Maximum principles for operator equations

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Contribution was based on the enclosed paper.

Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 329 (2007) 238–267
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Maximum principle for functional equations in the space of discontinuous functions of three variables

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Received 24 February 2006
Available online 24 July 2006
Submitted by Steven G. Krantz

Abstract

The paper is devoted to the maximum principles for functional equations in the space of measurable essentially bounded functions. The necessary and sufficient conditions for validity of corresponding maximum principles are obtained in a form of theorems about functional inequalities similar to the classical theorems about differential inequalities of the Vallee Poussin type. Assertions about the strong maximum principle are proposed. All results are also true for difference equations, which can be considered as a particular case of functional equations. The problems of validity of the maximum principles are reduced to nonoscillation properties and disconjugacy of functional equations. Note that zeros and nonoscillation of a solution in a space of discontinuous functions are defined in this paper. It is demonstrated that nonoscillation properties of functional equations are connected with the spectral radius of a corresponding operator acting in the space of essentially bounded functions. Simple sufficient conditions of nonoscillation, disconjugacy and validity of the maximum principles are proposed. The known nonoscillation results for equation in space of functions of one variable follow as a particular cases of these assertions. It should be noted that corresponding coefficient tests obtained on this basis cannot be improved. Various applications to nonoscillation, disconjugacy and the maximum principles for partial differential equations are proposed.

Keywords: Maximum principles; Positivity; Nonoscillation; Disconjugacy; Spectral radius; Functional inequalities

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.06.023

1. Introduction

In this paper the maximum principles for the functional equation

$$u(t,x,y) = (Tu)(t,x,y) + f(t,x,y),$$
$$(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty), \tag{1.1}$$

are studied. Here $T : L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is a linear continuous operator, $L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is the space of measurable essentially bounded functions $u : [0,+\infty) \times [0,+\infty) \times [0,+\infty) \to (-\infty,+\infty)$, $f \in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is a given function. A function $u \in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ satisfying Eq. (1.1) is called a solution of this equation.

The maximum principles are the classical topics in the theory of partial differential equations [18]. Let P be a linear partial differential operation and a function u be smooth enough. Consider the boundary value problem $Pu = f$ in a corresponding open domain M and $u_{\partial M} = \varphi$ on the boundary. One of the formulation of the maximum principle is the following: there exists a positive constant N such that $\|u\| \leqslant N(\|f\| + \|\varphi\|)$. This is the correct solvability of the boundary value problem. In other terminology this means that the Green's operator G of this boundary value problem and the solution z of the equation $Pu = 0$, satisfying the condition $u_{\partial M} = \varphi$, are bounded. Another form of the maximal principle is the following (see, for example, [13]): from a partial differential inequality $Pu \geqslant 0$ in M and the inequality $u_{\partial M} \geqslant 0$ on its boundary the nonnegativity of a function u in M follows. We can say in this case that the solution u of the boundary value problem $Pu = f$ in M, $u_{\partial M} = 0$ does not have the negative extremum for $f \geqslant 0$. Assertions of this type are also called the comparison theorems. Such assertions are reduced to nonnegativity of the Green's operator G and the solution z. In this case from the inequalities $Pv \geqslant Pu \geqslant Pw$ in M and $v_{\partial M} \geqslant u_{\partial M} \geqslant w_{\partial M}$ on the boundary the inequality $v \geqslant u \geqslant w$ in M follows. This allows us to estimate the solution u.

The strong maximum principle can be formulated in the following form: if in a connected domain M the inequalities $u \geqslant 0$ and $Pu \geqslant 0$ are satisfied, then either $u \equiv 0$ or $u > 0$ in M. The maximum principles in spaces of functions without classical assumption about smoothness have not been studied enough. The strong maximum principle was discussed in the recent paper by H. Brezis and A.C. Ponce [2], where, for example, a class of quasicontinuous functions was considered. They recall that a function $v : M \to (-\infty,+\infty)$ is quasicontinuous if there exists a sequence of open subsets $\{\omega_n\}$ of M such that the function v on the domain $M \backslash \omega_n$ is continuous for $n \geqslant 1$ and $cap\, \omega_n \to 0$ as $n \to \infty$. If $u \in L^1(M)$, where M is an open bounded set, $u \geqslant 0$ and the Laplacian of u is a Radon measure on M, then there exists quasicontinuous v such that $u = v$ a.e. in M [2].

Essential difficulties in the study of the maximum principles in spaces of discontinuous functions appear as a result of corresponding definitions of partial derivatives in these spaces. That is why one of the natural approaches can be as follows: to represent the derivatives as corresponding differences and to study the functional equation (1.1) in the spaces of discontinuous functions. Note that Eq. (1.1) will be considered in the space of measurable essentially bounded functions, which is very similar to the class of the quasicontinuous functions.

In this paper analogs of the maximum principles are obtained for functional equation (1.1). It is clear that the key problems are existence of bounded inverse operator $(I - T)^{-1}$, its positivity and a corresponding nonoscillation of solutions of Eq. (1.1). The difficulties in studying nonoscillation follows from the fact that we consider equation in the space of discontinuous functions. Below we define zeros of functions and nonoscillation in the space $L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$. Several theorems of our paper establish equivalence of this nonoscillation, assertions about corresponding functional inequalities and the fact that the spectral radius of the operator T is less than one.

Now let us explain why Eq. (1.1) is a very natural object. Various mathematical models with partial functional and integro-differential equations were examined by many authors (see, for example, [9,24]). In this paper we consider as examples parabolic and elliptic partial differential equations (i.e. $A \geqslant 0$, $B \geqslant 0$, $C \geqslant 0$)

$$\begin{aligned} &A(t,x,y)u_t'(t,x,y) \\ &\quad = B(t,x,y)u_{xx}''(t,x,y) + C(t,x,y)u_{yy}''(t,x,y) \\ &\qquad + p(t,x,y)u(t,x,y) + \int\limits_{h_1(t)}^{h_2(t)} \int\limits_{h_3(x)}^{h_4(x)} \int\limits_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi \\ &\quad = f(t,x,y), \quad (t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty), \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &A(t,x,y)u_{tt}''(t,x,y) + B(t,x,y)u_{xx}''(t,x,y) + C(t,x,y)u_{yy}''(t,x,y) \\ &\qquad + D(t,x,y)u_t'(t,x,y) + E(t,x,y)u_x'(t,x,y) + F(t,x,y)u_y'(t,x,y) \\ &\qquad + p(t,x,y)u(t,x,y) + \int\limits_{h_1(t)}^{h_2(t)} \int\limits_{h_3(x)}^{h_4(x)} \int\limits_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi \\ &\quad = f(t,x,y), \quad (t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty), \end{aligned} \tag{1.3}$$

where

$$u(t,x,y) = \varphi(t,x,y), \quad \text{for } (t,x,y) \notin [0,+\infty) \times [0,+\infty) \times [0,+\infty). \tag{1.4}$$

Let us represent partial derivatives in these equations in the difference form, then the following integro-functional equations can be obtained respectively

$$\begin{aligned} &\big\{A(t,x,y) + 2B(t,x,y) + 2C(t,x,y) - p(t,x,y)h^2\big\}u(t,x,y) \\ &\quad = A(t,x,y)u(t-h,x,y) + B(t,x,y)u(t,x+h,y) + B(t,x,y)u(t,x-h,y) \\ &\qquad + C(t,x,y)u(t,x,y+h) + C(t,x,y)u(t,x,y-h) \\ &\qquad + h^2 \int\limits_{h_1(t)}^{h_2(t)} \int\limits_{h_3(x)}^{h_4(x)} \int\limits_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi + h^2 f(t,x,y), \\ &\quad (t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty), \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\big\{2A(t,x,y) + 2B(t,x,y) + 2C(t,x,y) + hD(t,x,y) + hE(t,x,y) + hF(t,x,y) \\ &\qquad - h^2 p(t,x,y)\big\}u(t,x,y) \\ &\quad = A(t,x,y)u(t+h,x,y) + A(t,x,y)u(t-h,x,y) \\ &\qquad + B(t,x,y)u(t,x+h,y) + B(t,x,y)u(t,x-h,y) \end{aligned}$$

$$+ C(t,x,y)u(t,x,y+h) + C(t,x,y)u(t,x,y-h)$$
$$+ hD(t,x,y)u(t+h,x,y) + hE(t,x,y)u(t,x+h,y) + hF(t,x,y)u(t,x,y+h)$$
$$+ h^2 \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi + h^2 f(t,x,y),$$
$$(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty). \tag{1.6}$$

It is clear that the representation of the derivatives in the partial differential equations in the difference form leads us actually to an analysis of the functional equations

$$u(t,x,y) = \sum_{i=1}^{m} p_i(t,x,y)u\big(g_i(t), d_i(x), r_i(y)\big)$$
$$+ \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi,$$
$$(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty), \tag{1.7}$$

and

$$u(t,x,y) = \sum_{i=1}^{m} p_i(t,x,y)u\big(g_i(t), d_i(x), r_i(y)\big)$$
$$+ \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi + f(t,x,y),$$
$$(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty),$$

where

$$u(t,x,y) = \varphi(t,x,y), \quad \text{for } (t,x,y) \notin [0,+\infty) \times [0,+\infty) \times [0,+\infty).$$

Let us assume that $p_i : [0,+\infty) \times [0,+\infty) \times [0,+\infty) \to (-\infty,+\infty)$ $(i = 1,\dots,m)$ and $k : [0,+\infty) \times [0,+\infty) \times [0,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty) \times (-\infty,+\infty) \to (-\infty,+\infty)$ are measurable essentially bounded functions, g_i, d_i and r_i $(i = 1,\dots,m)$ are continuous functions, h_1–h_6 are continuous functions such that the differences $t - h_j(t)$ $(j = 1,2)$, $x - h_j(x)$ $(j = 3,4)$, $y - h_j(y)$ $(j = 5,6)$ and $t - g_i(t)$, $x - d_i(x)$, $y - r_i(y)$ $(i = 1,\dots,m)$ are bounded on $[0,+\infty)$, f and φ are measurable essentially bounded functions.

We can also approximate the integrals in Eqs. (1.5) and (1.6) by corresponding sums and consider the difference equations

$$\big\{A(t,x,y) + 2B(t,x,y) + 2C(t,x,y) - p(t,x,y)h^2\big\}u(t,x,y)$$
$$= A(t,x,y)u(t-h,x,y) + B(t,x,y)u(t,x+h,y) + B(t,x,y)u(t,x-h,y)$$
$$+ C(t,x,y)u(t,x,y+h) + C(t,x,y)u(t,x,y-h)$$
$$+ h^2 \sum_{s=h_1(t)}^{h_2(t)} \sum_{\theta=h_3(x)}^{h_4(x)} \sum_{\xi=h_5(y)}^{h_6(y)} h^3 k(t,x,y,s,\theta,\xi)u(s,\theta,\xi) + h^2 f(t,x,y),$$
$$t,x,y \in \{0,h,2h,\dots\}, \tag{1.5d}$$

and

$$\begin{aligned}
&\big\{2A(t,x,y)+2B(t,x,y)+2C(t,x,y)+hD(t,x,y)+hE(t,x,y)+hF(t,x,y)\\
&\quad -h^2p(t,x,y)\big\}u(t,x,y)\\
&\quad = A(t,x,y)u(t+h,x,y)+A(t,x,y)u(t-h,x,y)\\
&\qquad + B(t,x,y)u(t,x+h,y)+B(t,x,y)u(t,x-h,y)\\
&\qquad + C(t,x,y)u(t,x,y+h)+C(t,x,y)u(t,x,y-h)\\
&\qquad + hD(t,x,y)u(t+h,x,y)+hE(t,x,y)u(t,x+h,y)+hF(t,x,y)u(t,x,y+h)\\
&\qquad + h^2\sum_{s=h_1(t)}^{h_2(t)}\sum_{\theta=h_3(x)}^{h_4(x)}\sum_{\xi=h_5(y)}^{h_6(y)} h^3k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)+h^2f(t,x,y),\\
&\quad t,x,y\in\{0,h,2h,\ldots\},
\end{aligned} \tag{1.6d}$$

in the space $L^\infty_{h[0,\infty)\times[0,\infty)\times[0,\infty)}$ of bounded functions of three variables determined for all nonnegative h integer-valued t,x,y. The norm in this space is defined by the formula $\|u\|_h = \sup|u(t,x,y)|$. Analogously we define $\|\varphi\|_h = \sup|\varphi(t,x,y)|$. In the last paragraph of the paper we obtain sufficient conditions of the following property A: for every sufficiently small positive constant h and bounded functions f and φ problem (1.5d), (1.4) (or (1.6d), (1.4)) has a unique solution $u \in L^\infty_{h[0,\infty)\times[0,\infty)\times[0,\infty)}$ and there exists a positive constant N such that $\|u\|_h \leqslant N(\|f\|_h + \|\varphi\|_h)$, where N does not depend on h.

