Sturm-Liouville Problem on Unbounded Interval (joint work with Alois Kufner)

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Functional setting

Let $p > 1$ be a real number and let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as $\varphi(s) = |s|^{p-2}s$ for $s \neq 0$, $\varphi(0) = 0$. Let $r = r(t)$, $c = c(t)$ be continuous and positive functions on $[0, \infty)$. For $x = x(t)$ defined on $[0, \infty)$ denote $x(\infty) := \lim_{t\to\infty} x(t)$. We study the eigenvalue problem

$$\begin{cases} (r(t)\varphi(x'(t)))' + \lambda c(t)\varphi(x(t)) = 0, \quad t \geq 0, \\ x'(0) = 0,\ x(\infty) = 0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter.

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where $\lambda \in \mathbb{R}$ is a spectral parameter.

A function $x \in W^{1,p}_{\infty}(r)$ (the weighted Sobolev space being defined later) is called a *weak solution* of (1) if the integral identity

$$\int_0^\infty r(t)\varphi(x'(t))y'(t)\mathrm{d}t = \lambda \int_0^\infty c(t)\varphi(x(t))y(t)\mathrm{d}t \tag{2}$$

holds for all $y \in W^{1,p}_{\infty}(r)$ (with both integrals being finite).

Eigenvalues, eigenfunctions

The parameter λ is called an *eigenvalue* of (1) if this problem has a nontrivial (i.e. nonzero) weak solution (called an *eigenfunction* of (1)).

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(S.L.) Property for (1):

"The set of all eigenvalues of (1) forms an increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_1 > 0$ and

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Every eigenvalue λ_n, $n = 1, 2, \ldots$, is simple in the sense that there exists a unique normalized eigenfunction x_{λ_n} associated with λ_n. Moreover, the eigenfunction x_{λ_n} has precisely $n-1$ zeros in $(0,\infty)$. In particular, x_{λ_1} does not change sign in $(0,\infty)$. For $n \geq 3$, between two consecutive zeros of $x_{\lambda_{n-1}}$ in $(0,\infty)$ there is exactly one zero of x_{λ_n}."

Condition on weights

Our main result depends on the following condition on the weight functions (coefficients in the equation) r and c:

$$\lim_{t\to\infty}\Big(\int_0^t c(\tau)\mathrm{d}\tau\Big)^{1/p}\Big(\int_t^\infty r^{1-p'}(\tau)\mathrm{d}\tau\Big)^{1/p'}=0. \qquad (3)$$

Example

Let $p = 2, r \equiv c \equiv 1$. Then evidently there is no (weak) solution of

$$x''(t) + \lambda x(t) = 0, \quad x'(0) = 0, x(\infty) = 0$$

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Note that in this case we have

$$\Big(\int_0^t c(\tau) \mathrm{d}\tau \Big)^{1/p} \Big(\int_t^\infty r^{1-p'}(\tau) \mathrm{d}\tau \Big)^{1/p'} \equiv \infty, \quad t \in (0, \infty).$$

Example

Let $p = 2$, $r(t) = (t+1)^2$, $c(t) \equiv 1$. Then we have

$$\Big(\int_0^t c(\tau)\mathrm{d}\tau\Big)^{1/p}\Big(\int_t^\infty r^{1-p'}(\tau)\mathrm{d}\tau\Big)^{1/p'}$$
$$= \Big(\int_0^t \mathrm{d}\tau\Big)^{1/2}\Big(\int_t^\infty \frac{\mathrm{d}\tau}{(1+\tau)^2}\Big)^{1/2} = \Big(\frac{t}{1+t}\Big)^{1/2} \to 1$$

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The boundary value problem

$$((t+1)^2 x'(t))' + \lambda x(t) = 0, \quad x'(0) = 0, x(\infty) = 0 \tag{4}$$

has no eigenvalue.

Remark

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The foregoing examples indicate that if the condition (3) is violated, then the (S.L.) Property for (1) need not be satisfied.

