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Workshop on Differential Equations

Boundary Value Problems and Related Topics

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Initial and Boundary Value Quasilinear Problems of Higher Order

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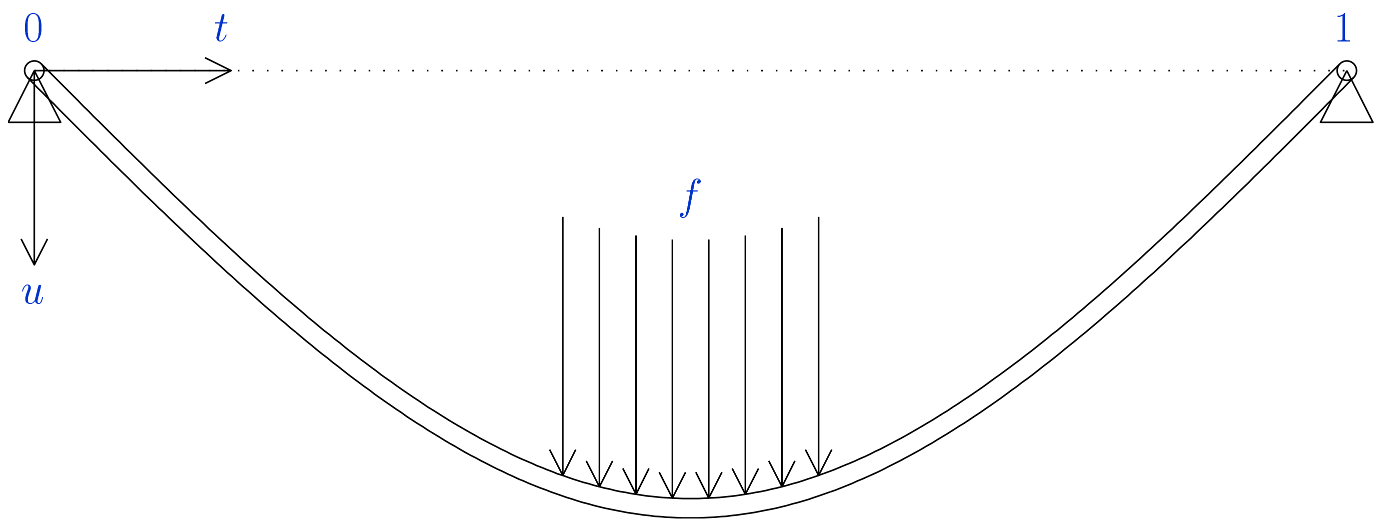
Contents

1. Introduction – motivation, goals
2. Initial value problems for the $2n$th-order quasilinear equations, in particular second-order
   - existence and uniqueness of the solution
   - continuous dependence of the solution on the initial conditions and parameters
3. Boundary value problems for the fourth-order quasilinear equations
   - simplicity, positivity of the eigenvalues
   - discreteness of the spectrum,
     isolation and nodal properties of the eigenfunctions,
     interlacing property,
     continuous dependence of the eigenvalues on p
   - global bifurcation theorem (Rabinowitz type),
     existence theorem in nonresonance

Beam: a fourth-order problem



semilinear fourth-order Navier problem (ends supported, not clamped)

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t)), & t \in [0,1], \\ u(0) = u''(0) = u(1) = u''(1) = 0 \end{cases}$$

Dirichlet problem (clamped ends)

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t)), & t \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{1}$$

linear Dirichlet eigenvalue problem

$$\begin{cases} u^{(4)}(t) = \lambda u(t), & t \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{2}$$

weak solution of (2): critical point of $\Phi_2\colon W_0^{2,2}(0,1) \to \mathbb{R}$,

$$\Phi_2(u) \coloneqq \frac{1}{2}\int_0^1 |u''(t)|^2\,\mathrm{d}t - \frac{\lambda}{2}\int_0^1 |u(t)|^2\,\mathrm{d}t.$$

critical points of $\Phi_p\colon W_0^{2,p}(0,1) \to \mathbb{R}$,

$$\Phi_p(u) \coloneqq \frac{1}{p}\int_0^1 |u''(t)|^p\,\mathrm{d}t - \frac{\lambda}{p}\int_0^1 |u(t)|^p\,\mathrm{d}t, \qquad p \in (1,\infty)$$