Equations (1.5d) and (1.6d) can be also regarded as corresponding difference schemes (see, for example, [20, Chapter III, Section 1.3]). In terms of the theory of the difference schemes validity of the property A means that corresponding difference schemes are stable in $L^\infty_{h[0,\infty)\times[0,\infty)\times[0,\infty)}$ (see, for example, [10, Chapter 5, Section 12]). It is known that stable difference schemes are especially valuable for applications (see, for example, [14, Chapter III, Section 13.2] and [19, Chapter III, Section 5]). Results about the property A for difference equation in the space of bounded subsequences of one variable and stability of a corresponding three point-difference scheme are proposed, for example, in the recent paper [5].

For the operator on the right-hand side of Eq. (1.7) to act on the space of measurable essentially bounded functions, we assume (see [7]) that for each one-dimensional numerical set M_1, the equalities $mes\, g_i^{-1}(M_1) = 0$ for $i = 1,\ldots,m$ follow from the equality $mes\, M_1 = 0$, for each one-dimensional numerical set M_2, the equalities $mes\, d_i^{-1}(M_2) = 0$ for $i = 1,\ldots,m$ follow from the equality $mes\, M_2 = 0$, and for each one-dimensional numerical set M_3, the equalities $mes\, r_i^{-1}(M_3) = 0$ for $i = 1,\ldots,m$ follow from the equality $mes\, M_3 = 0$. The necessary condition of the action of the operator $T: L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$, which is practically very close to a sufficient condition, is the following: there are no intervals $[\nu_1,\mu_1]$ such that at least one of the functions $g_i(t)$, $d_i(x)$ or $r_i(y)$ is equal to a constant.

Considering solutions in the space $L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$, we should define zeros of discontinuous functions.

Definition 1.1. The point (ν_1,ν_2,ν_3) is a zero of a function $u(t,x,y)$ if the following limit

$$\lim_{\varepsilon\to 0+}\Big\{\operatorname*{ess\,sup}_{(t,x,y)\in[\nu_1-\varepsilon,\nu_1+\varepsilon]\times[\nu_2-\varepsilon,\nu_2+\varepsilon]\times[\nu_3-\varepsilon,\nu_3+\varepsilon]} u(t,x,y) \times \operatorname*{ess\,inf}_{(t,x,y)\in[\nu_1-\varepsilon,\nu_1+\varepsilon]\times[\nu_2-\varepsilon,\nu_2+\varepsilon]\times[\nu_3-\varepsilon,\nu_3+\varepsilon]} u(t,x,y)\Big\}, \tag{1.8}$$

is nonpositive.

Obviously, if u is a continuous function, then the point (ν_1, ν_2, ν_3) is its zero, if and only if $u(\nu_1, \nu_2, \nu_3) = 0$. Zeros determined by this definition include those standard zeros, as well as points of the sign change. Note, for example, that the function $u(t, x, y) = \sin \frac{1}{x+t+y}$ has a zero at each point of the planes $x + t + y = \frac{1}{\pi n}$, $n = \pm 1, \pm 2, \pm 3, \ldots,$ and also (according to Definition 1.1) at each point of the plane $x + t + y = 0$.

Zeros of solutions in the space of discontinuous functions of one variable were first defined in a recent paper [6].

If a function $u(t, x, y)$ is considered for $(t, x, y) \in [0, +\infty) \times [0, +\infty) \times [0, +\infty)$, we define also the following generalized zeros.

Definition 1.2. We say that (∞, ν_2, ν_3) is a generalized zero of a function $u(t, x, y)$ if the following limit

$$\lim_{\nu_1 \to +\infty} \lim_{\varepsilon \to 0+} \left\{ \operatorname*{ess\,sup}_{(t,x,y) \in [\nu_1-\varepsilon, \nu_1+\varepsilon] \times [\nu_2-\varepsilon, \nu_2+\varepsilon] \times [\nu_3-\varepsilon, \nu_3+\varepsilon]} \left| u(t, x, y) \right| \right\}$$

is equal to zero.

Note that (∞, ν_2, ν_3) is a generalized zero if a function $u(t, x, y)$ tends to zero when $t \to +\infty$.

Definition 1.3. A function $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is nonoscillating with respect to t in the zone $(t, x, y) \in [0, +\infty) \times [\nu_2, \mu_2] \times [\nu_3, \mu_3]$ if there exists ν such that $u(t, x, y)$ does not have zeros for $(t, x, y) \in (\nu, \mu) \times [\nu_2, \mu_2] \times [\nu_3, \mu_3]$ for each $\mu > \nu$. If there exists a sequence $(t_n, x_n, y_n) \in (\nu, +\infty) \times [\nu_2, \mu_2] \times [\nu_3, \mu_3]$ of zeros of the function $u(t, x, y)$ such that $t_n \to +\infty$, we can say that the function $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ oscillates with respect to t in the zone $(t, x, y) \in [0, +\infty) \times [\nu_2, \mu_2] \times [\nu_3, \mu_3]$.

Definition 1.4. A function $u \in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ ($u \in L^\infty_{[\nu_1,\infty)\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$) is strongly positive if $u(t, x, y) > 0$ for almost all $(t, x, y) \in [\nu_1, \mu_1] \times [\nu_2, \mu_2] \times [\nu_3, \mu_3]$ ($(t, x, y) \in [\nu_1, +\infty) \times [\nu_2, \mu_2] \times [\nu_3, \mu_3]$), and this function u does not have zeros (zeros as well as generalized zeros) there.

Note that the function $|\sin t|$ is positive for $(t, x, y) \in [0, +\infty) \times [0, +\infty) \times [0, +\infty)$, but not strongly positive because of zeros at the points of the planes $t = \pi n$ $(n = 0, 1, 2, \ldots)$. The function $u(t, x, y)$ is strongly positive if and only if it is an internal element of a cone of positive essentially bounded functions (see [16]).

Now consider the following difference equation (more exactly to call this the recurrence relation)

$$u_{t,x,y} = \sum_{k=-m}^{m} \sum_{j=-m}^{m} \sum_{i=-m}^{m} p_{t,x,y} u_{t-k,x-j,y-i}, \quad t, x, y \in \{1, 2, 3, \ldots\}, \tag{1.9}$$

$$u_{t,x,y} = \varphi_{t,x,y}, \quad \text{if } t < 0 \text{ or } x < 0 \text{ or } y < 0, \tag{1.10}$$

and the functional equation

$$u(t, x, y) = \sum_{i=-m}^{m} \sum_{j=-m}^{m} \sum_{i=-m}^{m} p_{ijk}(t, x, y) u\big(g_i(t), d_j(x), r_k(y)\big),$$
$$(t, x, y) \in [0, +\infty) \times [0, +\infty) \times [0, +\infty), \tag{1.11}$$

with initial function (1.4).

If we set in Eq. (1.11) $g_i(t) = t - i$, $d_j(x) = x - j$, $r_k(y) = y - k$ where i, j and k are integers, and the coefficients $p_{ijk}(t, x, y)$ and the initial function $\varphi(t, x, y)$ are constants in each parallelepiped $[i, i+1) \times [j, j+1) \times [k, k+1)$, we obtain that the corresponding solutions $u(t, x, y)$ are constants ($u(t, x, y) = u_{ijk}$) in each such parallelepiped. In this case, the difference equation (1.9) can be actually considered as a particular case of functional equation (1.11). Obviously, each assertion obtained for functional equation (1.11) is also true for difference equation (1.9). Our approach allows us also to draw conclusions about the behavior of solutions of functional equation (1.11) using the corresponding properties of difference equation (1.9). It can be noted that the theory of difference equations was intensively developed during the last two decades (see, for example, [1,11,15,17]).

Oscillation properties of partial difference equations were studied, for example, in [3,4,22,23, 25]. Oscillation and nonoscillation of functional equations in spaces of functions of one variable were considered, for example, in [6,8,21].

Let us denote by $L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$, where $0 \leqslant \nu_1 < \mu_1$, $0 \leqslant \nu_2 < \mu_2$, $0 \leqslant \nu_3 < \mu_3$, the space of measurable essentially bounded functions $u : [\nu_1, \mu_1] \times [\nu_2, \mu_2] \times [\nu_3, \mu_3] \to (-\infty, +\infty)$. In order to study oscillatory properties of Eq. (1.1), we examine the following auxiliary equation:

$$
\begin{aligned}
&z(t, x, y) = (T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} z)(t, x, y) + f_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}(t, x, y),\\
&\quad (t, x, y) \in [\nu_1, \mu_1] \times [\nu_2, \mu_2] \times [\nu_3, \mu_3].
\end{aligned}
\tag{1.12}
$$

The operator $T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ is determined by the equality

$$
T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} = \xi_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} T \xi^{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3},
\tag{1.13}
$$

where the operator $\xi_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ narrows down the function $u(t, x, y)$ from $[0, +\infty) \times [0, +\infty) \times [0, +\infty)$ to $[\nu_1, \mu_1] \times [\nu_2, \mu_2] \times [\nu_3, \mu_3]$, while the operator $\xi^{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ expands the function $z(t, x, y)$ as follows

$$
\begin{aligned}
&\xi^{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} z(t, x, y)\\
&\quad = \begin{cases} z(t, x, y), & \nu_1 \leqslant t \leqslant \mu_1,\ \nu_2 \leqslant x \leqslant \mu_2,\ \nu_3 \leqslant y \leqslant \mu_3,\\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
\tag{1.14}
$$

and

$$
\begin{aligned}
&f_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}(t, x, y)\\
&\quad = T\big\{\big(1 - \sigma_{\nu_1,\mu_1}(t)\big)\big(1 - \sigma_{\nu_2,\mu_2}(x)\big)\big(1 - \sigma_{\nu_3,\mu_3}(y)\big)u\big\}(t, x, y) + f(t, x, y),\\
&\quad (t, x, y) \in [\nu_1, \mu_1] \times [\nu_2, \mu_2] \times [\nu_3, \mu_3],
\end{aligned}
\tag{1.15}
$$

where $u \in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is a solution of Eq. (1.1) and

$$
\sigma_{\nu,\mu}(t) = \begin{cases} 1, & \nu \leqslant t \leqslant \mu,\\ 0, & \text{otherwise.} \end{cases}
$$

Note that for Eq. (1.7) this operator

$$
T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]},
$$

is determined by the equality

$$(T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}z)(t,x,y) = \sum_{i=1}^{m} p_i(t,x)z\big(g_i(t),d_i(x),r_i(y)\big) + \int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)z(s,\theta,\xi)\,ds\,d\theta\,d\xi,$$
$$(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \tag{1.16}$$

where

$$z(t,x,y)=0 \quad \text{for } (t,x,y)\notin[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3].$$

Lemma 1.1. *Let a function $u\in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ be a solution of Eq.* (1.1). *Then the function $z\in L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ satisfying the equality*

$$z(t,x,y)=u(t,x,y), \qquad (t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \tag{1.17}$$

is a solution of Eq. (1.12).