Weighted spaces

Let $L^p(c)$ denote the weighted Lebesgue space of all functions $x = x(t)$ defined on $(0, \infty)$, for which

$$\|x\|_{p;c} := \Big(\int_0^\infty c(t)|x(t)|^p \mathrm{d}t \Big)^{1/p}$$

is finite. Then $L^p(c)$ equipped with the norm $\| \cdot \|_{p;c}$ is a uniformly convex Banach space.

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is finite. Then $L^p(c)$ equipped with the norm $\| \cdot \|_{p;c}$ is a uniformly convex Banach space.

Let $W^{1,p}_\infty(r)$ be the set of all absolutely continuous functions $x = x(t)$ defined on $[0, \infty)$ such that $x(\infty) = 0$ and

$$\|x\|_{1,p;r} := \left(\int_0^\infty r(t)|x'(t)|^p \mathrm{d}t \right)^{1/p}$$

is finite. Then $W^{1,p}_\infty(r)$ equipped with the norm $\| \cdot \|_{1,p;r}$ is a uniformly convex Banach space.

Weak solution

A function $x \in W^{1,p}_{\infty}(r)$ is called a *weak solution* of (1) if the integral identity

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Let us note that if $x \in W^{1,p}_{\infty}(r)$ is a weak solution of (1), then $r\varphi(x') \in C^1[0,\infty)$, the equation in (1) is satisfied at every point and $x'(0) = x(\infty) = 0$.

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The reason for considering weak solutions consists in reformulating (1) as an abstract nonlinear eigenvalue problem and finding the eigenvalues in a "constructive way" employing the tools of nonlinear functional analysis.

Compact embedding

In order to use compactness arguments (associated with the Palais-Smale condition) we need the compact embedding:

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This compactness result holds if and only if condition (3) is satisfied. It follows e.g. from the results collected in the book

[1] B. Opic and A. Kufner
Hardy-Type Inequalities,
Pitman Research Notes in Mathematics Series 279, Longman Scientific and Technical, Harlow 1990.

Theorem

The condition (3) is satisfied if and only if the (S.L.) Property for (1) holds.

Tools

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Variational characterization of eigenvalues.

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Oscillatory criteria for half-linear equations.

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Properties of the initial-value problem for half-linear equation.

Principal eigenvalue

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Let us assume (3). Then (1) has the least eigenvalue $\lambda_1 > 0$ which can be characterized as follows:

$$\lambda_1 = \min \frac{\int_0^\infty r(t)|x'(t)|^p \mathrm{d}t}{\int_0^\infty c(t)|x(t)|^p \mathrm{d}t}, \tag{7}$$

where the miminum is taken over all $x \in W_\infty^{1,p}(r)$, $x \neq 0$.

Higher eigenvalues

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Let $\mathcal{S} := \{x \in W_\infty^{1,p}(r) : \|x\|_{p;c} = 1\}$. Let $\mathcal{S}^k$ be the unit sphere in $\mathbb{R}^k$. For $k \in \mathbb{N}$ define a family of sets $\mathcal{F}_k := \{\mathcal{A} \subset \mathcal{S} : \mathcal{A} = -\mathcal{A}, \mathcal{A} = h(\mathcal{S}^k)$ with h a continuous odd mapping from $\mathbb{R}^k$ into $\mathcal{S}\}$.

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Let us consider the functional

$$I(x) := \|x\|^p_{1,p;r}.$$

The compact embedding (5) allows to prove that for any $k \in \mathbb{N}$,

$$\lambda_k = \min_{\mathcal{A} \in \mathcal{F}_k} \max_{x \in \mathcal{A}} I(x) \tag{8}$$

are eigenvalues of (1). The interested listener is referred to

[2] P. Drábek and S.B. Robinson
Resonance problems for the p-Laplacian,
Journal of Functional Analysis 169 (1999), 189-200.

for the proof which can be literally adapted in our situation.