— weak solutions of the *quasilinear* fourth-order Dirichlet eigenvalue problem

$$\begin{cases} \big(\psi_p(u''(t))\big)'' = \lambda\psi_p(u(t)), & t \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases}$$

where

$$\psi_p(s) \coloneqq \begin{cases} 0 & \text{for } s = 0, \\ |s|^{p-2}s & \text{for } s \neq 0 \end{cases}$$

$u \mapsto (\psi_p(u''))''$: p-biharmonic operator ($u^{(4)}$ for $p = 2$)

$u \mapsto (\psi_p(u'))'$: p-Laplacian (u'' for $p = 2$)

denote $p^* = \dfrac{p}{p-1}$; ψ_p and ψ_{p^*} inverse functions

Goals

- spectral properties (simplicity, discreteness) of the Dirichlet eigenvalue problem

$$\begin{cases} \big(\psi_p(u''(t))\big)'' = \lambda \psi_p(u(t)), \qquad t \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{3}$$

  and the Neumann eigenvalue problem (free ends)

$$\begin{cases} \big(\psi_p(u''(t))\big)'' = \lambda \psi_p(u(t)), \qquad t \in [0,1], \\ u''(0) = (\psi_p(u''))'(0) = u''(1) = (\psi_p(u''))'(1) = 0 \end{cases} \tag{4}$$

- existence theorem, global bifurcation of nontrivial solutions of perturbed problems (3) and (4)

Initial value problem

[Benedikt, J., "*Uniqueness theorem for p-biharmonic equations*", Electron. J. Differential Equations **2002** (2002), № 53, pp. 1–17]
[Benedikt, J., "*Uniqueness theorem for quasilinear $2n$th-order equations*", J. Math. Anal. Appl. **293** (2004), pp. 589–604]

linear second-order initial value problem

$$\begin{cases} -u''(t) = \lambda u(t), \qquad t \geq t_0, \\ u(t_0) = \alpha, \quad u'(t_0) = \beta, \end{cases} \tag{5}$$

$\lambda, \alpha, \beta \in \mathbb{R}$

existence: ≥ 1 solution of the initial value problem
uniqueness: ≤ 1 solution of the initial value problem

local: on $[t_0, t_0 + \varepsilon]$ for an $\varepsilon > 0$
global: on $[t_0, +\infty)$ or $[t_0, t_1]$ for arbitrary $t_1 \geq t_0$

global uniqueness of the solution $\Rightarrow$ continuous dependence of the solution on the initial conditions and parameters

(5) is equivalent to

$$\begin{cases} u_0'(t) = u_1(t), \qquad u_0(t_0) = \alpha, \\ u_1'(t) = -\lambda u_0(t), \quad u_1(t_0) = \beta, \end{cases} \qquad t \geq t_0$$

right-hand side is Lipschitz continuous

quasilinear second-order initial value problem

$$\begin{cases} -\big(\psi_p(u'(t))\big)' = \lambda \psi_q(u(t)), & t \geq t_0, \\ u(t_0) = \alpha, \quad \psi_p(u'(t_0)) = \beta, \end{cases} \tag{6}$$

$p, q > 1$, is equivalent to

$$\begin{cases} u_0'(t) = \psi_{p^*}(u_1(t)), & u_0(t_0) = \alpha, \\ u_1'(t) = -\lambda \psi_q(u_0(t)), & u_1(t_0) = \beta, \end{cases} \qquad t \geq t_0,$$

where $u_0 = u$, $u_1 = \psi_p(u')$

if $p > 2$ or $q < 2$, then the right-hand side is not Lipschitz continuous (neither locally):

$$p > 2 \quad \Rightarrow \quad \psi_{p^*}'(0) = \infty \qquad\qquad q < 2 \quad \Rightarrow \quad \psi_q'(0) = \infty$$

methods:

local existence: Schauder fixed-point theorem, Peano theorem
global existence: boundedness
local uniqueness: special estimates
global uniqueness: trivial consequence of the local uniqueness

local existence and uniqueness for (6) with $\lambda \geq 0$: Drábek, Manásevich (1999)

$p \geq q$ or $\lambda \geq 0 \Rightarrow$ global existence
$p < q$ and $\lambda < 0$: counterexample to global existence (blow-up)