Definition 1.5. The parallelepiped $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ is called a zone of disconjugacy if for each strongly positive f the equation

$$z(t,x,y)=(T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}z)(t,x,y)+f(t,x,y),$$
$$(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \tag{1.18}$$

has a unique solution z and this solution is strongly positive for $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$.

Essentially different definition of disconjugacy of difference equations was studied in the known paper by P. Hartman [12].

The problems of oscillation are reduced to estimates of the spectral radius of the operator $T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$. Note that the result of the action of the operator

$$T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$$

determined by equality (1.13) on continuous functions can be, in general, a discontinuous function. For example, consider the action of the operator $T_{\nu_1,\mu_1;0,\infty;0,\infty}$, where $\nu_1+1<\mu_1$, of the following simple form $(T_{\nu_1,\mu_1;0,\infty;0,\infty}z)(t,x,y)=z(t-1,x,y)$ on the function $z(t,x,y)=1$. As a result we obtain the discontinuous function

$$(T_{\nu_1,\mu_1;0,\infty;0,\infty}1)(t,x,y)=\begin{cases}0, & \nu_1\leqslant t\leqslant\nu_1+1,\ 0\leqslant x,y<\infty,\\ 1, & \nu_1+1<t,\ 0\leqslant x,y<\infty.\end{cases}$$

This is a reason to consider Eq. (1.1) in the space of discontinuous functions.

2. Comparison theorems and maximum principle

In this paragraph conditions of positivity of solutions and corresponding maximum principles for Eq. (1.1) will be obtained.

Theorem 2.1. *Let T be a positive operator, then the following assertions are equivalent*:

(1) *The spectral radius r of the operator $T : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is less than one.*
(2) *For each $f \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ there exists a unique solution $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq.* (1.1) *and this solution u is nonnegative* (*strongly positive*) *for f nonnegative* (*strongly positive*).
(3) *There exists a strongly positive function $v \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ such that the function $\varphi(t,x) \equiv v(t,x,y) - (Tv)(t,x,y)$ is strongly positive for $(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$.*

Remark 2.1. Assertion (2) is correct solvability of Eq. (1.1) and can be interpreted as a possible formulation of the maximum principle for functional equations.

Proof. We prove Theorem 2.1 using the following scheme: (1) → (2) → (3) → (1).

(1) → (2). If the spectral radius r of the operator $T : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is less than one, then a unique solution $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq. (1.1) can be written as $u(t,x,y) = \{(I-T)^{-1}f\}(t,x,y) = \{(I+T+T^2+T^3+\cdots)f\}(t,x,y)$. The positivity of the operator T implies that $u(t,x,y) \geqslant f(t,x,y) > 0$ for $(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$.

(2) → (3). Let us set $f(t,x,y) = 1$ for $(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$ in Eq. (1.1). According to the assertion (2), the solution v of the equation $u(t,x,y) = (Tu)(t,x,y) + 1$ is nonnegative. In this case it satisfies the inequality $v(t,x,y) \geqslant 1$, i.e. the function v is strongly positive. The function $\varphi(t,x,y) \equiv v(t,x,y) - (Tv)(t,x,y) = 1 > 0$.

The implication (3) → (1) is well known (see [16, Theorem 5.6, p. 86]). □

Example 2.1. Let us consider the equation

$$u(t,x,y) = 2u(t+1,x,y), \quad (t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty),$$

possessing the oscillating solution

$$u(t,x,y) = \begin{cases} 1/2^{m-1}, & m-1 \leqslant t \leqslant m - \frac{1}{2}, \\ -1/2^{m-1}, & m - \frac{1}{2} < t < m, \end{cases} \qquad m = 1,2,3,\ldots.$$

It may seem that the function $v(t,x,y) = \frac{1}{3^{[t]}}$, where $[t]$ is the integer part of t, satisfies the condition (3) of Theorem 2.1 and consequently the spectral radius of the operator $T : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ has to be less than one. Existence of nontrivial solution of the homogeneous equation contradicts to this fact.

Actually the function $v(t,x,y)$ is strongly positive and satisfies the inequality $\varphi(t,x,y) \equiv v(t,x,y) - (Tv)(t,x,y) > 0$, where $\varphi(t,x,y)$ is strongly positive in each parallelepiped $(t,x,y) \in [0,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]$, but, according to Definition 1.3, these functions are not strongly positive on $[0,+\infty) \times [0,+\infty) \times [0,+\infty)$ and consequently the function v does not satisfy the condition (3). Thus this example demonstrates that conditions of the strong positivity of the functions v and φ in the condition (3) of Theorem 2.1 are essential.

If we set $v = 1$ in the assertion (3) of Theorem 2.1 then for Eq. (1.7) the following result is obtained.

Theorem 2.2. *Let $p_i(t,x,y) \geqslant 0$ $(i = 1,\dots,m)$, $k(t,x,y,s,\theta,\xi) \geqslant 0$, and the following inequality be satisfied*

$$\operatorname*{ess\,sup}_{t,x,y\in[0,+\infty)} \left\{ \sum_{i=1}^{m} p_i(t,x,y)\sigma_{0,\infty}\big(g_i(t)\big)\sigma_{0,\infty}\big(d_i(x)\big)\sigma_{0,\infty}\big(r_i(y)\big) + \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\sigma_{0,\infty}(s)\sigma_{0,\infty}(\theta)\sigma_{0,\infty}(\xi)\,d\theta\,ds\,d\xi \right\} < 1, \tag{2.1}$$

then there exists a unique solution $u \in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of the equation

$$u(t,x,y) = \sum_{i=1}^{m} p_i(t,x,y)u\big(g_i(t),d_i(x),r_i(y)\big) + \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi + f(t,x,y),$$
$$(t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{2.2}$$
$$u(t,x,y) = \varphi(t,x,y), \quad \textit{for } (t,x,y) \notin [0,+\infty)\times[0,+\infty)\times[0,+\infty),$$

for each couple of essentially bounded functions f, φ and this solution is nonnegative if f and φ are nonnegative for $(t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty)$.

Remark 2.2. The condition of positivity of the operator T in Theorem 2.1 is essential for nonnegativity of solutions. Consider the equations with negative operators T:

$$u(t,x,y) = pu(t-1,x,y), \quad (t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty),$$

where p is negative, and

$$u(t,x,y) = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} k(t,s)u(s,\theta,\xi)\,ds, \quad (t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty),$$

where the kernel $k(t,s)$ is negative and in both cases

$$u(t,x,y) = \varphi(t,x,y), \quad \text{when } (t,x,y) \notin [0,+\infty)\times[0,+\infty)\times[0,+\infty).$$

Each nontrivial solution $u(t,x,y)$ of these equations changes its sign in the zones of the size greater than one with respect to t.

Example 2.2. Let us consider the equation

$$u(t,x,y) = \frac{1}{2}u(t+1,x,y), \quad (t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty).$$

The spectral radius of the operator $(Tu)(t,x,y) = \frac{1}{2}u(t+1,x,y)$ is less than one according to Theorem 2.2. The function z defined as

$$z(t,x,y) = \begin{cases} 2^{m-1}, & m-1 \leqslant t \leqslant m-\frac{1}{2}, \\ -2^{m-1}, & m-\frac{1}{2} < t \leqslant m, \end{cases} \qquad m = 1,2,3,\dots,$$

satisfies the equation. It seems that this example contradicts the equivalence of the assertions (1) and (2) in Theorem 2.1, but this is not true since assertion (2) claims nonnegativity of solutions only from the space of *bounded* functions $L^{\infty}_{[0,\infty)\times\{0,\infty)\times[0,\infty)}$, and the function z is unbounded on $[0,+\infty)\times[0,+\infty)\times[0,+\infty)$. The unique solution in the space $L^{\infty}_{[0,\infty)\times\{0,\infty)\times[0,\infty)}$ is $u(t,x,y)\equiv 0$ for $[0,+\infty)\times[0,+\infty)\times[0,+\infty)$.

Let us now formulate the assertion about comparison of solutions of Eq. (1) with two ordered right-hand sides.

Corollary 2.1. *If the operator* $T: L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ *be positive, its spectral radius* r *is less than one and the function* $\psi \in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ *satisfies the inequality* $\psi(t,x,y) \geqslant f(t,x,y)$ *for* $(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty)$*, then the solution* $u\in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ *of Eq.* (1.1) *and the solution* $v\in L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ *of the equation*

$$\begin{aligned} &v(t,x,y) = (Tv)(t,x,y) + \psi(t,x,y), \\ &\quad (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{2.3}$$

satisfy the inequality $u(t,x,y) \leqslant v(t,x,y)$ *for* $(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty)$.

In order to prove this assertion let us note that the difference $z = v - u$ satisfies the equation

$$\begin{aligned} &z(t,x,y) = Tz(t,x,y) + \psi(t,x,y) - f(t,x,y), \\ &\quad (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty). \end{aligned}$$

Due to the implication (1) → (2) of Theorem 2.1 we obtain the inequality $v - u \geqslant 0$.

Let us consider now Eq. (1.1) without assumption about positivity of the operator T. Assume that the operator T can be represented as the difference of two positive operators T^+ and T^-. The most interesting case for us is the operator $T: L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of the form

$$\begin{aligned} (Tu)(t,x,y) = {} & \sum_{i=1}^{m} p_i(t,x,y)\, u\big(g_i(t), d_i(x), r_i(y)\big) \\ & + \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\, u(s,\theta,\xi)\, ds\, d\theta\, d\xi, \\ & (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{2.4}$$

$$u(t,x,y) = 0, \quad \text{for } (t,x,y)\notin[0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{2.5}$$

the operators T^+ and T^- can be written as follows

$$\begin{aligned} \big(T^+u\big)(t,x,y) = {} & \sum_{i=1}^{m} p_i^+(t,x,y)\, u\big(g_i(t), d_i(x), r_i(y)\big) \\ & + \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k^+(t,x,y,s,\theta,\xi)\, u(s,\theta,\xi)\, ds\, d\theta\, d\xi, \\ & (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{2.6}$$

and

$$\big(T^-u\big)(t,x,y)=\sum_{i=1}^{m}p_i^-(t,x,y)u\big(g_i(t),d_i(x),r_i(y)\big)$$
$$+\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)}k^-(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi,$$
$$(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{2.7}$$

where

$$u(t,x,y)=0,\quad \text{for } (t,x,y)\notin[0,+\infty)\times[0,+\infty)\times[0,+\infty),$$

$p^+(t,x,y)=\max\{p(t,x,y),0\}$, $p^-(t,x,y)=\max\{-p(t,x,y),0\}$, and $k^+(t,x,y,s,\theta,\xi)=\max\{k(t,x,y,s,\theta,\xi),0\}$, $k^-(t,x,y,s,\theta,\xi)=\max\{-k(t,x,y,s,\theta,\xi),0\}$.

Define the operator $|T|$ as $|T|=T^++T^-$.

