Periodic solutions for difference equations with ϕ-Laplacian

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For $\mathbf{x} \in \mathbb{R}^p$ set $|\mathbf{x}|_1 = \sum_{k=1}^{p} |x_k|$, $\mathbf{x}^{\pm} = (x_1^{\pm}, \cdots, x_p^{\pm})$ and

$$Q(\mathbf{x}) = \frac{1}{p} \sum_{k=1}^{p} x_k.$$

If $\alpha, \beta \in \mathbb{R}^p$, we write $\alpha \leq \beta$ (resp. $\alpha < \beta$) if $\alpha_k \leq \beta_k$ for all $1 \leq k \leq p$ (resp. $\alpha_k < \beta_k$ for all $1 \leq k \leq p$).

Let $n \in \mathbb{N}, n \geq 4$ be fixed and $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Define

$$D\mathbf{x} = (Dx_1, \cdots, Dx_{n-1}) \in \mathbb{R}^{n-1}$$

by

$$Dx_k = x_{k+1} - x_k, \quad (1 \leq k \leq n-1)$$

and, if $\max_{1 \leq k \leq n-1} |Dx_k| < a$, define

$$D\phi(D\mathbf{x}) = (D\phi(Dx_2), \cdots, D\phi(Dx_{n-1})) \in \mathbb{R}^{n-2}$$

by

$$D\phi(Dx_k) = \phi(Dx_k) - \phi(Dx_{k-1}), \quad (2 \leq k \leq n-1).$$

If $f = (f_2, \cdots, f_{n-1})$ is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^{n-2}$ we define the continuous function $N_f : \mathbb{R}^n \to \mathbb{R}^{n-2}$ by

$$N_f(\mathbf{x}) = (f_2(x_2, Dx_2), \cdots, f_{n-1}(x_{n-1}, Dx_{n-1})).$$

Notice that,

$$QN_f(\mathbf{x}) = \frac{1}{n-2} \sum_{k=2}^{n-1} f_k(x_k, Dx_k) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

B_ρ denotes de open ball of center 0 and radius ρ.
We study the existence of the solutions for the problems

$$D\phi(D\mathbf{x}) + N_f(\mathbf{x}) = 0, \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}.$$

Forced equations

Assume that ϕ is *$singular$* or *$classical$*.

$$D\phi(D\mathbf{x}) = \mathbf{h}, \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}, \tag{1}$$

where $\mathbf{h} = (h_2, \cdots, h_{n-1}) \in \mathbb{R}^{n-2}$.

Forced equations

Assume that ϕ is *singular* or *classical*.

$$D\phi(D\mathbf{x}) = \mathbf{h}, \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}, \tag{2}$$

where $\mathbf{h} = (h_2, \cdots, h_{n-1}) \in \mathbb{R}^{n-2}$.

Lemma 2 *For each $\mathbf{h} = (h_2, \cdots, h_{n-1}) \in \mathbb{R}^{n-2}$ there exist a unique $\gamma := Q_\phi(\mathbf{h})$ such that*

$$2\phi^{-1}(\gamma) + \sum_{k=3}^{n-1} \phi^{-1}\Big(\sum_{j=2}^{k-1} h_j + \gamma\Big) = 0.$$

Moreover, the function Q_ϕ is continuous.

Proposition 1 *Forced periodic problem (1) is solvable iff*

$$\sum_{k=2}^{n-1} h_k = 0$$

holds in which case solutions of (1) are of the form

$$(\frac{x_2 + x_{n-1}}{2}, x_2, \cdots, x_{n-1}, \frac{x_2 + x_{n-1}}{2}),$$

where $x_2 \in \mathbb{R}$ *and*

$$x_k = x_2 + \sum_{j=3}^{k} \phi^{-1}(\sum_{l=2}^{j-1} h_l + Q_\phi(\mathbf{h})) \quad (3 \le k \le n-1).$$

When ϕ is *bounded*, then we need the following supplementary condition which is necessary and sufficient together with $Q(\mathbf{h}) = 0$.

$$\exists \gamma \in \,]-a, a[: \quad \sum_{j=2}^{k-1} h_j + \gamma \in \,]-a, a[\quad (3 \leq k \leq n-1).$$

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For example, this is the case when

$\left|\sum_{k=2}^{l-1} h_k\right| < a \quad (3 \leq l \leq n-1)$ or

$a \leq \sum_{k=2}^{l-1} h_k < 2a \quad (3 \leq l \leq n-1)$ or

$-2a < \sum_{k=2}^{l-1} h_k \leq a \quad (3 \leq l \leq n-1)$.