$p \leq q$ or $\lambda \geq 0 \Rightarrow$ global uniqueness
$p > q$ and $\lambda < 0$: counterexample to local uniqueness

generalization: $2n$th-order problem

$$\begin{cases} (-1)^n\big(\psi_p(u^{(n)}(t))\big)^{(n)} = \lambda\psi_q(u(t)), & t \ge t_0, \\ u^{(i)}(t_0) = \alpha_i, \quad (\psi_p(u^{(n)}))^{(j)}(t_0) = \beta_j, & i,j \in \{0,1,\dots,n-1\} \end{cases} \tag{7}$$

local existence and uniqueness for (7) with $n = 2$, $p = q$ and $\lambda > 0$: Drábek, Ôtani (2001)

$p \ge q$	**YES**	
$p < q$	$(-1)^n\lambda > 0$	**NO** (counterexample) – blow-up for certain initial conditions
	$\lambda = 0$	**YES** (trivial)
	$(-1)^n\lambda < 0$	**?** (**YES** for $n = 1$)

Global existence of the solution of (7)

$\sum_{i=0}^{n-1}\lvert\alpha_i\rvert + \sum_{j=0}^{n-1}\lvert\beta_j\rvert > 0$	**YES**		
$\alpha_0 = \cdots = \alpha_{n-1} =$ $= \beta_0 = \cdots = \beta_{n-1} = 0$	$p \le q$	**YES**	
	$p > q$	$(-1)^n\lambda > 0$	**NO** (counterexample)
		$\lambda = 0$	**YES** (trivial)
		$(-1)^n\lambda < 0$	**?** (**YES** for $n = 1$)

Local uniqueness of the solution of (7)

the most general problem

$$\begin{cases} (-1)^n\big(a(t)\psi_{p(t)}(u^{(n)}(t))\big)^{(n)} = b_1(t)\psi_{q_1(t)}(u^+(t)) - b_2(t)\psi_{q_2(t)}(u^-(t)), & t \geq t_0, \\ u^{(i)}(t_0) = \alpha_i,\ (a\psi_p(u^{(n)}))^{(j)}(t_0) = \beta_j, & i, j \in \{0, 1, \ldots, n-1\} \end{cases}$$

where $n \in \mathbb{N}$, $a, b_1, b_2, p, q_1, q_2 \in C$, $a > 0$, $p, q_1, q_2 > 1$,
$\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1} \in \mathbb{R}$,
$u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ (i.e., $u = u^+ - u^-$)

Boundary value problem

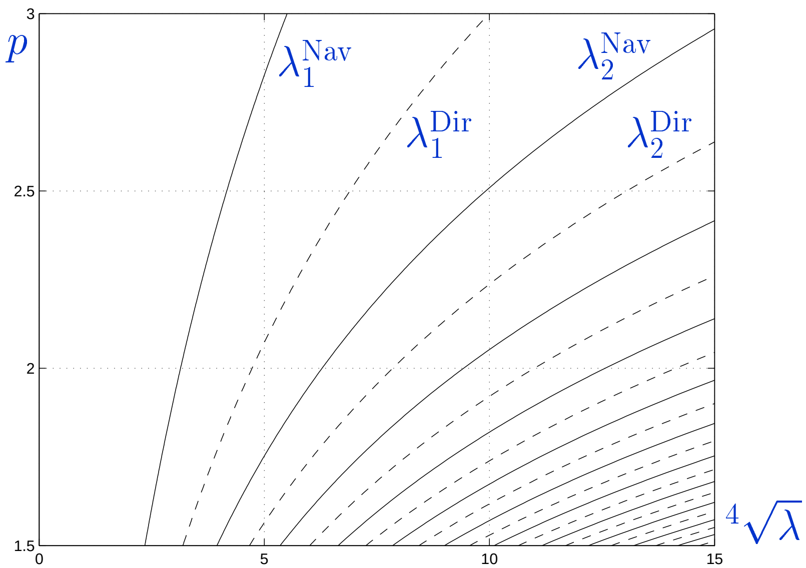
[Benedikt, J., "*On simplicity of spectra of p-biharmonic equations*",
Nonlinear Anal. **58** (2004), № 7–8, pp. 835–853.]
[Benedikt, J., "*On the discreteness of the spectra of the Dirichlet and Neumann p-biharmonic problem*",
Abstr. Appl. Anal. **293** (2004), pp. 589–604.]
[Benedikt, J., "*Continuous dependence of eigenvalues of p-biharmonic problems on p*"]

