Local analytic solutions to some nonhomogeneous problems with p-Laplacian

Gabriella Bognár

Department of Analysis
University of Miskolc
3515 Miskolc-Egyetemváros
Hungary

Hejnice, September 18. 2007.

We consider the solution of the nonlinear differential equation of the second order

$$\Delta_p u + (-1)^i |u|^{q-1} u = 0,\ u = u(x),\ x \in \mathbf{R}^n, \qquad \text{(E)}$$

where $n \geq 1$, p and q are positive real numbers, $i = 0,1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-1} \nabla u)$.

If $n = 1$: $\quad (\Phi_p(y'))' + (-1)^i \Phi_q(y) = 0,$

where for $r \in \{p, q\}$ $\quad \Phi_r(y) := \begin{cases} |y|^{r-1} y, & \text{for } y \in \mathbf{R} \setminus \{0\} \\ 0, & \text{for } y = 0. \end{cases}$

If $n > 1$ radially symmetric solutions:

$$\left(t^{n-1} \Phi_p(y')\right)' + (-1)^i t^{n-1} \Phi_q(y) = 0, \text{ on } (0,\ a).$$

$n = 1$:

For $p = q = 1$ $\quad y'' \pm y = 0,\quad y(0) = 1,\ y'(0) = 0,$ solution for '+' sign is $y = \cos x$.

for '-' sign is $y = \cosh x$.

$y'' \pm y = 0,\quad y(0) = 0,\ y'(0) = 1,$ solution for '+' sign is $y = \sin x$.

for '-' sign is $y = \sinh x$.

If $p = q$ $\quad y'' |y'|^{p-1} + y |y|^{p-1} = 0,\ y(0) = 0,\ y'(0) = 1,$ generalized sine $y = S_p(x)$.

-existence and uniqueness of radial solutions [W. Reichel-W. Walter, J. Inequal. Appl., (1997)]

-$n = 1$ and $i = 0$, the initial value problem has a unique solution on $\mathbf{R}$ [O. Dosly-P. Rehak Half-linear Differential Equations, Elsevier, 2005.], and [P. Drabek- R. Manásevich, Diff. Integral Equations, (1999)]), moreover, its solution can be given in closed form in terms of incomplete gamma functions

-$n = 2$, $p = q = 1$ then the solution (E) with $y(0) = 1$, $y'(0) = 0$ is $J_0(t)$, the Bessel function of first kind with zero order

-$n = 3$, $p = q = 1$ then the solution (E) with $y(0) = 1$, $y'(0) = 0$ is $j_0(t) = \sin t / t$, the spherical Bessel function of first kind with zero order

Existence of local analytic solution (Initial conditions: $y(0) = A,\ y'(0) = 0$)

Theorem *(Briot-Bouquet Theorem) Let us assume that in the system of equations*

$$\left.\begin{aligned} \xi \frac{dz_1}{d\xi} &= u_1(\xi, z_1(\xi), z_2(\xi)), \\ \xi \frac{dz_2}{d\xi} &= u_2(\xi, z_1(\xi), z_2(\xi)), \end{aligned}\right\}$$

functions u_1 *and* u_2 *are holomorphic functions of* ξ, $z_1(\xi)$, *and* $z_2(\xi)$ *near the origin, moreover* $u_1(0,0,0) = u_2(0,0,0) = 0$. *Then there exists a holomorphic solution satisfying the initial conditions* $z_1(0) = 0$, $z_2(0) = 0$ *if none of the eigenvalues of the matrix*

$$\begin{bmatrix} \left.\dfrac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} & \left.\dfrac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} \\ \left.\dfrac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} & \left.\dfrac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} \end{bmatrix}$$

is a positive integer.

$\rightarrow$ the existence of formal solutions $z_1 = \sum_{k=1}^{\infty} a_k \xi^k$ and $z_2 = \sum_{k=1}^{\infty} b_k \xi^k$,

and also that the formal solutions are convergent.