Theorem 2.3. *Let the spectral radius r of the operator $|T|:L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}\to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ be less than one. Then for each $f\in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ there exists a unique solution $u\in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq.* (1.1) *and it satisfies the inequality $|u(t,x,y)|\leqslant v(t,x,y)$, where the function $v(t,x,y)$ is such that $v(t,x,y)-(|T|v)(t,x,y)\geqslant|f(t,x,y)|$.*

Proof. If the spectral radius of the operator $|T|$ is less than one, then, according to Theorem 5.3 of monograph [16] (see p. 79), the spectral radius of the operator T is also less then one. A unique solution u of Eq. (1.1) in this case has the representation

$$u(t,x,y)=\big\{(I-T)^{-1}f\big\}(t,x,y)=\big\{\big(I+T+T^2+T^3+\cdots\big)f\big\}(t,x,y). \tag{2.8}$$

According to Theorem 2.1 there exist a unique solution v of the equation

$$v(t,x,y)=\big\{|T|v\big\}(t,x,y)+f(t,x,y),$$

and its representation is as follows

$$v(t,x,y)=\big\{\big(I-|T|\big)^{-1}f\big\}(t,x,y)=\big\{\big(I+|T|+|T|^2+|T|^3+\cdots\big)f\big\}(t,x,y). \tag{2.9}$$

Comparison of formulas (2.8) and (2.9) leads us to the conclusion that

$$\big|u(t,x,y)\big|\leqslant v(t,x,y). \qquad\Box$$

Remark 2.3. The condition on the spectral radius of the operator $|T|:L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}\to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ cannot be replaced by the condition that the spectral radii of the operators T^+ and T^- are less than one as the following example demonstrates. Consider the equation

$$u(t,x,y)=\frac{1}{2}u(t+1,x,y)-\frac{1}{2}u(t,x,y)+f(t,x,y). \tag{2.10}$$

In this case the operators T^+ and $T^-:L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}\to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ are defined by the formulas $(T^+u)(t,x,y)=\frac{1}{2}u(t+1,x,y)$ and $(T^-u)(t,x,y)=\frac{1}{2}u(t,x,y)$ respectively. If we set $v=1$ in assertion (3) of Theorem 2.1, we obtain that the spectral radii of the operators T^+ and T^- are less than one according to assertion (1) of this theorem. The operator $|T|:L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}\to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ in this case is $\{|T|u\}(t,x,y)=\frac{1}{2}u(t+1,x,y)+$

$\frac{1}{2}u(t,x,y)$, and its spectral radius is equal to one. The equation $v(t,x,y) - (|T|v)(t,x,y) = |f(t,x,y)|$ for the function $v(t,x,y)$ is the following

$$v(t,x,y) = \frac{1}{2}v(t+1,x,y) + \frac{1}{2}v(t,x,y) + \left|f(t,x,y)\right|. \tag{2.11}$$

Let us set $f(t,x,y)=0$ in Eqs. (2.10) and (2.11). Then the functions $u(t,x,y)=0$ and

$$v(t,x,y) = \begin{cases} 1, & m \leqslant t \leqslant m+\frac{1}{2}, \\ -1, & m+\frac{1}{2} < t < m+1, \end{cases} \qquad m = 0,1,2,\ldots,$$

are solutions of Eqs. (2.10) and (2.11) respectively, and the inequality $|u(t,x,y)| \leqslant v(t,x,y)$ is not fulfilled.

The following result proposes the correct solvability of Eq. (1.1). Denote $\|f\| = \operatorname{ess\,sup}_{(t,x,y)\in[0,\infty)\times[0,\infty)\times[0,\infty)} |f(t,x,y)|$.

Corollary 2.2. *Let the norm $\||T|\|$ of the operator $|T| : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ be less than one. Then for each $f \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ there exists a unique solution $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq.* (1.1) *and it satisfies the inequality $|u(t,x,y)| \leqslant \frac{1}{1-\||T|\|}\|f\|$.*

Note that in case of the negative operator T assertion about positivity of the solution u can be also obtained.

Theorem 2.4. *Let the operator $T : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ be negative and the spectral radius r of the operator $-T$ be less than one. Then for each $f \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ there exists a unique solution $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq.* (1.1) *and this solution is nonnegative if $f + Tf \geqslant 0$.*

The proof follows from the formula $u = (I-T)^{-1}f = f + Tf + T^2f + T^3f + T^4f + T^5f + \cdots = f + Tf + T^2(f+Tf) + T^4(f+Tf) + \cdots$.

Corollary 2.3. *If $\|T\| < 1$ and f is a positive constant, then there exists a unique solution $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq.* (1.1) *and this solution is strongly positive.*

The proof follows from Theorem 2.4 and the fact $1 > T1$.

Example 2.3. Consider the equation

$$u(t,x,y) = -\int_{t-1}^{t+1} k(t,s)u(s,x,y)\,ds + \varepsilon, \qquad (t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{2.12}$$

where

$$u(t,x,y) = 0, \quad \text{for } (t,x,y) \notin [0,+\infty)\times[0,+\infty)\times[0,+\infty),$$

ε is a positive constant and

$$0 \leqslant k(t,s) \leqslant e^{-\alpha(t-s)}, \quad (t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty). \tag{2.13}$$

If $\alpha > \frac{e^2-1}{e}$, then $\|T\| < 1$ and the unique solution u of Eq. (2.12) satisfies the inequality $u(t,x,y) \geqslant \frac{e\alpha - e^2+1}{e\alpha}\varepsilon > 0$.

Example 2.4. Consider now the equation

$$u(t,x,y) = -\frac{1}{2}u\big(t-g(t),x,y\big), \quad (t,x,y) \in [0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{2.14}$$

where

$$u(t,x,y) = \varphi(t,x,y), \quad \text{for } (t,x,y) \notin [0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{2.15}$$
$$g(t) = \begin{cases} \pi, & t\in[0,\pi), \\ \sin t, & t \in [\pi,+\infty), \end{cases}$$

and φ is strongly negative. It seems that the solution of this equation has to change sign in each zone of size greater than one with respect to t, but that is wrong. Actually Eq. (2.14) with the initial function (2.15) can be rewritten in the following equivalent form

$$\begin{aligned} &u(t,x,y) = -\frac{1}{2}u\big(t-g(t),x,y\big) + f(t,x,y), \\ &\quad (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{2.16}$$

where

$$u(t,x,y) = 0, \quad \text{for } (t,x,y) \notin [0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{2.17}$$

and

$$f(t,x,y) = \begin{cases} -\frac{1}{2}\varphi(t-g(t),x,y), & (t,x,y)\in[0,\pi)\times[0,+\infty)\times[0,+\infty), \\ 0, & (t,x,y)\in[\pi,+\infty)\times[0,+\infty)\times[0,+\infty). \end{cases}$$

All conditions of Theorem 2.4 are fulfilled for Eq. (2.16) with the initial function (2.17), including the fact that the function f satisfies the inequality $f + Tf \geqslant 0$. According to this theorem, the unique solution $u(t,x,y)$ will be nonnegative for $(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty)$.

There is no contradiction because $u(t,x,y) = f(t,x,y) > 0$ for $(t,x,y)\in[0,\pi)\times[0,+\infty)\times[0,+\infty)$ and $u(t,x,y)=0$ for $(t,x,y)\in[\pi,+\infty)\times[0,+\infty)\times[0,+\infty)$.

3. Nonoscillation and strong maximum principle in the unbounded zone $[0,+\infty)\times[0,+\infty)\times[0,+\infty)$

In the previous paragraph conditions of nonoscillation were obtained under the assumption that the right-hand side f is strongly positive. In this paragraph we consider more interesting cases when, for example, $f(t,x,y)$ is strongly positive only for $(t,x,y)\in[0,\varepsilon_0]\times[0,+\infty)\times[0,+\infty)$ and $f(t,x,y)=0$ for $(t,x,y)\in(\varepsilon_0,+\infty)\times[0,+\infty)\times[0,+\infty)$. We will demonstrate that without an additional assumption the strong maximum principle is not valid for Eq. (1.1) with the positive operator T.

In order to study nonoscillation with respect to increasing t we write the following representation of the operator T for each fixed ν:

$$\begin{aligned} &(Tu)(t,x,y) = \big(T^t_{0,\nu;0,\infty;0,\infty}u\big)(t,x,y) + \big(T^t_{\nu,\infty;0,\infty;0,\infty}u\big)(t,x,y), \\ &\quad (t,x,y)\in[\nu,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{3.1}$$

where the operators

$$T^t_{0,\nu;0,\infty;0,\infty} : L^\infty_{[0,\nu]\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$$

and

$$T^t_{\nu,\infty;0,\infty;0,\infty} : L^\infty_{[\nu,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$$

are determined as follows:

$$T^t_{0,\nu;0,\infty;0,\infty} = T\xi^{0,\nu,0,\infty;0,\infty}, \qquad T^t_{\nu,\infty;0,\infty;0,\infty} = T\xi^{\nu,\infty;0,\infty;0,\infty}, \tag{3.2}$$

the operator $\xi^{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ is determined by the formula (1.14).

For positive operator T we denote $\varepsilon_0 = \inf\{\nu\colon\ T^t_{0,\nu;0,\infty;0,\infty} 1 \neq 0$ for $(t,x,y) \in [\nu,\infty) \times [0,+\infty) \times [0,+\infty)\}$.

Definition 3.1. The positive operator T is called strongly positive with respect to increasing t if for all functions $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$, each of them is strongly positive for $(t,x,y) \in [0,\nu) \times [0,+\infty) \times [0,+\infty)$, there exists a positive number ε_1 such that for all numbers $\nu > \varepsilon_0$ the functions $(T^t_{0,\nu;0,\infty;0,\infty} u)(t,x,y)$ is strongly positive for $(t,x,y) \in [\nu,\nu+\varepsilon_1) \times [0,+\infty) \times [0,+\infty)$.

Analogously we can define strongly positive operators with respect to decreasing t, increasing and decreasing x and y.

Remark 3.1. Note that each advanced with respect to t operator T, for example, the operator $(Tu)(t,x,y) = \frac{1}{2}u(t+1,x,y)$, leads to the zero operators $T^t_{0,\nu;0,\infty;0,\infty}$ and is not strongly positive with respect to increasing t. The delay operator $(Tu)(t,x,y) = u(t-\sin^2 t,x,y)$ is not strongly positive since $(T^t_{0,\nu;0,\infty;0,\infty} u)(t,x,y) = 0$ for $\nu = \pi, 2\pi, 3\pi, \ldots$.

Remark 3.2. The operator $(Tu)(t,x,y) = u(t-1,x,y)$ is strongly positive with respect to increasing t with $\varepsilon_0 = \varepsilon_1 = 1$.

Remark 3.3. In the right-hand side of Eq. (1.7) we see the linear operator

$$T : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$$

defined by the formula

$$\begin{aligned} Tu(t,x,y) &= \sum_{i=1}^m p_i(t,x,y)u\big(g_i(t),d_i(x),r_i(y)\big) \\ &\quad + \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi, \\ &\quad (t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty), \\ u(t,x,y) &= 0, \quad \text{for } (t,x,y) \notin [0,+\infty) \times [0,+\infty) \times [0,+\infty), \end{aligned} \tag{3.3}$$

where $p_i(t,x,y) \geqslant 0$, $i = 1,\ldots,m$, $k(t,x,y,s,\theta,\xi) \geqslant 0$. In this case

$$\varepsilon_0 = \inf\Biggl\{\nu\colon \sum_{i=1}^{m} p_i(t,x,y)\sigma_{0,\infty}\bigl(g_i(t)\bigr) + \int\limits_{\max\{0,h_1(t)\}}^{\max\{0,h_2(t)\}} \int\limits_{h_3(x)}^{h_4(x)} \int\limits_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\,d\theta\,ds\,d\xi > 0,$$
$$(t,x,y) \in [\nu,\infty) \times [0,\infty) \times [0,\infty)\Biggr\},$$

$$\varepsilon_1 = \inf_{\nu\in(0,\infty)}\Biggl\{\varepsilon_\nu\colon \operatorname*{ess\,inf}_{(t,x,y)\in(\nu,\nu+\varepsilon_\nu)\times[0,\infty)\times[0,\infty)}\Biggl[\sum_{i=1}^{m} p_i(t,x,y)\sigma_{0,\nu}\bigl(g_i(t)\bigr) + \sigma_{0,\nu}\bigl(h_1(t)\bigr)\int\limits_{h_1(t)}^{t}\int\limits_{h_3(x)}^{h_4(x)}\int\limits_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\,d\theta\,ds\,d\xi\Biggr] > 0\Biggr\}. \tag{3.4}$$

The operator T, defined by formula (3.3), is strongly positive with respect to increasing t if $\varepsilon_1 > 0$.

