEDOROV V.E. Linear Sobolev Type Equations and Degenerate Semigroups of Operators. VSP, Utrecht – Boston, 2003.

4. FEDOROV V.E. Holomorphic solution semigroups for Sobolev-type equations in locally convex spaces. **Sb. Math.**, 2004, 195, no.8, 1205-1234.

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$$

$$R^L_\mu(M) = (\mu L - M)^{-1}L, \quad L^L_\mu(M) = L(\mu L - M)^{-1}$$

$$R^L_{(\mu,p)}(M) = \prod_{k=0}^{p} R^L_{\mu_k}(M), \quad L^L_{(\mu,p)}(M) = \prod_{k=0}^{p} L^L_{\mu_k}(M)$$

Definition 1. Operator M is strongly (L,p)-sectorial, if

(i) $\exists a \in \mathbb{R}\ \exists \theta \in (\pi/2, \pi)\ S_{a,\theta} \equiv \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta\} \subset \rho^L(M)$;

(ii) $\exists K \in \mathbb{R}_+\ \forall \mu_k \in S_{a,\theta},\ k = \overline{0,p}$,

$$\max\left\{\|R^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{U})}, \|L^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{F})}\right\} \leq \frac{K}{\prod\limits_{k=0}^{p} |\mu_k - a|};$$

(iii) there exists a dense subspace $\overset{\circ}{\mathfrak{F}}$ in $\mathfrak{F}$ such that

$$\|M(\lambda L - M)^{-1} L^L_{(\mu,p)}(M) f\|_{\mathfrak{F}} \leq \frac{\mathrm{const}(f)}{|\lambda - a| \prod\limits_{k=0}^{p} |\mu_k - a|} \quad \forall f \in \overset{\circ}{\mathfrak{F}}$$

for all $\lambda, \mu_0, \mu_1, ..., \mu_p \in S_{a,\theta}$;

(iv) for all $\lambda, \mu_0, \mu_1, ..., \mu_p \in S_{a,\theta}$

$$\|R^L_{(\mu,p)}(M)(\lambda L - M)^{-1}\|_{\mathcal{L}(\mathfrak{F};\mathfrak{U})} \leq \frac{K}{|\lambda - a| \prod\limits_{k=0}^{p} |\mu_k - a|}.$$

Theorem 1 [3]. *Let operator M be strongly (L,p)-sectorial. Then*

(i) $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$;

(ii) $L_k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $M_k \in Cl(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;

(iii) *there exist operators* $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ *и* $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$;

(iv) *operator* $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{U}^0)$ *is nilpotent of the power not greater than* p;

(v) *there exists strongly continuous in origin and analytic semigroup* $\{U(t) \in \mathcal{L}(\mathfrak{U}) : |\arg t| < \theta - \pi/2\}$ *of the equation* $L\dot{u} = Mu$;

(vi) *operator* $L_1^{-1} M_1 \in Cl(\mathfrak{U}^1)$ *generates nondegenerate analytic semigroup* $\{U_1(t) = U(t)\Big|_{\mathfrak{U}^1} \in \mathcal{L}(\mathfrak{U}^1) : |\arg t| < \theta - \pi/2\}$.

Remark 1. Denote by P (Q) the projector along $\mathfrak{U}^0$ on $\mathfrak{U}^1$ (along $\mathfrak{F}^0$ on $\mathfrak{F}^1$). It is fulfilled $QL = LP$, $QMu = MPu$ for all $u \in \operatorname{dom} M$. The operator semigroup of the equation $L\dot{u} = Mu$ in the case $\ker L \neq \{0\}$ has a nontrivial kernel coinciding with $\mathfrak{U}^0$ and containing all M-adjoint vectors of operator L with height $\leq p$.