Theorem *For any* $p \in (0, +\infty)$, $q \in (0, +\infty)$, $i = 0,1$, $n \in \mathbf{N}$ *the initial value problem of (E) with* $y(0) = A$, $y'(0) = 0$ *has an unique analytic solution of the form* $y(t) = Q\left(t^{1+1/p}\right)$ *in* $(0, a)$, *where* Q *is a holomorphic solution to*

$$Q'' = \frac{(-1)^{i+1}}{p\,(1+1/p)^{p+1}} t^{-\frac{p+1}{p}} \frac{\Phi_q(Q)}{|Q'|^{p-1}} - \frac{n}{p+1} t^{-(1+1/p)} Q'$$

near zero satisfying $Q(0) = A$, $Q'(0) = (-1)^{i+1} \frac{p}{p+1} \left(|A|^{q-1} A / n\right)^{\frac{1}{p}}$.

Let us take solution of (E) in the form

$$y(t) = Q\left(t^{\alpha}\right),$$

where function $Q \in C^2(0,a)$ and α is a positive constant. Substituting $y(t) = Q\left(t^{\alpha}\right)$ into (E) we get that Q satisfies

$$Q''\left(t^{\alpha}\right) = \frac{(-1)^{i+1}}{p\,\alpha^{p+1}} t^{-(\alpha-1)(p+1)} \frac{\Phi_q(Q)}{|Q'|^{p-1}} - \frac{n-1+p(\alpha-1)}{p\,\alpha} t^{-\alpha} Q'$$

and taking $\xi = t^{\alpha}$ we have

$$Q''(\xi) = \frac{(-1)^{i+1}}{p\,\alpha^{p+1}} \xi^{-\frac{(\alpha-1)}{\alpha}(p+1)} \frac{\Phi_q(Q)}{|Q'|^{p-1}} - \frac{n-1+p(\alpha-1)}{p\,\alpha} \xi^{-1} Q'.$$

Here we introduce function Q in the form

$$Q(\xi) = \gamma_0 + \gamma_1 \xi + z(\xi),$$

where $z \in C^2(0,a)$, $z(0) = 0$, $z'(0) = 0$, therefore Q must have the properties $Q(0) = \gamma_0$, $Q'(0) = \gamma_1$, $Q'(\xi) = \gamma_1 + z'(\xi)$, $Q''(\xi) = z''(\xi)$. We restate

$$\left.\begin{aligned} z_1(\xi) &= z(\xi) \\ z_2(\xi) &= z'(\xi) \end{aligned}\right\} \text{ with } \left.\begin{aligned} z_1(0) &= 0 \\ z_2(0) &= 0 \end{aligned}\right\},$$

therefore

$$z''(\xi) = \frac{(-1)^{i+1}}{p\,\alpha^{p+1}}\,\xi^{-\frac{(\alpha-1)}{\alpha}(p+1)}\,\frac{\Phi_q\left(\gamma_0+\gamma_1\xi+z(\xi)\right)}{\left|\gamma_1+z'(\xi)\right|^{p-1}} - \frac{n-1+p(\alpha-1)}{p\,\alpha}\,\xi^{-1}\left(\gamma_1+z'(\xi)\right).$$

We generate the system of equations

$$\left.\begin{array}{l} u_1(\xi,\ z_1(\xi),\ z_2(\xi)) = \xi\ z_1'(\xi) \\ u_2(\xi,\ z_1(\xi),\ z_2(\xi)) = \xi\ z_2'(\xi) \end{array}\right\}$$

as follows

$$\left.\begin{array}{l} u_1(\xi,\ z_1(\xi),\ z_2(\xi)) = \xi\ z_2 \\ u_2(\xi,\ z_1(\xi),\ z_2(\xi)) = \dfrac{(-1)^{i+1}}{p\,\alpha^{p+1}}\,\xi^{\frac{1-p(\alpha-1)}{\alpha}}\,\dfrac{\Phi_q\left(\gamma_0+\gamma_1\xi+z_1(\xi)\right)}{\left|\gamma_1+z_2(\xi)\right|^{p-1}} \\ \qquad - \dfrac{n-1+p(\alpha-1)}{p\,\alpha}\left(\gamma_1+z_2(\xi)\right) \end{array}\right\}.$$

