Positive Solutions of Discrete Equations

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Introduction and the Problem Considered

We use the following notation: for integers s, q, $s \le q$, we define

$$\mathbb{Z}_s^q := \{s, s+1, \ldots, q\}.$$

Using notation $\mathbb{Z}_s^q$, we suppose $s \le q$.
The subject of our study is a linear scalar discrete equation of k-th order

$$\Delta x(n) = -\sum_{i=0}^{k} p_i(n) x(n-i), \tag{1}$$

where

$$p_0 \colon \mathbb{Z}_a^\infty \to \mathbb{R}, \; p_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}_+ := [0, \infty),$$

$i = 1, \ldots, k$, $k \ge 1$, a is an integer, and $n \in \mathbb{Z}_a^\infty$.

Let

$$\varphi\colon \mathbb{Z}_{a-k}^{a} \longrightarrow \mathbb{R}.$$

Together with (1), we consider an initial problem: determine a solution $x = x(n)$ of equation (1) satisfying the initial conditions

$$x(n) = \varphi(n),\ \ n \in \mathbb{Z}_{a-k}^{a}. \tag{2}$$

A solution of initial problem (1), (2) is defined as an infinite sequence of numbers $\{x^n\}_{n=-k}^{\infty}$ with $x^n = x(a+n)$, i.e.,

$$\{x^{-k} = \varphi(a-k), \dots, x^0 = \varphi(a), \quad x^1 = x(a+1), \dots, x^n = x(a+n), \dots\}$$

such that for any $n \in \mathbb{Z}_a^{\infty}$ equality (1) holds. If it will be convenient, we denote the solution $x = x(n)$ of the initial problem (1), (2) as

$$x(n) = x(n; a, \varphi).$$

Our aim is to find sufficient conditions with respect to the right-hand side of equation (1) in order to guarantee the existence of at least one initial function

$$x(n) = \varphi^*(n),\ n \in \mathbb{Z}_{a-k}^{a} \tag{3}$$

with

$$\varphi^* \colon \mathbb{Z}_{a-k}^{a} \longrightarrow \mathbb{R}^+ := (0, \infty)$$

such that the solution $x^* = x^*(n; a, \varphi^*)$ of the initial problem (1), (2) (with $\varphi \equiv \varphi^*$) remains positive on $\mathbb{Z}_{a-k}^{\infty}$.

Nonlinear Preliminaries

Let us consider the scalar discrete equation

$$\Delta u(n) = f(n, u(n), u(n-1), \ldots, u(n-k)), \tag{4}$$

where $f\colon \mathbb{Z}_a^\infty \times \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$ and $k \geq 1$ is an integer.
Let $\varphi\colon \mathbb{Z}_{a-k}^a \longrightarrow \mathbb{R}$ be a given function. Together with discrete equation (4) we consider an initial problem: we are seeking for the solution $u = u(n)$, $n \in \mathbb{Z}_{a-k}^\infty$ of (4) satisfying initial conditions

$$u(n) = \varphi(n), \; n \in \mathbb{Z}_{a-k}^a. \tag{5}$$

The notion of a solution of the initial problem (4), (5) can be adapted easily from the previous section. The existence and uniqueness of the solution of the initial problem (4), (5) is obvious as well. If f is continuous, then the initial problem (4), (5) depends continuously on the initial data.

Let functions

$$\boldsymbol{b}, \boldsymbol{c}\colon \mathbb{Z}_{a-k}^{\infty} \longrightarrow \mathbb{R}$$

be given such that

$$\boldsymbol{b}(\boldsymbol{n}) < \boldsymbol{c}(\boldsymbol{n}), \;\; \boldsymbol{n} \in \mathbb{Z}_{a-k}^{\infty}.$$

For $\boldsymbol{n} \in \mathbb{Z}_{a-k}^{\infty}$ we define sets

$$\boldsymbol{\omega}(\boldsymbol{n}) := \{(\boldsymbol{n}, \boldsymbol{t})\colon \boldsymbol{t} \in \mathbb{R}, \boldsymbol{b}(\boldsymbol{n}) < \boldsymbol{t} < \boldsymbol{c}(\boldsymbol{n})\}$$

and

$$\boldsymbol{\omega}^*(\boldsymbol{n}) := \{(\boldsymbol{t})\colon \boldsymbol{t} \in \mathbb{R}, \boldsymbol{b}(\boldsymbol{n}) < \boldsymbol{t} < \boldsymbol{c}(\boldsymbol{n})\}.$$