Theorem 3.1. *Let the operator $T : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ be strongly positive with respect to increasing t its spectral radius r be less than one, and the right-hand side $f(t,x,y)$ be positive (strongly positive) for $(t,x,y) \in [0,\varepsilon_0] \times [0,+\infty) \times [0,+\infty)$ and nonnegative for others. Then there exists a unique solution $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq. (1.1) for each essentially bounded f and this solution is positive (nonoscillating) for $(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$.*

Proof. Note that the inequality $r < 1$ for the spectral radius of the operator T, according to Theorem 2.1, implies that the spectral radius $r_{\nu,\infty}$ of the operator $T_{\nu,\infty,0,\infty;0,\infty}$ is also less than one. Actually the function v in assertion (3) satisfies also the condition that $\varphi(t,x,y) \equiv v(t,x,y) - (T_{\nu,\infty;0,\infty;0,\infty}v)(t,x,y)$ is strongly positive for $(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$. The equivalence of assertions (1) and (3) implies now that $r_{\nu,\infty} < 1$.

The solution $u \in L^\infty_{[0,+\infty)\times[0,+\infty)\times[0,+\infty)}$ for $(t,x,y) \in [\nu,+\infty) \times [0,+\infty) \times [0,+\infty)$ satisfies (according to Lemma 1.1) Eq. (1.12), where $\mu_1 = +\infty$, $\mu_2 = +\infty$, $\mu_3 = +\infty$, $\nu_2 = \nu_3 = 0$. Without loss of generality let us suppose that the solution $u(t,x,y)$ is positive (strongly positive) for $(t,x,y) \in [0,\mu) \times [0,+\infty) \times [0,+\infty)$ and $u(\mu,x,y) = 0$ (in sense of Definition 1.1). Let us fix ν such that $\nu_1 \equiv \nu + \varepsilon_1 > \mu$. The condition that the operator T is strongly positive with respect to increasing t implies that $f_{\nu,\infty;0,\infty;0,\infty}(t,x,y)$ is positive (strongly positive) for $(t,x,y) \in [\nu,\nu_1] \times [0,+\infty) \times [0,+\infty)$. Now

$$\begin{aligned}
u(t,x,y) &= \bigl\{(I - T_{\nu,\infty;0,\infty;0,\infty})^{-1} f_{\nu,\infty;0,\infty;0,\infty}\bigr\}(t,x,y) \\
&= \bigl\{\bigl(I + T_{\nu,\infty;0,\infty;0,\infty} + T^2_{\nu,\infty;0,\infty;0,\infty} + T^3_{\nu,\infty;0,\infty;0,\infty} + \cdots\bigr) \\
&\quad \times f_{\nu,\infty;0,\infty;0,\infty}\bigr\}(t,x,y) \\
&\geqslant f_{\nu,\infty;0,\infty;0,\infty}(t,x,y) > 0
\end{aligned}$$

for $t \in [\nu,\nu_1] \times [0,+\infty) \times [0,+\infty)$. This contradiction to the assumption $u(\mu,x,y) = 0$ completes the proof. □

Example 3.1. The condition of strong positivity of the operator T with respect to increasing t is essential as the following equation

$$u(t,x,y)=\frac{1}{2}u(t+1,x,y)+f(t,x,y),\quad (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty),$$

demonstrates. Let us set $f(t,x,y)=1$ for $0\leqslant t\leqslant 1$ and $f(t,x,y)=0$ for $t\geqslant 1$. Here the operator $(Tu)(t,x,y)=\frac{1}{2}u(t+1,x,y)$ is not strongly positive with respect to increasing t. The unique solution in the space of bounded in essential functions $L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ satisfies the equality $u(t,x,y)\equiv 0$ for $(t,x,y)\in[1,+\infty)\times[0,+\infty)\times[0,+\infty)$. Note that there exist *unbounded* positive functions satisfying this equation, for example, $u(t,x,y)=2^{m-1}$ for $m\leqslant t<m+1, t,x,y\in[0,+\infty)$, $m=1,2,3,\dots,$ as well as oscillating *unbounded* functions satisfying this equation (see Example 2.2).

Corollary 3.1. *If $T:L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}\to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is positive operator, $\|T\|<1$ and $f\in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is nonnegative* (*nonpositive*)*, then a solution $u\in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq.* (1.1) *is nonnegative* (*nonpositive*)*. If, in addition, $f(t,x,y)$ is positive* (*strongly positive*) *for $(t,x,y)\in[0,\varepsilon_0]\times[0,+\infty)\times[0,+\infty)$, and the operator T is strongly positive with respect to increasing t, then a solution is positive* (*nonoscillating*) *in the zone $(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty)$.*

In order to prove Corollary 3.1 we set $v=1$ in the assertion (3) of Theorem 2.1 and then apply Theorem 3.1.

Theorem 3.2. *Let $p_i(t,x,y)\geqslant 0$ $(i=1,\dots,m)$, $k(t,x,y,s,\theta,\xi)\geqslant 0$, the initial function φ be strongly positive, the number ε_1 defined by formula* (3.4) *be positive and the following inequality be satisfied*

$$\operatorname*{ess\,sup}_{t,x,y\in[0,+\infty)}\left\{\sum_{i=1}^{m}p_i(t,x,y)\sigma_{0,\infty}\big(g_i(t)\big)\sigma_{0,\infty}\big(d_i(x)\big)\sigma_{0,\infty}\big(r_i(y)\big)\right.$$
$$\left.+\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)}k(t,x,y,s,\theta,\xi)\sigma_{0,\infty}(s)\sigma_{0,\infty}(\theta)\sigma_{0,\infty}(\xi)\,d\theta\,ds\,d\xi\right\}<1, \tag{3.5}$$

then the solution $u\in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of Eq. (1.7) *is nonoscillating in the zone $(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty)$.*

The proof follows from Corollary 3.1.

Remark 3.4. The condition that the operator T is strongly positive with respect to increasing t in Theorem 3.1 (in the case of Eq. (1.7)—the condition $\varepsilon_1>0$ in Theorem 3.2) is essential. In the case of the equation

$$u(t,x,y)=\frac{1}{2}u\big(t-\sin^2 t,x,y\big),\quad (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \tag{3.6}$$

the spectral radius of the nonstrongly positive operator $Tu(t)=\frac{1}{2}u(t-\sin^2 t,x,y)$ is less than one (in order to prove this we can set $v=1$ the assertion (3) of Theorem 2.1), but the unique solution is $u(t,x,y)\equiv 0$. On the basis of this example the more general assertion can be obtained.

Theorem 3.3. *If the following inequality*

$$\operatorname*{ess\,sup}_{t,x,y\in[0,+\infty)}\left\{\sum_{i=1}^{m}\left|p_i(t,x,y)\right| + \int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)}\left|k(t,x,y,s,\theta,\xi)\right| \right.$$
$$\left. \times\, \sigma_{0,\infty}(s)\sigma_{0,\infty}(\theta)\sigma_{0,\infty}(\xi)\,d\theta\,ds\,d\xi\right\} < 1, \tag{3.7}$$

is satisfied, the initial function $\varphi(t,x,y)=0$ for $x<0$ or $y<0$ and there exists a point t_0 such that $g_i(t)\geqslant t_0$ $(i=1,\dots,m)$ and $h_1(t)\geqslant t_0$ for $t\geqslant t_0$, then the unique solution of Eq. (1.7) *in the space $L^{\infty}_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ satisfies the equality $u(t,x,y)\equiv 0$ for $[t_0,+\infty)\times[0,+\infty)\times[0,+\infty)$.*

Remark 3.5. Equation (1.1) considered in the space of measurable essentially bounded functions, can be written in the form $Pu=f$, where $P=I-T$. Theorem 3.3 demonstrates that the strong maximum principle for Eq. (1.1) is not, generally speaking, valid, i.e. the equality $u(t,x,y)\equiv 0$ for $(t,x,y)\in M_1\subset M$, where $mes\, M_1$ is positive, does not imply the equality $u(t,x,y)\equiv 0$ for $(t,x,y)\in M$.

Example 3.2. Let us consider the equation

$$\begin{aligned} &u(t,x,y)=b(t)u(t+1,x,y)+c(t)u(t-1,x,y)+f(t,x,y),\\ &\quad (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{3.8}$$

where $u(t,x,y)=0$ for $(t,x,y)\notin[0,+\infty)\times[0,+\infty)\times[0,+\infty)$

$$b(t)=\begin{cases}\frac{1}{3}, & 0\leqslant t\leqslant\mu_1,\\ \frac{1}{2}, & \mu_1\leqslant t,\end{cases}$$
$$c(t)=\begin{cases}\frac{1}{3}, & 0\leqslant t\leqslant\mu_1,\\ 0, & \mu_1\leqslant t,\end{cases}$$
$$f(t,x,y)=\begin{cases}\frac{1}{3}, & 0\leqslant t\leqslant\mu_1,\\ 0, & \mu_1\leqslant t.\end{cases}$$

The operator $(Tu)(t,x,y)=b(t)u(t+1,x,y)+c(t)u(t-1,x,y)$ is not strongly positive with respect to increasing t, and its spectral radius is less than one. The unique solution of Eq. (3.8) is the following

$$u(t,x,y)=\begin{cases}1, & 0\leqslant t\leqslant\mu_1,\\ 0, & \mu_1\leqslant t.\end{cases}$$

Now it is clear that the similar property appears in the case of differential equations if the equation decreases its order in a corresponding domain. Let us consider the differential operator P of the following form

$$(Pu)(t,x,y)=\begin{cases}-u''_{tt}(t,x,y)+u(t,x,y),\\ \quad 0\leqslant t\leqslant\mu_1,\ 0\leqslant x\leqslant\mu_2,\ 0\leqslant y\leqslant\mu_3,\\ u'_t(t,x,y)+u(t,x,y),\\ \quad \mu_1\leqslant t\leqslant\mu_0,\ 0\leqslant x\leqslant\mu_2,\ 0\leqslant y\leqslant\mu_3,\end{cases} \tag{3.9}$$

in the parallelepiped $M = [0, \mu_0] \times [0, \mu_2] \times [0, \mu_3]$. If we write the derivatives in the difference form with $h = 1$, we get Eq. (3.8).