In order to satisfy conditions $u_1(0,0,0)=0$ and $u_2(0,0,0)=0$ we must get zero for the power of ξ :

$$\frac{1-p(\alpha-1)}{\alpha}=0,$$

i.e., $\alpha=\frac{1}{p}+1$. To ensure $u_2(0,0,0)=0$ we have the connection

$$n|\gamma_1|^{p-1}\gamma_1+\left(\frac{p}{p+1}\right)^p(-1)^i|\gamma_0|^{q-1}\gamma_0=0,$$

i.e.,

$$\gamma_1=(-1)^{i+1}\frac{p}{p+1}\left(\frac{\Phi_q(\gamma_0)}{n}\right)^{\frac{1}{p}}.$$

For u_1 and u_2 we find that

$$\left.\frac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} = 0, \qquad \left.\frac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} = 0,$$

$$\left.\frac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} = -\frac{p^p q\,|\gamma_0|^{q-1}}{(p+1)^{p+1}|\gamma_1|^{p-1}}, \qquad \left.\frac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} = -\frac{n\;p}{p+1}.$$

Therefore the eigenvalues of the matrix

$$\begin{bmatrix} \partial u_1/\partial z_1 & \partial u_1/\partial z_2 \\ \partial u_2/\partial z_1 & \partial u_2/\partial z_2 \end{bmatrix}$$

at $(0,0,0)$ are 0 and $-np/(p+1)$.

Corollary *From Theorem 2 it follows that solution* $y(t)$ *for(E) has an expansion of the form* $y(t) = \sum_{k=0}^{\infty} a_k \, t^{k\left(\frac{1}{p}+1\right)}$ *satisfying* $y(0) = A$ *and* $y'(0) = 0.$

Determination of local solution

We seek a solution of (E) with $y(0) = 1,\ y'(0) = 0$ in the form

$$y(t) = a_0 + a_1\, t^{\frac{1}{p}+1} + a_2\, t^{2\left(\frac{1}{p}+1\right)} + \dots .$$

Differentiation yields

$$y'(t) = t^{\frac{1}{p}} \left[a_1\left(\frac{1}{p}+1\right) + 2a_2\left(\frac{1}{p}+1\right) t^{\frac{1}{p}+1} + 3a_3\left(\frac{1}{p}+1\right) t^{2\left(\frac{1}{p}+1\right)} + \dots \right],$$

thus we get

$$\left(y'(t)\right)^p = t \left[a_1\left(\frac{1}{p}+1\right) + 2a_2\left(\frac{1}{p}+1\right) t^{\frac{1}{p}+1} + 3a_3\left(\frac{1}{p}+1\right) t^{2\left(\frac{1}{p}+1\right)} + \dots \right]^p .$$

...

$$y^{q}(t)=\left(a_0+a_1\,t^{\frac{1}{p}+1}+a_2\,t^{2\left(\frac{1}{p}+1\right)}+\ldots\right)^{q}$$

$$=A_0+A_1\,t^{\frac{1}{p}+1}+A_2\,t^{2\left(\frac{1}{p}+1\right)}+\ldots$$

$$\left(y'(t)\right)^{p}=t\left[a_1\left(\frac{1}{p}+1\right)+2a_2\left(\frac{1}{p}+1\right)\,t^{\frac{1}{p}+1}+3a_3\left(\frac{1}{p}+1\right)\,t^{2\left(\frac{1}{p}+1\right)}+\ldots\right]^{p}$$

$$=t\left[B_0+B_1\,t^{\frac{1}{p}+1}+B_2\,t^{2\left(\frac{1}{p}+1\right)}+\ldots\right],$$

where coefficients A_i and B_i can be expressed in terms of a_i ($i=0,1,\ldots$).

