UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Monotone Iterative Techniques for Discontinuous Functional Impulsive Equations

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CZECH REPUBLIC

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$$\frac{d}{dt}\phi_p(u'(t)) = \frac{d}{dt}(|u'(t)|^{p-2}u'(t)) = f(t, u(t), u'(t)), \ t \in [0, T].$$

for some $p > 1$.

Preliminaries

ϕ – Laplacian Equations

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- C. De Coster, P. Habets, *Two – point boundary value problems. Lower and upper solutions.* **Mathematics in Science and Engineering, 205, Elsevier, Amsterdam,** 2006.

Preliminaries

References

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- A. Cabada, J. Tomeček, Existence of extremal solutions for nonlinear discontinuous impulsive functional ϕ-Laplacian equations with nonlinear discontinuous functional boundary conditionss, *Mem. Differential Equations Math. Phys.* **40**, *(2007), 1 – 16.*

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we give existence results for the non impulsive problem.

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$$\begin{aligned} C_P^m \;=\; & \{u : [0, T] \to \mathbb{R} \;:\; \text{ for all } k = 0, \dots, p,\, u \in C^m(J_k), \\ & \text{there exist } u^{(l)}(t_k^+),\; k = 1, \dots, p \;\; u^{(l)}(t_k^-) \equiv u^{(l)}(t_k), \\ & k = 1, \dots, p+1;\; l = 0, \dots, m\} \end{aligned}$$

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and

$$W_P^{m,q} = \{u : [0,T] \to \mathbb{R} \;:\; u_{|J_k} \in W^{m,q}(J_k), k = 0, \dots, p\}$$

for $m \in \mathbb{N} \cup \{0\}$ and $1 \le q \le \infty$.

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$$(\phi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \quad \text{for a. e. } t \in I_0.$$

Carathéodory Case

Definitions and Hypotheses

For all $k = 1, \ldots, p$, functions $I_k(\alpha(t_k), \cdot)$ are one–to–one and there exist $\alpha'(t_k^-)$ and $\alpha'(t_k^+)$ satisfying

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Theorem 1 Assume that hypotheses $(H_1) - (H_6)$ hold.

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Then there exists at least one solution $u \in [\alpha, \beta]$ of the problem (P_1)

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has a unique solution $u \in C_P^1$, such that $\phi \circ u' \in W_P^{1,1}$.

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Here, for each $k = 0, \ldots, p$, τ_k is the unique solution of the equation

$$B_k - A_k = \int_{t_k}^{t_{k+1}} \bar{\phi}^{-1}\left(\int_{t_k}^{z} \tilde{f}(s)ds + \tau\right)dz.$$

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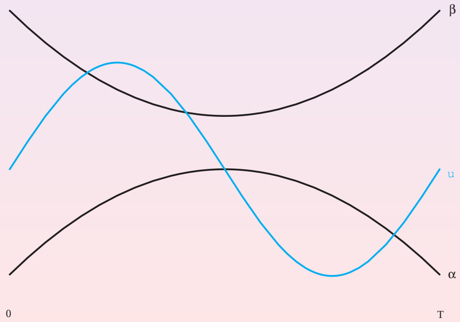

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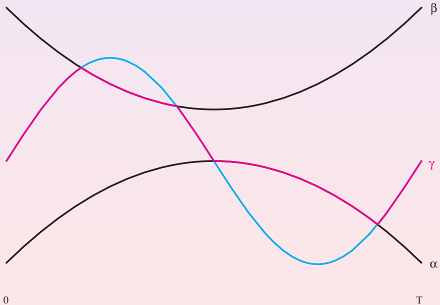

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Preliminaries

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Considered Problem

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Existence Results

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Discontinuous Case

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- M. X. Wang, A. Cabada, J. J. Nieto, Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions, *Ann. Polon. Math.* **58**, *(1993), 221 – 235.*

Proof of Theorem 1

Given $K > 0$ the Nagumo Constant, we define:

$$x \in \mathbb{R} \longmapsto \bar{\phi}(x) = \begin{cases} x - K + \phi(K), & \text{for } x > K, \\ \phi(x), & \text{for } -K \leq x \leq K, \\ x + K + \phi(-K), & \text{for } x < -K. \end{cases}$$

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(\bar{\phi}(u'(t)))' = \tilde{f}_u(t), & \text{a. e. } t \in [0,T]\backslash P,\\
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$$B_{k-1}(u)=\gamma(t_k,u(t_k)+M_k(u(t_k),u(t_k^+),u'(t_k),u'(t_k^+),u)).$$

$$A_k(u)=\gamma(t_k^+,u(t_k^+)+I_k(u(t_k),u(t_k^+))),$$

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$$(PM)\left\{\begin{array}{lcl} (\bar{\phi}(u'(t)))' & = & \tilde{f}_u(t), \quad \text{a. e. } t \in [0,T]\setminus P, \\ u(t_k) & = & B_{k-1}(u), \quad k=1,\dots,p, \\ u(t_k^+) & = & A_k(u) \quad k=1,\dots,p, \\ u(0) & = & A_0(u), \\ u(T) & = & B_p(u). \end{array}\right.$$

$$A_0(u)=\gamma(0,u(0)+g_1(u(0),u(T))),$$

$$B_p(u)=\gamma(T,u(T)+g_2(u(0),u(T),u'(0),u'(T),u)),$$

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$\tau_k(u)$ is the unique solution of equation

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- The proof holds from comparison results and degree theory.