A continuum of solutions

Assume that ϕ is *$singular$*. Let us introduce the vector space

$$\widetilde{V}^{n-2} = \{\widetilde{\mathbf{x}} \in \mathbb{R}^n : \widetilde{x}_1 = \widetilde{x}_n,\ D\widetilde{x}_1 = D\widetilde{x}_{n-1},\ \widetilde{x}_2 = 0\}$$

endowed with the orientation of $\mathbb{R}^n$ and the norm $\|\widetilde{\mathbf{x}}\| := \max_{3 \le k \le n-1} |\widetilde{x}_k|$. Its elements correspond to the elements of $\mathbb{R}^n$ of the form $(\frac{\widetilde{x}_{n-1}}{2}, 0, \widetilde{x}_3, \cdots, \widetilde{x}_{n-1}, \frac{\widetilde{x}_{n-1}}{2})$. If $(s, \widetilde{x}) \in \mathbb{R} \times \widetilde{V}^{n-2}$ is a solution of the problem

$$D\phi(D\widetilde{x}_k) = f_k(s + \widetilde{x}_k, D\widetilde{x}_k) - \frac{1}{n-2} \sum_{j=2}^{n-1} f_j(s + \widetilde{x}_j, D\widetilde{x}_j)$$

$$(2 \le k \le n-1) \qquad (3)$$

then $\mathbf{x} = s + \widetilde{\mathbf{x}}$ is a solution of the problem

$$D\phi(D\mathbf{x}) + N_f(\mathbf{x}) = QN_f(\mathbf{x}), \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}.$$

For each fixed $s \in \mathbb{R}$, problem (3) is equivalent to the fixed point problem in $\widetilde{V}^{n-2} : \widetilde{\mathbf{x}} = \widetilde{\mathcal{P}}(s, \widetilde{\mathbf{x}})$, where $\widetilde{\mathcal{P}}(s, \widetilde{\mathbf{x}}) = \widetilde{\mathbf{y}}$ and, for $3 \leq k \leq n-1$,

$$\begin{aligned}\widetilde{y}_k &= \sum_{j=3}^{k} \phi^{-1}\{\sum_{l=2}^{j-1}[f_l(s+\widetilde{x}_l, D\widetilde{x}_l) - \frac{1}{n-2}\sum_{m=2}^{n-1} f_m(s+\widetilde{x}_m, D\widetilde{x}_m)] \\ &+ Q_\phi([f_m(s+\widetilde{x}_m, D\widetilde{x}_m) - \frac{1}{n-2}\sum_{l=2}^{n-1} f_l(s+\widetilde{x}_l, D\widetilde{x}_l)]_{m=2}^{n-1})\}.\end{aligned}$$

Lemma 3 *The set $\mathcal{S}$ of the solutions $(s, \widetilde{\mathbf{x}}) \in \mathbb{R} \times \widetilde{V}^{n-2}$ of problem*

$$\widetilde{\mathbf{x}} = \widetilde{\mathcal{P}}(s, \widetilde{\mathbf{x}})$$

contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and projection on $\widetilde{V}^{n-2}$ is contained in the ball $B_{(n-3)a}$.

Villari-type nonlinearities

Theorem 1 *Assume that* ϕ *is* $singular$ *and that there exists* $R > 0$ *and* $\epsilon \in \{-1, 1\}$ *such that*

$$\epsilon \sum_{k=2}^{n-1} f_k(x_k, Dx_k) \geq 0 \quad \textit{if} \quad \min_{2\leq k\leq n-1} x_k \geq R, \max_{2\leq k\leq n-2} |Dx_k| < a,$$

$$\epsilon \sum_{k=2}^{n-1} f_k(x_k, Dx_k) \leq 0 \quad \textit{if} \quad \max_{2\leq k\leq n-1} x_k \leq -R, \max_{2\leq k\leq n-2} |Dx_k| < a.$$

Then

$$D\phi(D\mathbf{x}) + N_f(\mathbf{x}) = 0, \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}.$$

has at least one solution.