The unique continuous solution of the boundary value problem

$$Pu(t,x,y) = f(t,x,y), \quad (t,x,y) \in [0,\mu_0] \times [0,\mu_2] \times [0,\mu_3], \tag{3.10}$$

and $u = 0$ on the boundaries of M, where

$$f(t,x,y) = \begin{cases} 1, & 0 \leqslant t \leqslant \mu_1,\ 0 \leqslant x \leqslant \mu_2,\ 0 \leqslant y \leqslant \mu_3, \\ 0, & \mu_1 \leqslant t,\ 0 \leqslant x \leqslant \mu_2,\ 0 \leqslant y \leqslant \mu_3, \end{cases} \tag{3.11}$$

is

$$u(t,x,y) = \begin{cases} -\frac{1}{1+e^{\mu_1}}(e^t + e^{\mu_1}e^{-t}) + 1, & 0 \leqslant t \leqslant \mu_1,\ 0 \leqslant x \leqslant \mu_2,\ 0 \leqslant y \leqslant \mu_3, \\ 0, & \mu_1 \leqslant t \leqslant \mu_0,\ 0 \leqslant x \leqslant \mu_2,\ 0 \leqslant y \leqslant \mu_3. \end{cases} \tag{3.12}$$

4. Disconjugacy and nonoscillation in parallelepiped

Consider the functional equation on the bounded domain

$$\begin{aligned} &u(t,x,y) = (T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}u)(t,x,y) + f(t,x,y), \\ &\quad (t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ be a positive operator, then the following assertions are equivalent*:

(1) *The spectral radius r of the operator*

$$T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$$

is less than one.

(2) *For each nonnegative* (*strongly positive*) $f \in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ *there exists a unique solution* $u \in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ *of Eq.* (4.1) *and this solution u is nonnegative* (*strongly positive*).

(3) *There exists a strongly positive function* $v \in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ *such that the function* $\varphi(t,x,y) \equiv v(t,x,y) - (T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}v)(t,x,y)$ *is strongly positive for* $(t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]$.

The proof is analogous to the proof of Theorem 2.1.

Definition 4.1. Denote for a positive operator T the following constants:

$$\begin{aligned}
\nu_1^{t+} &= \inf\{\nu\colon\ T1_{\nu_1,\nu;\nu_2,\mu_2;\nu_3,\mu_3} \neq 0 \text{ for } (t,x,y) \in [\nu,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]\}, \\
\nu_2^{x+} &= \inf\{\nu\colon\ T1_{\nu_1,\mu_1;\nu_2,\nu_2;\nu_3,\mu_3} \neq 0 \text{ for } (t,x,y) \in [\nu_1,\mu_1] \times [\nu,\mu_2] \times [\nu_3,\mu_3]\}, \\
\nu_3^{y+} &= \inf\{\nu\colon\ T1_{\nu_1,\mu_1;\nu_2,\mu_2;\nu,\mu_3} \neq 0 \text{ for } (t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu,\mu_3]\}, \\
\mu_1^{t-} &= \inf\{\nu\colon\ T1_{\mu,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} \neq 0 \text{ for } (t,x,y) \in [\nu_1,\mu] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]\}, \\
\mu_2^{x-} &= \inf\{\nu\colon\ T1_{\nu_1,\mu_1;\mu_2,\mu_2;\nu_3,\mu_3} \neq 0 \text{ for } (t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu] \times [\nu_3,\mu_3]\}, \\
\mu_3^{y-} &= \inf\{\nu\colon\ T1_{\nu_1,\mu_1;\nu_2,\mu_2;\mu,\mu_3} \neq 0 \text{ for } (t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu]\}, \\
&\varepsilon(\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3) \\
&\quad = \max\{\nu_1^{t+} - \nu_1, \nu_2^{x+} - \nu_2, \nu_3^{y+} - \nu_3, \mu_1 - \mu_1^{t-}, \mu_2 - \mu_2^{x-}, \mu_3 - \mu_3^{y-}\},
\end{aligned}$$

$$\varepsilon = \sup\big\{\varepsilon(\alpha_1,\beta_1;\alpha_2,\beta_2;\alpha_3,\beta_3),\\ [\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times[\alpha_3,\beta_3]\subset[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]\big\},$$

where

$$1_{\alpha_1,\beta_1;\alpha_2,\beta_2;\alpha_3,\beta_3}(t,x,y)=\begin{cases}1, & \alpha_1\leqslant t\leqslant\beta_1,\ \alpha_2\leqslant x\leqslant\beta_2,\ \alpha_3\leqslant y\leqslant\beta_3,\\ 0, & \text{otherwise.}\end{cases}$$

In the more interesting case, when the right-hand side f is only nonnegative, we propose the following assertion.

Theorem 4.2. *Let the following three conditions be fulfilled*:

(1) *the spectral radius of the operator*

$$T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}: L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}\to L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$$

is less than one,

(2) *the operator $T: L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}\to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ is strongly positive with respect to at least one of the six directions*: *or increasing t, or decreasing t, or increasing x, or decreasing x, or increasing y, or decreasing y,*

(3) *the right-hand side $f(t,x,y)$ is strongly positive for $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]/[\nu_1+\varepsilon(\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3),\mu_1-\varepsilon(\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3)]\times[\nu_2+\varepsilon(\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3),\mu_2-\varepsilon(\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3)]\times[\nu_3+\varepsilon(\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3),\mu_3-\varepsilon(\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3)]$.*

Then the solution $u\in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ of Eq. (4.1) *is strongly positive in the parallelepiped $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$.*

The proof is analogous to the proof of Theorem 3.1.

Remark 4.1. Assertion (2) of Theorem 4.1 claims that $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ is a zone of disconjugacy of Eq. (4.1). Theorem 4.2 actually obtains nonoscillation of solution u in the parallelepiped. These two assertions can be also interpreted as possible formulations of the maximum principles [18] for functional equations.

Consider now the following particular cases of Eq. (4.1)

$$\begin{aligned} b(t,x,y)u(t,x,y) &= a(t,x,y)u(t+1,x,y)+c(t,x,y)u(t-1,x,y)\\ &\quad + d(t,x,y)u(t,x+1)+e(t,x,y)u(t,x-1,y)\\ &\quad + h(t,x,y)u(t,x,y+1)+g(t,x,y)u(t,x,y-1)+f(t,x,y),\\ &(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \end{aligned}\tag{4.2}$$

and

$$\begin{aligned} b_0u(t,x,y) &= a_0u(t+1,x,y)+c_0u(t-1,x,y)\\ &\quad + d_0u(t,x+1)+e_0u(t,x-1,y)\\ &\quad + h_0u(t,x,y+1)+g_0u(t,x,y-1)+f(t,x,y),\\ &(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \end{aligned}\tag{4.3}$$

with the initial function

$$u(t,x,y)=\varphi(t,x,y),\quad \text{if } (t,x,y)\notin[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3].\tag{4.4}$$

Here $a(t,x,y), b(t,x,y), c(t,x,y), d(t,x,y), e(t,x,y), h(t,x,y)$ and $g(t,x,y)$ are positive essentially bounded functions, and $a_0, b_0, c_0, d_0, e_0, h_0$ and g_0 are positive constants.

For Eqs. (4.2) and (4.3) the operators

$$T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$$

and

$$T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^{\infty}_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$$

can be written as follows

$$\begin{aligned}(T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}u)(t,x,y) &= \frac{a(t,x,y)}{b(t,x,y)}u(t+1,x,y) + \frac{c(t,x,y)}{b(t,x,y)}u(t-1,x,y) \\ &\quad + \frac{d(t,x,y)}{b(t,x,y)}u(t,x+1,y) + \frac{e(t,x,y)}{b(t,x,y)}u(t,x-1,y) \\ &\quad + \frac{h(t,x,y)}{b(t,x,y)}u(t,x,y+1) + \frac{g(t,x,y)}{b(t,x,y)}u(t,x,y-1), \qquad (4.5)\end{aligned}$$

$$\begin{aligned}\big(T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}u\big)(t,x,y) &= \frac{a_0}{b_0}u(t+1,x,y) + \frac{c_0}{b_0}u(t-1,x,y) \\ &\quad + \frac{d_0}{b_0}u(t,x+1,y) + \frac{e_0}{b_0}u(t,x-1,y) \\ &\quad + \frac{h_0}{b_0}u(t,x,y+1) + \frac{g_0}{b_0}u(t,x,y-1), \qquad (4.6)\end{aligned}$$

where

$$u(t,x,y) = 0, \quad \text{if } (t,x,y) \notin [\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \tag{4.7}$$

respectively.

Let us set $v = \beta^y\beta^x\beta^t$ in the assertion (3) of Theorem 4.1. If the inequality

$$(a_0+d_0+h_0)\beta^2 - b_0\beta + c_0 + e_0 + g_0 < 0, \tag{4.8}$$

is satisfied then the assertion (3) of Theorem 4.1 is fulfilled for Eq. (4.3). Last inequality is satisfied if and only if

$$b_0^2 > 4(a_0+d_0+h_0)(c_0+e_0+g_0). \tag{4.9}$$

The inequality (4.9) implies that the spectral radius of the operator $T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ is less than one. According to assertion (2) of Theorem 4.1 $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ is a zone of disconjugacy.

Let us check that other conditions of Theorem 4.2 are also fulfilled. The operator $T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ is strongly positive. The function $f(t,x,y)$ is greater than the function $\varphi(t,x,y)$ multiplied by the corresponding positive constant. According to Theorem 4.2 the solution u is positive in $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$.

We have proven the following assertion.

Theorem 4.3. *If inequality* (4.9) *is satisfied, then each parallelepiped* $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ *is a zone of disconjugacy of Eq.* (4.3) *and for strongly positive* $\varphi(t,x,y)$ *the solution of the homogeneous equation*

$$\begin{aligned} b_0 u(t,x,y) &= a_0 u(t+1,x,y) + c_0 u(t-1,x,y) \\ &\quad + d_0 u(t,x+1) + e_0 u(t,x-1,y) \\ &\quad + h_0 u(t,x,y+1) + g_0 u(t,x,y-1), \\ &(t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3], \\ u(t,x,y) &= \varphi(t,x,y), \quad \text{if } (t,x,y) \notin [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3], \end{aligned} \tag{4.10}$$

is strongly positive in this parallelepiped.

Remark 4.2. The inequality (4.9) cannot be improved. If in Eq. (4.3) the coefficients d_0, e_0, h_0 and g_0 are zeros, we actually consider this equation in the space of functions of one variable. The inequality

$$b_0^2 < 4a_0 c_0, \tag{4.11}$$

implies oscillation of all solutions in this case [8].

Theorem 4.4. *Let the inequality* (4.9) *be satisfied and*

$$\begin{array}{llll} a(t,x,y) \leqslant a_0, & b(t,x,y) \geqslant b_0, & c(t,x,y) \leqslant c_0, & \\ d(t,x,y) \leqslant d_0, & e(t,x,y) \leqslant e_0, & h(t,x,y) \leqslant h_0, & g(t,x,y) \leqslant g_0, \end{array} \tag{4.12}$$

then each parallelepiped $[\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]$ *is a zone of disconjugacy of Eq.* (4.2) *and for strongly positive* $\varphi(t,x,y)$ *the solution of the homogeneous equation*

$$\begin{aligned} b(t,x,y)u(t,x,y) &= a(t,x,y)u(t+1,x,y) + c(t,x,y)u(t-1,x,y) \\ &\quad + d(t,x,y)u(t,x+1) + e(t,x,y)u(t,x-1,y) \\ &\quad + h(t,x,y)u(t,x,y+1) + g(t,x,y)u(t,x,y-1), \\ &(t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3], \\ u(t,x,y) &= \varphi(t,x,y), \quad \text{if } (t,x,y) \notin [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3], \end{aligned} \tag{4.13}$$

is strongly positive in this parallelepiped.