5. HENRY D. Geometric Theory of Semilinear Parabolic Equations. Springer – Verlag, Berlin – Heidelberg – New York, 1981.

For generator $A \in Cl(\mathfrak{U}^1)$ of nondegenerate analytic semigroup construct an continuously invertible operator $A_1 = A - bI$ with sufficiently great $b \in \mathbb{R}$. Then subspace $\mathfrak{U}^1_\alpha \equiv \mathrm{dom} A_1^\alpha \subset \mathfrak{U}^1$ is Banach space with the norm $\|u\|_\alpha = \|A_1^\alpha u\|_{\mathfrak{U}^1}$, $\alpha \geq 0$.

Let operator B mapping open set $U \subset \mathbb{R} \times \mathfrak{U}^1_\alpha$, $\alpha \in [0, 1)$, to $\mathfrak{U}^1$ and satisfies the local Hölder condition with respect to t and the local Lipschitz condition with respect to u on U. In other words, for any $(t_1, u_1) \in U$ there exists its neighbourhood $V \subset U$ such that for all $(t, u), (s, v) \in V$ $\|B(t, u) - B(s, v)\|_{\mathfrak{U}^1} \leq C(|t - s|^\theta + \|u - v\|_\alpha)$, where $C, \theta \in \mathbb{R}_+$.

DEFINITION 2. Let a function $u \in C([t_0,T);\mathfrak{U}^1)$ satisfies the condition (1), for all $t \in (t_0,T)$ it is fulfilled that $(t,u(t)) \in U$, $u(t) \in \mathrm{dom}A$, there exists a derivative $\dot{u}(t)$, mapping $t \to B(t,u(t))$ satisfies the local Hölder condition,

$$\int_{t_0}^{t} \frac{\|B(t,u(t))\|_{\mathfrak{U}^1}}{(t-s)^\alpha} ds \to 0 \quad \text{as} \quad t \to t_0+$$

and for (t_0,T) the equation

$$\dot{u}(t) = Au(t) + B(t,u(t)) \tag{3}$$

is fulfilled. Then u is called the solution of the Cauchy problem for the equation (3).

Theorem 2 [5]. *Let operator A is sectorial, mapping $B : U \to \mathfrak{U}^1$ satisfies the local Hölder condition with respect to t and the local Lipschitz condition with respect to u on open set $U \subset \mathbb{R} \times \mathfrak{U}^1_\alpha$, $\alpha \in [0,1)$. Then for any $(t_0,u_0) \in U$ there exists $T = T(t_0,u_0) > t_0$ such that the problem* (1), (3) *has a unique solution on* (t_0,T).

ОПРЕДЕЛЕНИЕ 3. Let operator $N : U \to \mathfrak{F}$ is defined on set $U \subset \mathbb{R} \times \mathfrak{U}$. Let a function $u \in C([t_0, T); \mathfrak{U})$ satisfies the condition (1), for all $t \in (t_0, T)$ it is fulfilled that $(t, u(t)) \in U$, $u(t) \in \mathrm{dom} M$, there exists a derivative $\dot{u}(t)$, mapping $t \to N(t, u(t))$ satisfies the local Hölder condition,

$$\int\limits_{t_0}^{t} \frac{\|N(t, u(t))\|_{\mathfrak{F}}}{(t-s)^{\alpha}} ds \to 0 \quad \text{as} \quad t \to t_0+$$

and for (t_0, T) the equation (2) is fulfilled. Then u is called the solution of the Cauchy problem for the equation (2).

Theorem 3. *Let operator M is strongly (L, p)-sectorial, operator $N : U \to \mathfrak{F}$ is defined on set $U \subset \mathbb{R} \times \mathfrak{U}^0 \oplus \mathfrak{U}^1_{\alpha}$, $\alpha \in [0, 1)$, satisfies the local Hölder condition with respect to t and the local Lipschitz condition with respect to u on open set $V = U \cap \mathbb{R} \times \mathfrak{U}^1_{\alpha}$ in the space $\mathbb{R} \times \mathfrak{U}^1_{\alpha}$, besides, $\mathrm{im} N \subset \mathfrak{F}^1$. Then for any $(t_0, u_0) \in V$ there exists $T = T(t_0, u_0) > t_0$ such that the problem (1), (2) has a unique solution on (t_0, T).*