Example 1 *If* $\mathbf{e} = (e_2, \cdots, e_{n-1}), c \in \mathbb{R} \setminus 0$, $d \in \mathbb{R}$, $q \geq 0$ *and* $p > 1$, *then the problem*

$$D\left(\frac{Dx_k}{\sqrt{1-(Dx_k)^2}}\right) + d|Dx_k|^q + c|x_k|^{p-1}x_k = e_k \quad (2 \leq k \leq n-1)$$

$$x_1 = x_n, \quad Dx_1 = Dx_{n-1},$$

has at least one solution.

Corollary 1 *Assume that* ϕ *is* $singular$. *Let* $h_k : \mathbb{R}^2 \to \mathbb{R}$ $(2 \leq k \leq n-1)$ *be bounded on* $\mathbb{R}\times]-a, a[$. *Then, for each* $\mu \neq 0$, *the following problem has at least one solution:*

$$D\phi(Dx_k) + \mu x_k = h_k(x_k, Dx_k) \quad (2 \leq k \leq n-1)$$

$$x_1 = x_n, \quad Dx_1 = Dx_{n-1}.$$

Lemma 4 *Assume that* $\phi :]-b, b[\to]-a, a[$ *where* $0 < a, b \leq \infty$. *Let* $\mathbf{x}$ *be a solution of*

$$D\phi(D\mathbf{x}) + N_f(\mathbf{x}) = 0, \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}. \tag{4}$$

and assume that there exists $\mathbf{c} = (c_2, \ldots, c_{n-1}) \in \mathbb{R}^{n-2}$ *such that* $|\mathbf{c}^-|_1 < a$ *and*

$$f_k(u, v) \geq c_k, \ \forall (u, v) \in \mathbb{R}^2, \ 2 \leq k \leq n-1. \tag{5}$$

holds. Then

$$\max_{2 \leq k \leq n-1} |Dx_k| \leq M_\phi,$$

where

$$M_\phi = \max\{|\phi^{-1}(|\mathbf{c}^-|_1)|, |\phi^{-1}(-|\mathbf{c}^-|_1)|\}.$$

Theorem 2 *Assume that* $\phi : \mathbb{R} \to]-a, a[$ *where* $0 < a \leq \infty$. *Assume also that that there exists* $\mathbf{c}$ *like in Lemma 4. If there exist* $M' > M_\phi, R > 0$ *and* $\epsilon \in \{-1, 1\}$ *such that*

$$\epsilon \sum_{k=2}^{n-1} f_k(x_k, Dx_k) \geq 0 \quad \textit{if} \quad \min_{2 \leq k \leq n-1} x_k \geq R, \max_{1 \leq k \leq n-1} |Dx_k| < M',$$

$$\epsilon \sum_{k=2}^{n-1} f_k(x_k, Dx_k) \leq 0 \quad \textit{if} \quad \max_{2 \leq k \leq n-1} x_k \leq -R, \max_{1 \leq k \leq n-1} |Dx_k| < M',$$

then problem (4) has at least one solution.

(B - H.B. Thompson)

Corollary 2 *Assume that* $\phi : \mathbb{R} \to \,]-a, a[$ *where* $0 < a \leq \infty$. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous, bounded function such that* $g \leq 0$, $\mathbf{h} = (h_2, \cdots, h_{n-1})$, $|\mathbf{h}^-|_1 < a$ *and the following Landesman-Lazer type condition is satisfied*

$$\limsup_{u \to -\infty} g(u) < \frac{1}{n-2} \sum_{k=2}^{n-1} h_k < \liminf_{u \to \infty} g(u).$$

Then the problem

$$D\phi(Dx_k) + g(x_k) = h_k \quad (2 \leq k \leq n-1),$$
$$x_1 = x_n, \quad Dx_1 = Dx_{n-1},$$

has at least one solution.