We will use the J. C. P. Miller formula:

$$A_k = \frac{1}{k}\sum_{j=0}^{k-1}[(k-j)q-j]A_j a_{k-j}, \qquad k>0$$

$$B_k = \frac{p}{a_1 k(p+1)}\sum_{j=0}^{k-1}[(k-j)p-j]B_j a_{k-j+1}\left[(k-j+1)\left(\frac{1}{p}+1\right)\right]$$

From the differential equation we have:

$$t^{n-1}\left[B_0 n + B_1\left(n+\frac{1}{p}+1\right)t^{\frac{1}{p}+1} + B_2\left(n+2\left(\frac{1}{p}+1\right)\right)t^{2\left(\frac{1}{p}+1\right)} + \ldots\right]$$

$$+(-1)^i t^{n-1}\left[A_0 + A_1\, t^{\frac{1}{p}+1} + A_2\, t^{2\left(\frac{1}{p}+1\right)} + \ldots\right] = 0$$

From where for the coefficients: $\quad B_k\left(n+k\left(\frac{1}{p}+1\right)\right)+(-1)^i A_k = 0 \quad k \geq 0$

From initial condition $y(0)=1$ we get $a_0 = 1$, $A_0 = 1$, therefore

$$B_0 = (-1)^{i+1}\frac{1}{n}$$

From $B_1(n+\frac{1}{p}+1)+(-1)^i A_1 = 0$, we find

$$B_0 = \left[a_1\left(\frac{1}{p}+1\right)\right]^p,$$

thus

$$a_1 = \frac{p}{p+1}\left[(-1)^i\frac{1}{n}\right]^{\frac{1}{p}}.$$

The evaluation of coefficients a_k, $k \geq 1$ from $B_k(n+k\left(\frac{1}{p}+1\right))+(-1)^i A_k = 0$ we can perform by MAPLE.

Example: `n:=2; i:=0; p:=0.5; q:=1`

$$\left(t\ \Phi_{0.5}(y')\right)' + t\ \Phi_1(y) = 0, \text{ on } (0,\ a)$$

$$y(0) = 1,\ \ y'(0) = 0.$$

Solution:

$y(t) = 1 - 0.2222222222\ t^3 + 0.0370370370\ t^6 - 0.0047031158\ t^9 + 0.0005443421\ t^{12}$
$-0.0000580631\ t^{15} + 0.0000058931\ t^{18} - 0.0000005750\ t^{21} + 0.0000000544\ t^{24}$
$-0.0000000050\ t^{27}$

Existence of local analytic solution (Initial conditions: $y(0)=0,\ y'(0)=B$)

Theorem *For any* $p\in(0,\ +\infty),\ q\in(0,\ +\infty),\ i=0,1,\quad n=1$ *the initial value problem of (E) with* $y(0)=0,\ y'(0)=B$ *has an unique analytic solution of the form* $y(t)=t\,Q\left(t^{q+1}\right)$ *in* $(0,\ a)$, *where* Q *is a holomorphic solution to*

$$Q''=-\frac{t^{-(q+1)}}{(q+1)}\left[(q+2)Q'+(-1)^i\frac{\Phi_q(Q)}{p(q+1)\Phi_{p-1}(Q+(q+1)t^{q+1}Q')}\right]$$

near zero satisfying $Q(0)=B,\quad Q'(0)=(-1)^{i+1}\frac{1}{p(q+1)(q+2)}B^{q-p+1}$.

Example: **n:=1; i:=0; p:=3; q:=2**

$$\left(\Phi_3(y')\right)'+\Phi_2(y)=0,\ \text{on}\ (0,\ a)$$

$$y(0)=0,\quad y'(0)=1.$$

Solution:

$$y(t)=t\,(1-0.0625t^3-0.0066964t^6-0.0017439t^9-0.0006940t^{12}-0.00036558t^{15}$$

$$-0.00020711t^{18}-0.00012588t^{21}-0.00008100t^{24})$$

Case: $p = q$

$$\left(\Phi_p(y')\right)' + (-1)^i \Phi_p(y) = 0$$

$$y(0) = 0, \quad y'(0) = 1$$

$$y(t) = t\left(\beta_0 + \beta_1 t^{p+1} + \beta_2 t^{2(p+1)} + \beta_3 t^{3(p+1)} + \beta_4 t^{4(p+1)} + \ldots\right)$$

$$y(t) = t\left(\beta_0 + \beta_1 t^{p+1} + \beta_2 t^{2(p+1)} + \beta_3 t^{3(p+1)} + \beta_4 t^{4(p+1)} + \ldots\right), \qquad i = 0,$$

$$\beta_0 = 1$$

$$\beta_1 = -\frac{1}{(p+1)(p+2)}$$

$$\beta_2 = -\frac{p^2 - 2}{2(p+1)^2(p+2)(2p+3)}$$

$$\beta_3 = -\frac{4p^5 + 9p^4 - 6p^3 - 19p^2 + 3p + 12}{6(p+1)^3(p+2)^2(2p+3)(3p+4)}$$

$$\beta_4 = -\frac{36p^8 + 164p^7 + 167p^6 - 220p^5 - 400p^4 + 76p^3 + 288p^2 - 24p - 96}{24(p+1)^4(p+2)^3(2p+3)(3p+4)(4p+5)}$$