PROOF. Acting by operator $L_1^{-1}Q$ get

$$\dot{v} = L_1^{-1}M_1 v + L_1^{-1}QN(t, v + w), \tag{4}$$

where $Pu(t) = v(t)$, $(I - P)u(t) = w(t)$. Acting by $M_0^{-1}(I - Q)$ get the equation

$$H\,\dot{w} = w + M_0^{-1}(I - Q)N(t, v + w). \tag{5}$$

The problem (1), (2) is reduced to the Cauchy problem $v(t_0) = Pu_0$, $w(t_0) = (I - P)u_0$ for the system (4), (5).

If $\mathrm{im}N \subset \mathfrak{F}^1$ then $(I - Q)N = 0$ and the eqution (5) has a form

$$H\dot{w}(t) = w(t)$$

and has only trivial solution $w \equiv 0$ because H is nilpotent. The Cauchy problem $w(t_0) = (I - P)u_0$ for it has a solution if $u_0 \in \mathfrak{U}_\alpha^1$, i. e. $(t_0, u_0) \in V$. Then the equation (5) has a form

$$\dot{v} = L_1^{-1}M_1 v + L_1^{-1}QN(t, v). \tag{6}$$

In the next theorem consider the case $N(t, u) \equiv N(t, Pu)$.

Theorem 4. *Let operator M is strongly $(L,0)$-sectorial, operator $N : U \to \mathfrak{F}$ is continuously differentiable with respect to (t,u) in the sense of Frechet on open set $U \subset \mathbb{R}\times\mathfrak{U}^0\oplus\mathfrak{U}^1_\alpha$, $\alpha\in[0,1)$. Besides, suppose that for all $(t,u)\in U$, $w\in\mathfrak{U}^0$ the relations $(t,u+w)\in U$, $N(t,u)=N(t,u+w)$ are fulfilled. Then for any $(t_0,u_0)\in U$ such that*

$$(I-P)u_0 = -M_0^{-1}(I-Q)N(t_0,Pu_0), \tag{7}$$

there exists $T=T(t_0,u_0)>t_0$ such that the problem (1), (2) has a unique solution on (t_0,T).

PROOF. We have $N(t,u)\equiv N(t,Pu)$, $(t_0,Pu_0)\in U$. Acting as before get the system of the equation (6) and the equation

$$0 = w + M_0^{-1}(I-Q)N(t,v), \tag{8}$$

because $H=0$, when $p=0$. Express the solution $w(t)=-M_0^{-1}(I-Q)N(t,v(t))$ of the equation (8) through the solution of the equation (6). It will be the solution of the Cauchy problem $w(0)=(I-P)u_0$ if the condition (8) is satisfied.

Let $a, b, \alpha, \beta, \lambda \in \mathbb{R}$, $a < b$. Consider the problem

$$u(x,t_0) = u_0(x), \quad x \in (a,b), \tag{9}$$

$$\frac{\partial u}{\partial n}(a,t) + \lambda u(a,t) = \frac{\partial u}{\partial n}(b,t) + \lambda u(b,t) =$$
$$\frac{\partial v}{\partial n}(a,t) + \lambda v(a,t) = \frac{\partial v}{\partial n}(b,t) + \lambda v(b,t) = 0, \quad t \in (t_0,T), \tag{10}$$

$$u_t(x,t) = u_{xx}(x,t) - v_{xx}(x,t) + f(t,x,u(x,t),v(x,t)), \quad (x,t) \in (a,b)\times(t_0,T), \tag{11}$$

$$v_{xx}(x,t) + \beta v(x,t) + \alpha u(x,t) = 0, \quad (x,t) \in (a,b)\times(t_0,T). \tag{12}$$

Unknown functions are $u(x,t)$, $v(x,t)$.