Corollary 3 *Assume that* $\phi : \mathbb{R} \to]-a, a[$ *where* $0 < a \leq \infty$. *Let* $p > 0, a_k > 0, b_k \in \mathbb{R}\ (2 \leq k \leq n-1)$ *such that* $\sum_{k=2}^{n-1} b_k^+ < a$. *Then the periodic problem*

$$D\phi(Dx_k) + a_k(x_k^+)^p = b_k \quad (2 \leq k \leq n-1)$$
$$x_1 = x_n, \quad Dx_1 = Dx_{n-1},$$

has at least one solution if and only if $\sum_{k=2}^{n-1} b_k \geq 0$. *When* $a_k < 0, b_k \in \mathbb{R}\ (2 \leq k \leq n-1)$ *such that* $\sum_{k=2}^{n-1} b_k^- < a$, *problem bellow has at least one solution if and only if* $\sum_{k=2}^{n-1} b_k \leq 0$.

Upper and lower solutions

Theorem 3 *If*

$$D\phi(D\mathbf{x}) + N_f(\mathbf{x}) = 0, \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}.$$

has a $\textit{lower solution}$ α *and an* $\textit{upper solution}$ β *such that* $\alpha \leq \beta$*, then the problem bellow has a solution* $\mathbf{x}$ *such that* $\alpha \leq \mathbf{x} \leq \beta$*. (In the classical or bounded case the nonlinearity does not depend on the derivative).*

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has a $\mathit{lower\ solution}$ α *and an* $\mathit{upper\ solution}$ β *such that* $\alpha \leq \beta$, *then the problem bellow has a solution* $\mathbf{x}$ *such that* $\alpha \leq \mathbf{x} \leq \beta$. *(In the classical or bounded case the nonlinearity does not depend on the derivative).*

Let $\gamma_k : \mathbb{R} \to \mathbb{R}$, $(2 \leq k \leq n-1)$ be the continuous functions

$$\gamma_k(x) = \begin{cases} \beta_k, & x > \beta_k \\ x, & \alpha_k \leq x \leq \beta_k \\ \alpha_k, & x < \alpha_k, \end{cases}$$

and define $F_k(x,y) = f_k(\gamma_k(x), y), \quad (2 \leq k \leq n-1).$

We consider the modified problem

$$D\phi(Dx_k) - x_k + [F_k(x_k, Dx_k) + \gamma_k(x_k)] = 0 \quad (2 \le k \le n-1),$$
$$x_1 = x_n, \quad Dx_1 = Dx_{n-1},$$

We can show that if $\mathbf{x}$ is a solution of the modified problem then $\alpha \le \mathbf{x} \le \beta$ and hence $\mathbf{x}$ is a solution of the initial problem.

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We can show that if $\mathbf{x}$ is a solution of the modified problem then $\alpha \leq \mathbf{x} \leq \beta$ and hence $\mathbf{x}$ is a solution of the initial problem.
Assume that ϕ is $singular$. Using $Corollary$ 1, it follows that the modified problem has at least one solution.

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We can show that if $\mathbf{x}$ is a solution of the modified problem then $\alpha \le \mathbf{x} \le \beta$ and hence $\mathbf{x}$ is a solution of the initial problem.
Assume that ϕ is *classical* or *bounded.* In this case we apply the *Brouwer degree* to the homotopy $(0 \le \lambda \le 1)$

$$\lambda D\phi(Dx_k) - x_k + \lambda[F_k(x_k) + \gamma_k(x_k)] \quad (2 \le k \le n-1).$$

Using Lemmas 3, 4 and Theorem 3 we deduce the following result.

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Theorem 6 *Assume that ϕ is $singular$. If*

$$D\phi(D\mathbf{x}) + N_f(\mathbf{x}) = 0, \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}.$$

has a $lower\ solution$ α and an $upper\ solution$ β then the problem bellow has a solution $\mathbf{x}$ The result holds also in the $classical$ or $bounded$ case, if the supplementary condition (5) is satisfied.

In the classical case, the theorem above is a discrete version of a result due to *I. Rachůnková* and *M. Tvrdý*.

References

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