Proof. Inequality (4.9) implies that the spectral radius of the operator $T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ determined by the formula (4.6) is less than one. By Theorem 4.1 there exists a strongly positive function $v \in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ such that the function $\varphi(t,x,y) \equiv v(t,x,y) - (T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} v)(t,x,y)$ is strongly positive for $(t,x,y) \in [\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]$. The inequalities (4.12) imply that $(T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} v)(t,x,y) \geqslant (T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} v)(t,x,y)$, where the operator $T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}$ is determined by the formula (4.5). For this function v the function $v(t,x,y) - (T^0_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} v)(t,x,y)$ is strongly positive. According to the assertion (1) of Theorem 4.1, the spectral radius of the operator $T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ is less than one. According to the assertion (2) of Theorem 4.1, we obtain that $[\nu_1,\mu_1] \times [\nu_2,\mu_2] \times [\nu_3,\mu_3]$ is a zone of disconjugacy of Eq. (4.2). All conditions of Theorem 4.2 are fulfilled. According to this theorem we get that the solution u of Eq. (4.13) is strongly positive. □

5. The strong maximum principle in parallelepiped

Theorem 5.1. *Let the operator $T : L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)} \to L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ be strongly positive with respect to increasing and decreasing t, x, y, and the spectral radius of the operator $T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ determined by equality* (1.13) *be less than one. If the inequalities $u(t,x,y) \geqslant 0$ and $f(t,x,y) \equiv u - T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}u \geqslant 0$ are satisfied in $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$, while $f(t,x,y)$ is strongly positive in a cube $[t_1,t_2]\times[x_1,x_2]\times[y_1,y_2] \subset [\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ of the volume greater than $\varepsilon\times\varepsilon\times\varepsilon$ (where ε was introduced in Definition* 4.1*), then u is strongly positive in the parallelepiped $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$.*

Proof. Let us suppose that the function $f(t,x,y)$ is strongly positive for $[t_1,t_2]\times[x_1,x_2]\times[y_1,y_2]$ and zero otherwise in the parallelepiped $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$. The condition that the spectral radius of the operator $T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3} : L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]} \to L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ is less than one implies that the solution $u(t,x,y)$ is nonnegative in $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$. From the positivity of $f(t,x,y)$ for $[t_1,t_2]\times[x_1,x_2]\times[y_1,y_2]$ and nonnegativity of $u(t,x,y)$ we obtain that the solution $u(t,x,y) = (T_{\nu_1,\mu_1;\nu_2,\mu_2;\nu_3,\mu_3}u)(t,x,y) + f(t,x,y)$ is strongly positive in $[t_1,t_2]\times[x_1,x_2]\times[y_1,y_2]$.

Let us prove that $u(t,x,y)$ is strongly positive in the parallelepiped $[\nu_1,\mu_1]\times[x_1,x_2]\times[y_1,y_2]$. Suppose that $u(t,x,y) > 0$ for $(t,x,y) \in [t_1,\mu]\times[x_1,x_2]\times[y_1,y_2]$ and $u(\mu, x_0, y_0) = 0$ (in the sense of Definition 1.1). Let us fix s_0 such that $s_1 \equiv s_0 + \varepsilon_1 > \mu$, where ε_1 is determined in Definition 3.1. The condition that the operator T is strongly positive with respect to increasing t guarantees that $f_{s_0,s_1,x_1,x_2,y_1,y_2}(t,x,y)$ is strongly positive for $(t,x,y) \in [s_0,s_1]\times[x_1,x_2]\times[y_1,y_2]$. Now $u(t,x,y) = \{(I - T_{s_0,s_1,x_1,x_2,y_1,y_2})^{-1} f_{s_0,s_1,x_1,x_2,y_1,y_2}(t,x,y) = \{(I + T_{s_0,s_1,x_1,x_2,y_1,y_2} + T^2_{s_0,s_1,x_1,x_2,y_1,y_2} + T^3_{s_0,s_1,x_1,x_2,y_1,y_2} + \cdots) f_{s_0,s_1,x_1,x_2,y_1,y_2}(t,x,y) \geqslant f_{s_0,s_1,x_1,x_2,y_1,y_2}(t,x,y) > 0$ for $(t,x,y) \in [s_0,s_1]\times[x_1,x_2]\times[y_1,y_2]$. This contradicts the assumption $u(\mu,x_0,y_0) = 0$. We have proven that $u(t,x,y)$ is strongly positive for $(t,x,y) \in [t_1,\mu_1]\times[x_1,x_2]\times[y_1,y_2]$. Using the condition about strong positivity of the operator T with respect to decreasing t, we can prove analogously that $u(t,x,y)$ is strongly positive for $(t,x,y) \in [\nu_1,t_1]\times[x_1,x_2]\times[y_1,y_2]$, then, using the conditions that the operator T is strongly positive with respect to increasing and decreasing x and y, we can prove that $u(t,x,y)$ is strongly positive for $[t_1,t_2]\times[\nu_2,\mu_2]\times[y_1,y_2]$ and $[t_1,t_2]\times[x_1,x_2]\times[\nu_3,\mu_3]$ respectively. Then the same idea to continue positivity allows us to obtain that $u(t,x,y)$ is strongly positive in the parallelepiped $[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$. □

Remark 5.1. The condition that the sizes of the cube, where the function f is strongly positive, are $\varepsilon\times\varepsilon\times\varepsilon$, is essential as the following example demonstrates.

Let us consider the equation

$$\begin{aligned} b_0 u(t,x,y) = {} & a_0 u(t+1,x,y) + c_0 u(t-1,x,y) \\ & + d_0 u(t,x+1) + e_0 u(t,x-1,y) \\ & + h_0 u(t,x,y+1) + g_0 u(t,x,y-1) + f(t,x,y), \\ & (t,x,y) \in [0,2-\theta]\times[0,2-\theta]\times[0,2-\theta], \end{aligned} \tag{5.1}$$

where

$$u(t,x,y)=0, \quad \text{if } (t,x,y)\notin[0,2-\theta]\times[0,2-\theta]\times[0,2-\theta], \tag{5.2}$$

$$f(t,x,y)=\begin{cases}1, & 1-\theta\leqslant t\leqslant 1,\ 1-\theta\leqslant x\leqslant 1,\ 1-\theta\leqslant y\leqslant 1,\\ 0, & \text{otherwise,}\end{cases} \tag{5.3}$$

and $0<\theta<1$. The corresponding operator $T_{0,2+\theta;0,2+\theta;0,2+\theta}: L^{\infty}_{[0,2+\theta]\times[0,2+\theta]\times[0,2+\theta]} \to L^{\infty}_{[0,2+\theta]\times[0,2+\theta]\times[0,2+\theta]}$ is defined by equality (4.6) and satisfies the condition about strong positivity with respect to increasing and decreasing t,x,y. Assume that the coefficients satisfy the inequality (4.9), then the spectral radius of the operator $T_{0,2+\theta;0,2+\theta;0,2+\theta}$ is less than one. It is clear that ε defined by Definition 4.1 is equal to 1. Thus the function f is strongly positive in the cube of sizes $\theta\times\theta\times\theta$ less than $1\times1\times1$. All other conditions of Theorem 5.1 are fulfilled. The unique solution is

$$u(t,x,y)=\begin{cases}\frac{1}{b_0}, & 1-\theta\leqslant t\leqslant 1, 1-\theta\leqslant x\leqslant 1,\ 1-\theta\leqslant y\leqslant 1,\\ 0, & \text{otherwise,}\end{cases} \tag{5.4}$$

and the strong maximum principle is not true.

6. Applications to partial differential equations

Consider the partial differential equations

$$\begin{aligned} A(t,x,y)u'_t(t,x,y) &= B(t,x,y)u''_{xx}(t,x,y)+C(t,x,y)u''_{yy}(t,x,y)\\ &\quad + p(t,x,y)u(t,x,y)\\ &\quad + \int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi,\\ &(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &A(t,x,y)u''_{tt}(t,x,y)+B(t,x,y)u''_{xx}(t,x,y)+C(t,x,y)u''_{yy}(t,x,y)\\ &\quad + D(t,x,y)u'_t(t,x,y)+E(t,x,y)u'_x(t,x,y)+F(t,x,y)u'_y(t,x,y)\\ &\quad + p(t,x,y)u(t,x,y)+\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi=0,\\ &\quad (t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty), \end{aligned} \tag{6.2}$$

where

$$u(t,x,y)=\varphi(t,x,y), \quad \text{for } (t,x,y)\notin[0,+\infty)\times[0,+\infty)\times[0,+\infty). \tag{6.3}$$

We assume that the functions

$$A,B,C,D,E,F:[0,+\infty)\times[0,+\infty)\times[0,+\infty)\to(-\infty,+\infty)$$

and

$$\begin{aligned} k:&[0,+\infty)\times[0,+\infty)\times[0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty)\\ &\to(-\infty,+\infty) \end{aligned}$$

are nonnegative.

If we represent the partial derivatives in these equations in the difference form, then the following integro-functional equations can be obtained respectively

$$\begin{aligned}
&\bigl\{A(t,x,y)+2B(t,x,y)+2C(t,x,y)-p(t,x,y)h^2\bigr\}u(t,x,y)\\
&\quad= A(t,x,y)u(t-h,x,y)+B(t,x,y)u(t,x+h,y)+B(t,x,y)u(t,x-h,y)\\
&\qquad+C(t,x,y)u(t,x,y+h)+C(t,x,y)u(t,x,y-h)\\
&\qquad+h^2\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi,\\
&\quad(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty),
\end{aligned}\tag{6.4}$$

and

$$\begin{aligned}
&\bigl\{2A(t,x,y)+2B(t,x,y)+2C(t,x,y)+hD(t,x,y)+hE(t,x,y)+hF(t,x,y)\\
&\qquad-h^2p(t,x,y)\bigr\}u(t,x,y)\\
&\quad= A(t,x,y)u(t+h,x,y)+A(t,x,y)u(t-h,x,y)\\
&\qquad+B(t,x,y)u(t,x+h,y)+B(t,x,y)u(t,x-h,y)\\
&\qquad+C(t,x,y)u(t,x,y+h)+C(t,x,y)u(t,x,y-h)\\
&\qquad+hD(t,x,y)u(t+h,x,y)+hE(t,x,y)u(t,x+h,y)+hF(t,x,y)u(t,x,y+h)\\
&\qquad+h^2\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi,\\
&\quad(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty).
\end{aligned}\tag{6.5}$$

The corresponding operators T will be of the following forms

$$\begin{aligned}
(Tu)(t,x,y)={}&\frac{1}{\{A(t,x,y)+2B(t,x,y)+2C(t,x,y)-p(t,x,y)h^2\}}\\
&\times\Bigl\{A(t,x,y)u(t-h,x,y)+B(t,x,y)u(t,x+h,y)\\
&+B(t,x,y)u(t,x-h,y)+C(t,x,y)u(t,x,y+h)\\
&+C(t,x,y)u(t,x,y-h)\\
&+h^2\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi\Bigr\},\\
&(t,x,y)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty),
\end{aligned}\tag{6.6}$$

and

$$\begin{aligned}
&(Tu)(t,x,y)\\
&\quad=\frac{1}{\{2A(t,x,y)+2B(t,x,y)+2C(t,x,y)+hD(t,x,y)+hE(t,x,y)+hF(t,x,y)-h^2p(t,x,y)\}}\\
&\qquad\times\Bigl\{A(t,x,y)u(t+h,x,y)+A(t,x,y)u(t-h,x,y)\\
&\qquad+B(t,x,y)u(t,x+h,y)+B(t,x,y)u(t,x-h,y)
\end{aligned}$$

$$
\begin{aligned}
&+ C(t,x,y)u(t,x,y+h) + C(t,x,y)u(t,x,y-h) \\
&+ hD(t,x,y)u(t+h,x,y) + hE(t,x,y)u(t,x+h,y) \\
&+ hF(t,x,y)u(t,x,y+h) \\
&+ h^2 \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi \Biggr\}, \\
&(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty),
\end{aligned} \tag{6.7}
$$

respectively, where

$$u(t,x,y) = 0, \quad \text{for } (t,x,y) \notin [0,+\infty) \times [0,+\infty) \times [0,+\infty).$$

Theorem 6.1. *Let $A \geqslant 0$, $B \geqslant 0$, $C \geqslant 0$, $k \geqslant 0$ and the following inequality*

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\,ds\,d\theta\,d\xi \\
&< \operatorname*{ess\,inf}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \bigl\{-p(t,x,y)\bigr\},
\end{aligned} \tag{6.8}
$$

be satisfied. Then the following assertions are true:

(1) *for each essentially bounded f and ϕ there exists a unique solution $u \in L^\infty_{[0,\infty)\times[0,\infty)\times[0,\infty)}$ of each of Eqs.* (1.5), (1.4) *and* (1.6), (1.4) *and their solutions are nonnegative* (*strongly positive*) *for f and φ nonnegative* (*strongly positive*);
(2) *if the function φ is strongly positive for $(t,x,y) \notin [0,+\infty) \times [0,+\infty) \times [0,+\infty)$ and at least one of the functions A or B or C is strongly positive, then the solutions $u(t,x,y)$ of each of Eqs.* (6.4) *and* (6.5) *are nonoscillating for $(t,x,y) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$*;
(3) *for every sufficiently small positive constant h and bounded functions f and φ each problem* (1.5d), (1.4) *and* (1.6d), (1.4) *has a unique solution $u \in L^\infty_{h[0,\infty)\times[0,\infty)\times[0,\infty)}$ and there exists a positive constant N such that $\|u\|_h \leqslant N(\|f\|_h + \|\varphi\|_h)$, where N does not depend on h.*

Proof. Inequality (6.8) implies that the spectral radii of the operators T determined by equalities (6.6) and (6.7) are less than one. Now the first assertion follows from Theorem 2.1 and the second—from Theorem 3.2.