REMARK 2. The linear part of the system (11) – (12) coincides with the linear part of the system of phase space equations.

Take $\mathfrak{U} = \mathfrak{F} = (L_2(a,b))^2$,

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x^2} \\ \alpha & \beta + \frac{\partial^2}{\partial x^2} \end{pmatrix}, \quad N(t,u,v) = \begin{pmatrix} f(t,\cdot,u(\cdot),v(\cdot)) \\ 0 \end{pmatrix},$$

$H^2_{\frac{\partial}{\partial n}+\lambda}(a,b) = \{w \in H^2(a,b) : \left(\frac{\partial}{\partial n} + \lambda\right) w(a) = \left(\frac{\partial}{\partial n} + \lambda\right) w(b) = 0\}$, $\mathrm{dom} M = (H^2_{\frac{\partial}{\partial n}+\lambda}(\Omega))^2$.

Denote $Aw = w_{xx}$, $\mathrm{dom} A = H^2_{\frac{\partial}{\partial n}+\lambda}(a,b) \subset L_2(a,b)$. Using the sectorial operator A construct operator A_1 and spaces $\mathcal{H}_\alpha = D(A_1^\alpha)$, $\alpha \geq 0$.

Before it was shown that in the case of $-\beta \notin \sigma(A)$ operator M is strongly $(L,0)$-sectorial,

$$P = \begin{pmatrix} I & 0 \\ -\alpha(\beta + A)^{-1} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & A(\beta + A)^{-1} \\ 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{U}^0 = \ker P = \{0\} \times L_2(a,b)$, $\mathfrak{U}^1 = \mathrm{im} P = \{(u, -\alpha(\beta + A)^{-1}u) \in (L_2(a,b))^2 : u \in L_2(a,b)\}$ is isomorphic to $L_2(a,b) \times \{0\}$, $\mathfrak{F}^1 = \mathrm{im} Q = \{(u + A(\beta + A)^{-1}v, 0) \in (L_2(a,b))^2 : (u,v) \in (L_2(a,b))^2\} = L_2(a,b) \times \{0\}$.

Theorem 5. *Let* $-\beta \notin \sigma(A)$, *function* $f : \mathbb{R} \times [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is measurable with respect to* x, *satisfies the local Hölder condition with respect to* t *and the local Lipschitz condition with respect to* u *uniformly with respect to* x *for* $v = 0$, *besides,* $|f(t,x,u,v)| \leq h(x)g(t,|u|,|v|)$, *where* $h \in L_2(a,b)$, *function* g *is continuous and increasing with respect to second and third arguments. Then for any* $(t_0, u_0) \in \mathbb{R} \times \mathcal{H}_\alpha$, $\alpha \in (1/2, 1)$, *there exists* $T = T(t_0, u_0) > t_0$, *such that the problem* $(9) - (12)$ *has a unique solution on* (t_0, T).

Consider the problem (9), (10) for the similar system

$$u_t(x,t) = u_{xx}(x,t) - v_{xx}(x,t) + f(t, u(x,t)), \quad (x,t) \in (a,b) \times (t_0, T), \quad (13)$$

$$v_{xx}(x,t) + \beta v(x,t) + \alpha u(x,t) + g(t, u(x,t)) = 0, \quad (x,t) \in (a,b) \times (t_0, T). \quad (14)$$

Here

$$N(t,u,v) = \begin{pmatrix} f(t,u) \\ g(t,u) \end{pmatrix}.$$

Theorem 6. *Let $-\beta \notin \sigma(A)$, functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable with respect to (t,u). Then for any $(t_0, u_0) \in \mathbb{R} \times \mathcal{H}_\alpha$, $\alpha \in (1/2, 1)$, there exists $T = T(t_0, u_0) > t_0$ such that the problem (9), (10), (13), (14) has a unique solution on (t_0, T).*