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We prove that given u_1, $u_2 \in S$, then there exist u_3, $u_4 \in S$ such that

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J. A. Cid, On extremal fixed points in Schauder's theorem with applications to differential equations. *Bull. Belg. Math. Soc. Simon Stevin* **11** (2004), 15 – 20.

□

Extremal Solutions can be obtained for problem (P_2)

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In this case f, g_1, g_2, I_k and M_k can be discontinuous on the second variable.

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Let $[\alpha, \beta] \subset C_P^0$ and $G : [\alpha, \beta] \to [\alpha, \beta]$ nondecreasing.

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S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, **Marcel Dekker, New York,** 1994.

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Discontinuous Case

Monotone Iterative Techniques

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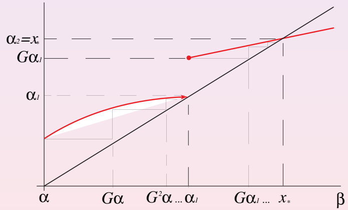

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$$(P_v) \begin{cases} (\phi(u'(t)))' = f(t, v, u(t), u'(t)) & \text{a. e. } t \in [0, T] \setminus P, \\ g_1(u(0), v) = 0, & \\ g_2(u(T), v) = 0, & \\ I_k(u(t_k), v) = 0, & k = 1, \dots, p, \\ M_k(u(t_k^+), v) = 0, & k = 1, \dots, p. \end{cases}$$

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$$(E)\begin{cases} u''(t) &= F([u(\xi)])\,|u'(t)|, \qquad \text{for all } t \in (0,2)\backslash\{1\},\\ u(0) &= A,\\ u(1) &= u(\eta),\\ u^2(1^+) &= u(\rho),\\ u(2) &= B, \end{cases}$$

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with $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = -\frac{x^3}{1+x^2}.$$

We prove that this problem has exactly two positive solutions.

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The expressions of such solutions are given by

$$u_*(t) = u^*(t) = A, \quad \text{for all } t \in [0, 1].$$

$$u_*^*(t) = \sqrt{A} + (B - \sqrt{A})H(t, x_*^*), \qquad \text{for all } t \in (1, 2]$$

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$$H(t,x) = \begin{cases} \dfrac{e^{-F([x])\,(t-1)} - 1}{e^{-F([x])} - 1} & \text{if } x \geq 1, \\[2ex] t - 1 & \text{if } 0 \leq x < 1, \end{cases}$$

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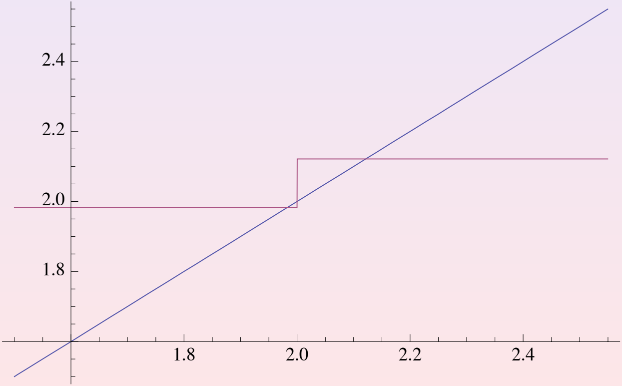
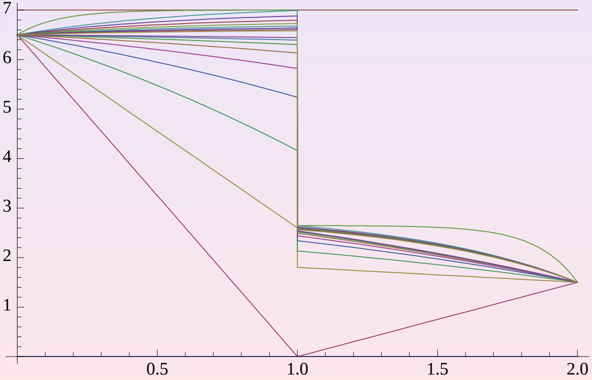
$$g(y) = \begin{cases} \dfrac{e^{-F(y)\,(\xi-1)} - 1}{e^{-F(y)} - 1} & \text{for all } y > 0, \\[2ex] \xi - 1 & \text{if } y = 0. \end{cases}$$

$$x = \bar{G}(x),$$




7
6
5
4
3
2
1
0.5
1.0
1.5
2.0

THANKS FOR YOUR ATTENTION!!