Let us prove assertion (3). Inequality (6.8) implies that the norms of the operators (6.6) and (6.7) are less than one, and consequently their spectral radii are less than one for each h.

Let us denote

$$
\begin{aligned}
K = {}& \operatorname*{ess\,inf}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \bigl\{-p(t,x,y)\bigr\} \\
&- \operatorname*{ess\,sup}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\,ds\,d\theta\,d\xi,
\end{aligned}
$$

and

$$R_0 = \operatorname*{ess\,sup}_{[0,\infty)\times[0,\infty)\times[0,\infty)} \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\,ds\,d\theta\,d\xi.$$

The operators T determined by formulas (6.6) and (6.7) are positive. Now it is clear that the solutions $u(t,x,y)$ of Eqs. (1.5d) and (1.6d) for sufficiently small h satisfy the inequality $\|u\|_h \leqslant \max\{1, \frac{2}{K}(R_0+1)\}(\|f\|_h + \|\varphi\|_h)$. $\square$

Consider the equations

$$\begin{aligned} A(t,x,y)u_t'(t,x,y) &= B(t,x,y)u_{xx}''(t,x,y) + C(t,x,y)u_{yy}''(t,x,y) \\ &\quad + p(t,x,y)u(t,x,y) \\ &\quad + \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi + f(t,x,y), \\ (t,x,y) &\in [\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} &A(t,x,y)u_{tt}''(t,x,y) + B(t,x,y)u_{xx}''(t,x,y) + C(t,x,y)u_{yy}''(t,x,y) \\ &\quad + D(t,x,y)u_t'(t,x,y) + E(t,x,y)u_x'(t,x,y) + F(t,x,y)u_y'(t,x,y) \\ &\quad + p(t,x,y)u(t,x,y) + \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi \\ &= f(t,x,y), \quad (t,x,y) \in [\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \end{aligned} \tag{6.10}$$

where

$$u(t,x,y) = \varphi(t,x,y), \quad \text{if } (t,x,y) \notin [\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]. \tag{6.11}$$

The representation of the partial derivatives in these equations in the difference form leads to the equations

$$\begin{aligned} &\{A(t,x,y) + 2B(t,x,y) + 2C(t,x,y) - p(t,x,y)h^2\}u(t,x,y) \\ &\quad = A(t,x,y)u(t-h,x,y) + B(t,x,y)u(t,x+h,y) + B(t,x,y)u(t,x-h,y) \\ &\qquad + C(t,x,y)u(t,x,y+h) + C(t,x,y)u(t,x,y-h) \\ &\qquad + h^2 \int_{h_1(t)}^{h_2(t)} \int_{h_3(x)}^{h_4(x)} \int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi + h^2 f(t,x,y), \\ &\quad (t,x,y) \in [\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \end{aligned} \tag{6.12}$$

and

$$\begin{aligned}
&\big\{2A(t,x,y)+2B(t,x,y)+2C(t,x,y)+hD(t,x,y)+hE(t,x,y)+hF(t,x,y)\\
&\quad -h^2p(t,x,y)\big\}u(t,x,y)\\
&= A(t,x,y)u(t+h,x,y)+A(t,x,y)u(t-h,x,y)\\
&\quad + B(t,x,y)u(t,x+h,y)+B(t,x,y)u(t,x-h,y)\\
&\quad + C(t,x,y)u(t,x,y+h)+C(t,x,y)u(t,x,y-h)+hD(t,x,y)u(t+h,x,y)\\
&\quad + hE(t,x,y)u(t,x+h,y)+hF(t,x,y)u(t,x,y+h)\\
&\quad + h^2\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi + h^2 f(t,x,y),\\
&(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3],
\end{aligned}\tag{6.13}$$

respectively.

Now we can replace condition (6.8) in Theorem 6.1.

Theorem 6.2. *If $A \geqslant 0$, $B \geqslant 0$, $C \geqslant 0$, $k \geqslant 0$ and there exists a positive number β that*

$$\begin{aligned}
&(A+B+C+hD+hE+hF)\beta^2-\big(2A+2B+2C+hD+hE+hF-ph^2\big)\beta\\
&\quad +(A+B+C)+h^2\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)\beta^s\beta^\theta\beta^\xi\,ds\,d\theta\,d\xi<0,\\
&(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3],
\end{aligned}\tag{6.14}$$

then the following assertions are true:

(1) *for each $f \in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$ there exists a unique solution*

$$u \in L^\infty_{[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]}$$

of Eqs. (6.13) *with the condition* (6.11) *in each parallelepiped $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ and this solution u is nonnegative* (*strongly positive*) *for f and φ nonnegative* (*for f strongly positive and φ nonnegative*);

(2) *if the function φ is strongly positive for $(t,x,y)\notin[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ and at least one of the functions A or B or C is strongly positive, then the solutions $u(t,x,y)$ of the homogeneous equation*

$$\begin{aligned}
&\big\{2A(t,x,y)+2B(t,x,y)+2C(t,x,y)+hD(t,x,y)+hE(t,x,y)\\
&\quad +hF(t,x,y)-h^2p(t,x,y)\big\}u(t,x,y)\\
&= A(t,x,y)u(t+h,x,y)+A(t,x,y)u(t-h,x,y)+B(t,x,y)u(t,x+h,y)\\
&\quad + B(t,x,y)u(t,x-h,y)+C(t,x,y)u(t,x,y+h)+C(t,x,y)u(t,x,y-h)\\
&\quad + hD(t,x,y)u(t+h,x,y)+hE(t,x,y)u(t,x+h,y)+hF(t,x,y)u(t,x,y+h)\\
&\quad + h^2\int_{h_1(t)}^{h_2(t)}\int_{h_3(x)}^{h_4(x)}\int_{h_5(y)}^{h_6(y)} k(t,x,y,s,\theta,\xi)u(s,\theta,\xi)\,ds\,d\theta\,d\xi,\\
&(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3],
\end{aligned}$$

where

$$u(t,x,y)=\varphi(t,x,y), \quad \textit{if } (t,x,y)\notin[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3],$$

is strongly positive in each parallelepiped $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$.

Proof. Let us set $v=\beta^y\beta^x\beta^t$ in assertion (3) of Theorem 2.1. According to assertion (1) of Theorem 2.1 the spectral radius of the operator T, defined by formula (6.7) on the parallelepiped $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ with the initial function

$$u(t,x,y)=0, \quad \text{if } (t,x,y)\notin[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3], \tag{6.15}$$

is less than one. Now references to assertion (2) of Theorems 2.1 and 3.2 complete the proof. □

Theorem 6.3. *If* $A\geqslant 0$, $B\geqslant 0$, $C\geqslant 0$, $k\geqslant 0$ *and there exists such a positive number* β *that*

$$\begin{aligned}&(B+C)\beta^2-\bigl(2A+2B+2C-ph^2\bigr)\beta+(A+B+C)\\&\quad+h^2\int\limits_{h_1(t)}^{h_2(t)}\int\limits_{h_3(x)}^{h_4(x)}\int\limits_{h_5(y)}^{h_6(y)}k(t,x,y,s,\theta,\xi)\beta^s\beta^\theta\beta^\xi\,ds\,d\theta\,d\xi<0,\end{aligned} \tag{6.16}$$

for $(t,x,y)\notin[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$. *Then assertions* (1) *and* (2) *of Theorem* 6.2 *are true for Eq.* (6.12) *with the condition* (6.11) *in each parallelepiped* $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$.

The proof is analogous to the proof of Theorem 6.2.

Consider now the partial differential equation

$$\begin{aligned}&A_0u''_{tt}(t,x,y)+B_0u''_{xx}(t,x,y)+C_0u''_{yy}(t,x,y)+p_0u(t,x,y)\\&\quad+D_0u'_t(t,x,y)+E_0u'_x(t,x,y)+F_0(t,x,y)u'_y(t,x,y)=f(t,x,y)\end{aligned} \tag{6.17}$$

for $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ with nonnegative coefficients A_0,B_0,C_0,D_0,E_0,F_0. Its difference analog is the following

$$\begin{aligned}&\bigl\{2A_0+2B_0+2C_0+hD_0+hE_0+hF_0-h^2p_0\bigr\}u(t,x,y)\\&\quad=A_0u(t+h,x,y)+A_0u(t-h,x,y)+B_0u(t,x+h,y)+B_0u(t,x-h,y)\\&\qquad+C_0u(t,x,y+h)+C_0u(t,x,y-h)+hD(t,x,y)u(t+h,x,y)\\&\qquad+hE(t,x,y)u(t,x+h,y)+hF(t,x,y)u(t,x,y+h)+h^2f(t,x,y),\\&\quad(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3],\end{aligned} \tag{6.18}$$

$$u(t,x,y)=\varphi(t,x,y), \quad \text{if } (t,x,y)\notin[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]. \tag{6.19}$$

Theorem 6.4. *Let the following inequality be fulfilled*

$$h^2p_0-2p_0h+\bigl\{(D_0+E_0+F_0)^2-4p_0(A_0+B_0+C_0)\bigr\}>0. \tag{6.20}$$

Then assertions (1) *and* (2) *of Theorem* 6.2 *are true for Eq.* (6.18) *with the initial condition* (6.19) *in* $(t,x,y)\in[\nu_1,\mu_1]\times[\nu_2,\mu_2]\times[\nu_3,\mu_3]$ *and each finite parallelepiped is a disconjugacy zone of Eq.* (6.18).

Remark 6.1. Let us assume that h is small enough, then inequality (6.20) is fulfilled if

$$(D_0 + E_0 + F_0)^2 > 4p_0(A_0 + B_0 + C_0). \tag{6.21}$$

If the coefficient p_0 is nonpositive, then each finite parallelepiped is a disconjugacy zone of Eq. (6.18). This is necessary and sufficient for disconjugacy of partial differential equation (6.17) in each finite parallelepiped in the case when $D_0 = E_0 = F_0 = 0$.

In the case $B_0 = C_0 = E_0 = F_0 = 0$ Eq. (6.17) becomes ordinary differential, and the inequality $D_0^2 \geqslant 4p_0 A_0$ is necessary and sufficient for nonoscillation.

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