Kratochvíl, Nečas (1971): eigenvalues of the Dirichlet problem (3) for $p \geq 2$ form a sequence $0 < \lambda_1(p) < \lambda_2(p) < \cdots \to +\infty$, the set of corresponding eigenfunctions is discrete, there exist at most finitely many linearly independent eigenfunctions, corresponding to one eigenvalue

Pinkus (1985): there exists a sequence of positive simple eigenvalues of the Dirichlet problem (3), the eigenfunction, corresponding to the nth eigenvalue has precisely $n-1$ zeros in $(0,1)$

Dirichlet problem:

- the eigenvalues of (3) form a sequence $0 < \lambda_1^{\mathrm{Dir}}(p) < \lambda_2^{\mathrm{Dir}}(p) < \cdots \to +\infty$,
- every $\lambda_n^{\mathrm{Dir}}(p)$ is simple, isolated, the corresponding eigenfunction has $n-1$ zeros in $(0,1)$,
- the functions $p \mapsto \lambda_n^{\mathrm{Dir}}(p)$, $n \in \mathbb{N}$, are continuous on $(1,\infty)$,
- the set of all eigenfunctions is discrete in $C^2[0,1]$,
- $\lambda_n^{\mathrm{Nav}}(p) < \lambda_n^{\mathrm{Dir}}(p) < \lambda_{n+1}^{\mathrm{Nav}}(p)$ for all $n \in \mathbb{N}$ and $p \in (1,\infty)$



interlacing property

Neumann problem:

- the eigenvalues of (4) form a sequence $0 = \lambda_0^{\mathrm{Neu}}(p) < \lambda_1^{\mathrm{Neu}}(p) < \cdots \to +\infty$,
- every $\lambda_n^{\mathrm{Neu}}(p)$, $n > 0$, is simple and isolated, the corresponding eigenfunction has exactly $n+1$ zeros in $(0,1)$, $\lambda_0^{\mathrm{Neu}}(p) = 0$ is not simple,
- the functions $p \mapsto \lambda_n^{\mathrm{Neu}}(p)$, $n \geq 0$, are continuous on $(1,\infty)$,
- the set of all eigenfunctions is discrete in $C^2[0,1]$,
- $\lambda_n^{\mathrm{Nav}}(p) < \lambda_n^{\mathrm{Neu}}(p) < \lambda_{n+1}^{\mathrm{Nav}}(p)$ for all $n \in \mathbb{N}$ and $p \in (1,\infty)$

Moreover, for $n \in \mathbb{N}$ we have $\lambda_n^{\mathrm{Neu}}(p) = \left(\lambda_n^{\mathrm{Dir}}(p^*)\right)^{p-1}$.

nonhomogeneous Dirichlet problem:

$$\begin{cases} \big(\psi_p(u''(t))\big)'' = \lambda \psi_q(u(t)), & t \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases}$$

- if u, v are solutions and $u''(0) = v''(0)$, then $u \equiv v$

$2n$th-order Dirichlet problem:

$$\begin{cases} (-1)^n \big(\psi_p(u^{(n)}(t))\big)^{(n)} = \lambda \psi_p(u(t)), & t \in [0,1], \\ u^{(i)}(0) = u^{(i)}(1) = 0, \quad i \in \{0, \dots, n-1\} \end{cases}$$

- every eigenvalue is positive and simple

$2n$th-order Neumann problem:

$$\begin{cases} (-1)^n \big(\psi_p(u^{(n)}(t))\big)^{(n)} = \lambda \psi_p(u(t)), & t \in [0,1], \\ u^{(i)}(0) = u^{(i)}(1) = 0, \quad i \in \{n, \dots, 2n-1\} \end{cases}$$

- every eigenvalue is nonnegative and every positive eigenvalue is simple

Global bifurcation

[Benedikt, J., "*Global bifurcation result for Dirichlet and Neumann p-biharmonic problem*"]