Remarks

(i) Generally $$\beta_n = \frac{P_n(p)}{n!(p+1)^n (p+2)^{\gamma_{n1}} (2p+3)^{\gamma_{n2}} \ldots [n(p+1)+1]^{\gamma_{nn}}},$$

where $P_n(p)$ is a polynomial in p with integer coefficients of degree $n-2+\gamma_{n1}+\gamma_{n2}+\ldots+\gamma_{nn}$,

$$\gamma_{nj} = \begin{cases} \left[\frac{n-1}{j}\right], & if \quad j < n, \\ 1, & if \quad j = n. \end{cases}$$

(ii) The coefficient of $P_n(p)$ concerning the maximal power in p is divisible by $((n-1)!)^2$.

(iii) Coefficients β_n are rational functions of p with poles at $-1, -2, -\frac{3}{2}, \ldots, -\frac{n-1}{n}$.

Exponential expansions

$$y''|y'|^{p-1} - y|y|^{p-1} = 0, \quad p > 0,$$
$$y(0) = 0, \ \ y'(0) = 1 \text{ or } y(0) = 1, \ \ y'(0) = 0$$

The first integral

$$|y'|^{p+1} - |y|^{p+1} = C, \text{ where } C = 1 \text{ for } Sh_p \text{ and } C = -1 \text{ for } Ch_p.$$

Exponential expansion of the form: $y = Ae^x\left(1 + a_1\left(Ae^x\right)^{-(p+1)} + a_2\left(Ae^x\right)^{-2(p+1)} + \ldots\right)$

$$y' = Ae^x\left(1 + (-p)a_1\left(Ae^x\right)^{-(p+1)} + (-1-2p)a_2\left(Ae^x\right)^{-2(p+1)} + \ldots\right)$$

Substitution $\left(Ae^x\right)^{-(p+1)} = u$

$$y^p = u^{-1}\left(1 + a_1u + a_2u^2 + \ldots\right)^{p+1} = u^{-1}\left(1 + \alpha_1u + \alpha_2u^2 + \ldots\right)$$

$$y'^p = u^{-1}\left(1 + (-p)a_1u + (-1-2p)a_2u^2 + \ldots\right)^{p+1} = u^{-1}\left(1 + \beta_1u + \beta_2u^2 + \ldots\right)$$

From the diff.equ.: $\alpha_k = \beta_\mathrm{k},\ k > 1$ and

$\alpha_1 = \beta_1 - 1$ for Sh_p and $\alpha_1 = \beta_1 + 1$ for Ch_p

Determination of α_n and β_n for $n = 0,1,2,\ldots$ by J.C.P. Miller formula:

$$\alpha_0 = 1, \quad \alpha_n = \frac{1}{n}\sum_{k=0}^{n-1}\left[(p+1)(n-k)-k\right]\alpha_k\, a_{n-k}, \qquad n > 1$$

$$\beta_0 = 1, \quad \beta_n = \frac{1}{n}\sum_{k=0}^{n-1}\left[(p+1)(n-k)-k\right]\beta_k\, a_{n-k}\left[1-(p+1)(n-k)\right], \qquad n > 1$$

Recursive formula

$$a_n = \pm\frac{1}{n^2(p+1)^2}\left[(p+1)(n-1)-1\right] a_{n-1}(p+1-np)$$

$$-\frac{1}{n^2(p+1)}\sum_{k=2}^{n-1}\left[(p+1)(n-k)-k\right]\alpha_k\, a_{n-k}(n-k) \quad \mathit{for}\ n > 1, \quad \mathit{and} \quad a_1 = \pm\left(-\frac{1}{(p+1)^2}\right)$$