Proposition

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Let (3) be satisfied. The sequence $\{\lambda_k\}_{k=1}^{\infty}$ defined by (8) forms the sequence of eigenvalues of (1) and

$$\lim_{n\to\infty} \lambda_n = \infty.$$

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Warning!

We do not know that this is the entire set of **all** eigenvalues yet!!! It remains to show that the sequence $\{\lambda_k\}_{k=1}^{\infty}$ exhausts the set of all eigenvalues of (1).

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To prove this fact is made possible by using purely ODE techniques.

Existence and uniqueness

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Let $t_0 \in [0, \infty)$, $A, B \in \mathbb{R}$. Consider the initial value problem

$$\begin{cases} (r(t)\varphi(x'(t)))' + \lambda c(t)\varphi(x(t)) = 0, \quad t > t_0, \\ x(t_0) = A, \quad x'(t_0) = B. \end{cases} \tag{9}$$

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Existence, uniqueness, global extensibility of the solution up to ∞ follows from

[3] A. Elbert
A half-linear second order differential equation,
Colloq. Math. Soc. János Bolyai 30 (1979), 153-180.

Disconjugacy and comparison of zeros

Disconjugacy and comparison of zeros

In order to compare zeros of different eigenfunctions we use the disconjugacy criteria for half-linear equations:

[4] O. Došlý
Oscillation criteria for half-linear second order differential equation,
Hiroshima J. Math. 28 (1998), 507-521.

Oscillatory criteria

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Došlý [4]:

Let

$$\limsup_{t\to\infty}\Big(\int_t^\infty r^{1-p'}(\tau)\mathrm{d}\tau\Big)^{p-1}\Big(\int_0^t c(\tau)\mathrm{d}\tau\Big) < \frac{(p-1)^{p-1}}{\lambda p^p}.$$

Then the equation in (9) is non oscillatory.

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Then the equation in (9) is non oscillatory.

Let

$$\liminf_{t\to\infty}\Big(\int_t^\infty r^{1-p'}(\tau)\mathrm{d}\tau\Big)^{p-1}\Big(\int_0^t c(\tau)\mathrm{d}\tau\Big) > \frac{(p-1)^{p-1}}{\lambda p^p}.$$

Then the equation in (9) is oscillatory.

Corollary

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Let us assume (3). Then the equation in (9) is nonoscillatory for any $\lambda \in \mathbb{R}$. Let us assume that (3) does not hold, i.e., there is a sequence $t_n \to \infty$ (as $n \to \infty$) such that

$$\liminf_{t_n \to \infty} \Big(\int_0^{t_n} c(\tau)\mathrm{d}\tau \Big)^{1/p} \Big(\int_{t_n}^{\infty} r^{1-p'}(\tau)\mathrm{d}\tau \Big)^{1/p'} > 0.$$

Then there exists $\lambda_0 > 0$ such that the equation in (9) is oscillatory provided $\lambda \geq \lambda_0$.

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Idea of the proof:

Assume that (3) is violated. Then according to the assertion above there exists $\lambda_0 > 0$ such that the equation in (9) is oscillatory provided $\lambda \geq \lambda_0$. But then taking the eigenvalue $\lambda_n \geq \lambda_0$ the corresponding eigenfunction does not vanish on $[a, b]$ with a and b large enough. This would imply that (9) is disconjugate on $[a, b]$ which contradicts its oscillatory behavior.

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Idea of the proof:

With the sequence of "variational eigenvalues" $\{\lambda_n\}_{n=1}^{\infty}$ and corresponding eigenfunctions x_{λ_n} in hands we employ comparison, disconjugacy and oscillatory criteria and proceed to show that besides of x_{λ_n} there are no other eigenfunctions of (1). The sequence $\{\lambda_n\}_{n=1}^{\infty}$ thus exhausts the set of **all eigenvalues** of (1).

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