Dirichlet problem:

$$\begin{cases} \left(\psi_p(u''(t))\right)'' = \lambda\psi_p(u(t)) + g(t,\lambda,u(t),u'(t),u''(t)), & t\in[0,1],\\ u(0)=u'(0)=u(1)=u'(1)=0, \end{cases} \tag{8}$$

λ bifurcation parameter, $g\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ continuous,

$$g(t,\lambda,s_0,s_1,s_2) = o\left((|s_0|+|s_1|+|s_2|)^{p-1}\right), \quad |s_0|+|s_1|+|s_2|\to 0, \tag{9}$$

uniformly for $t\in[0,1]$ and λ from any bounded subset of $\mathbb{R}$.

Then every point $(\lambda_n^{\mathrm{Dir}}(p),0)\in\mathbb{R}\times C^2[0,1]$, $n\in\mathbb{N}$, belongs to a component of the closure of the set of all nontrivial solutions (λ,u) of (8), which is either unbounded in $\mathbb{R}\times C^2[0,1]$, or it contains an even number of points $(\lambda_n^{\mathrm{Dir}}(p),0)$, $n\in\mathbb{N}$.

Neumann problem:

$$\begin{cases} \left(\psi_p(u''(t))\right)'' = \lambda\psi_p(u(t)) + g(t,\lambda,u(t),u'(t),u''(t)), & t\in[0,1],\\ u''(0)=(\psi_p(u''))'(0)=u''(1)=(\psi_p(u''))'(1)=0, \end{cases} \tag{10}$$

$g\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ continuous and satisfies (9).
Then every point $(\lambda_n^{\mathrm{Neu}}(p),0)\in\mathbb{R}\times C^2[0,1]$, $n>0$, belongs to a component of the closure of the set of all nontrivial solutions (λ,u) of (10), which is either unbounded in $\mathbb{R}\times C^2[0,1]$, or it contains an even number of points $(\lambda_n^{\mathrm{Neu}}(p),0)$, $n>0$.

Existence in nonresonance

Dirichlet problem:

$$\begin{cases} \big(\psi_p(u''(t))\big)'' = f(t,u(t),u'(t),u''(t)), & t\in[0,1],\\ u(0)=u'(0)=u(1)=u'(1)=0, \end{cases} \tag{11}$$

$f\colon[0,1]\times\mathbb{R}^3\to\mathbb{R}$ continuous.

If for a $\lambda^{\mathrm{L}}\in\mathbb{R}$, not an eigenvalue of (3),

$$f(t,s_0,s_1,s_2)-\lambda^{\mathrm{L}}\psi_p(s_0)=o\left((|s_0|+|s_1|+|s_2|)^{p-1}\right), \quad |s_0|+|s_1|+|s_2|\to\infty, \quad \text{unif. for } t\in[0,1], \tag{12}$$

then (11) has a solution.

If, moreover,

$$f(t,s_0,s_1,s_2)-\lambda^{\mathrm{S}}\psi_p(s_0)=o\left((|s_0|+|s_1|+|s_2|)^{p-1}\right), \quad |s_0|+|s_1|+|s_2|\to 0, \quad \text{unif. for } t\in[0,1], \tag{13}$$

for a $\lambda^{\mathrm{S}}\in\mathbb{R}$, neither an eigenvalue of (3), and the number of the eigenvalues of (3) between λ^{L} and λ^{S} is odd, then (11) has the trivial and a nontrivial solution.

Neumann problem:

$$\begin{cases} \big(\psi_p(u''(t))\big)'' = f(t,u(t),u'(t),u''(t)), & t\in[0,1],\\ u''(0)=(\psi_p(u''))'(0)=u''(1)=(\psi_p(u''))'(1)=0, \end{cases} \tag{14}$$

$f\colon[0,1]\times\mathbb{R}^3\to\mathbb{R}$ continuous.

If (12) for a $\lambda^{\mathrm{L}}\in\mathbb{R}$, not an eigenvalue of (4), then (14) has a solution.

If, moreover, (13) for a $\lambda^{\mathrm{S}}\in\mathbb{R}$, neither an eigenvalue of (4), and the number of *positive* eigenvalues of (4) between λ^{L} and λ^{S} is odd, then (14) has the trivial and a nontrivial solution.