+ sign for Sh_p and - sign for Ch_p.

$$y = Ae^x\left(1 + a_1\left(Ae^x\right)^{-(p+1)} + a_2\left(Ae^x\right)^{-2(p+1)} + \ldots\right) \qquad A = ?$$

For Sh_p: $\quad |y'|^{p+1} - |y|^{p+1} = 1, \;\; y' > 0, \;\; y > 0 \qquad x = \int_0^y \frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}}$

$$x = \int_0^1 \frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}} + \int_1^y \left[\frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}} - \frac{d\sigma}{\sigma}\right] + \int_1^y \frac{d\sigma}{\sigma}, \qquad x = \ln y - \mathrm{B} + o(1), \text{ where}$$

$$B = -\int_0^1 \frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}} - \int_1^\infty \left[\frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}} - \frac{d\sigma}{\sigma}\right] = \frac{1}{p+1}\left(\frac{\Gamma'\left(\frac{1}{p+1}\right)}{\Gamma\left(\frac{1}{p+1}\right)} - \frac{\Gamma'(1)}{\Gamma(1)}\right)$$

$M = -\frac{\Gamma'(1)}{\Gamma(1)} \approx 0.577215665\ldots$ (Euler-Mascheroni or Stirling number)

$$A = e^B$$

For Ch_p: $\quad |y'|^{p+1} - |y|^{p+1} = -1, \quad y' > 0, \quad y > 0 \qquad x = \int_0^y \frac{d\sigma}{(\sigma^{p+1} - 1)^{1/(p+1)}}$

$x = \int_1^y \left[\frac{d\sigma}{(\sigma^{p+1} - 1)^{1/(p+1)}} - \frac{d\sigma}{\sigma} \right] + \int_1^y \frac{d\sigma}{\sigma}, \qquad x = \ln y - \overline{\mathrm{B}} + o(1)$, where

$$\overline{B} = -\int_1^\infty \left[\frac{d\sigma}{(\sigma^{p+1} - 1)^{1/(p+1)}} - \frac{d\sigma}{\sigma} \right] = \frac{1}{p+1} \left(\frac{\Gamma'\left(\frac{p}{p+1}\right)}{\Gamma\left(\frac{p}{p+1}\right)} - \frac{\Gamma'(1)}{\Gamma(1)} \right)$$

$$\overline{A} = e^{\overline{B}}$$

If $y(x,\tilde{A}) = \tilde{A}e^{x}\left(1 + a_1\left(\tilde{A}e^{x}\right)^{-(p+1)} + a_2\left(\tilde{A}e^{x}\right)^{-2(p+1)} + \dots\right)$ is the solution of $y'^{p+1} - y^{p+1} = 1$, then $\tilde{y}(x,\tilde{A}) = \tilde{A}e^{x}\left(1 - a_1\left(\tilde{A}e^{x}\right)^{-(p+1)} + a_2\left(\tilde{A}e^{x}\right)^{-2(p+1)} - \dots\right)$ is the solution of $y'^{p+1} - y^{p+1} = -1$.

$$\tilde{y}(x,\tilde{A}) = e^{-i\frac{\pi}{p+1}} y(x + i\frac{\pi}{p+1}, \tilde{A})$$

Formula (Connections between Sh_p and Ch_p)

$$Ch_p(x) = e^{-i\frac{\pi}{p+1}} Sh_p(x + \overline{B} - B + i\frac{\pi}{p+1}),$$

$$Sh_p(x) = e^{i\frac{\pi}{p+1}} Ch_p(x - \overline{B} + B - i\frac{\pi}{p+1})$$

$p = 1$:

$$\cosh x = \frac{1}{i}\sinh\left(x + i\frac{\pi}{2}\right),$$

$$\sinh x = i\sinh\left(x - i\frac{\pi}{2}\right)$$

only in this case $B = \overline{B}$.

Radius of convergence: an upper bound for R

$$\frac{1}{R} = \overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}} \geq e^{-\rho \min(B,\overline{B})}$$

In terms of ρ this is

$$\rho \geq \min(B, \overline{B}).$$