Except this we define

$$\boldsymbol{\Omega} := \{(\boldsymbol{n}, \boldsymbol{t})\colon \boldsymbol{n} \in \mathbb{Z}_{a-k}^{\infty}, (\boldsymbol{n}, \boldsymbol{t}) \in \boldsymbol{\omega}(\boldsymbol{n})\}.$$

We will formulate an auxiliary nonlinear result on existence of a solution $u = u(n)$, $n \in \mathbb{Z}_{a-k}^{\infty}$ of (4) with the graph

$$\{(n, u(n))\}_{n=a-k}^{\infty}$$

remaining in Ω. It means, in other words, that under certain assumptions there exists at least one initial function φ such that

$$b(n) < \varphi(n) < c(n) \tag{6}$$

for $n \in \mathbb{Z}_{a-k}^{a}$ and

$$b(n) < u(n; a, \varphi) < c(n) \tag{7}$$

for every $n \in \mathbb{Z}_{a-k}^{\infty}$.

From inequalities (7) we can deduce the existence of a positive solution of the equation (4) if our sufficient conditions will be valid for the choice:

$$b(n) \equiv 0, \;\; c(n) > 0,$$

$n \in \mathbb{Z}_{a-k}^{\infty}$. This idea will be applied to the equation (1).

Now we are ready to formulate a nonlinear result, necessary for our investigation, concerning the existence of a solution of (4) with the graph lying in the set Ω.

Baštinec, J., Diblík J., Zhang, B.G.: Existence of bounded solutions of discrete delayed equations. *Proceedings of the Sixth International Conference on Difference Equations,* CRC, Boca Raton, FL, 359–366, 2004.

Theorem 1. *Let the function* $f\colon \mathbb{Z}_a^\infty \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ *be continuous. If, moreover, inequalities*

$$f(n, b(n), u_1, \dots, u_k) - b(n+1) + b(n) < 0, \tag{8}$$

$$f(n, c(n), u_1, \dots, u_k) - c(n+1) + c(n) > 0 \tag{9}$$

hold for every $n \in \mathbb{Z}_a^\infty$ *and every*

$$u_1 \in \omega^*(n-1), \dots, u_k \in \omega^*(n-k),$$

then there exists an initial problem

$$u(n) = \varphi(n),\ n \in \mathbb{Z}_{a-k}^a \tag{10}$$

with $\varphi\colon \mathbb{Z}_{a-k}^a \longrightarrow \mathbb{R}$*,* $\varphi(n) \in \omega^*(n)$*,* $n \in \mathbb{Z}_{a-k}^a$ *such that the corresponding solution* $u = u(n, a, \varphi)$ *of equation* (4) *satisfies*

$$b(n) < u(n; a, \varphi) < c(n) \tag{11}$$

for every $n \in \mathbb{Z}_{a-k}^\infty$*.*

Results

Theorem 2. *Let*

$$\sum_{i=1}^{k} p_i(n) > 0 \tag{12}$$

for any $n \in \mathbb{Z}_{a-k}^{\infty}$.
Then, for the existence of a positive solution $x = x(n)$ *of* (1), *the existence of a function*

$$\nu\colon \mathbb{Z}_{a-k}^{\infty} \longrightarrow \mathbb{R}^{+}$$

such that

$$\Delta\nu(n) \leq -\sum_{i=0}^{k} p_i(n)\nu(n-i) \tag{13}$$

for $n \in \mathbb{Z}_{a}^{+\infty}$ *is sufficient and necessary. Moreover* $x(n) < \nu(n)$ *holds on* $\mathbb{Z}_{a-k}^{\infty}$.

PROOF.

NECESSITY. It is obvious since it is possible to put $\nu \equiv x$, where x is a positive solution of (1).

SUFFICIENCY. We will use Theorem 1 with

$$f(n, u(n), u(n-1), \dots, u(n-k)) := -\sum_{i=0}^{k} p_i(n) u(n-i),$$

$$b(n) := 0,$$

$$c(n) := \nu(n).$$

In such case $\omega^*(n) \equiv \{(t)\colon t \in \mathbb{R}, 0 < t < \nu(n)\}$.
We verify inequalities (8), (9). With respect to (8) we have

$$f(n, b(n), u_1, \ldots, u_k) - b(n+1) + b(n) = f(n, 0, u_1, \ldots, u_k) = -\sum_{i=1}^{k} p_i(n) u_i.$$

It is easy to see that $u_i > 0$ if $u_i \in \omega^*(n-i)$, $i = 1, \ldots, k$. Then (we use (12) as well)

$$f(n, b(n), u_1, \ldots, u_k) - b(n+1) + b(n) \le -\sum_{i=1}^{k} p_i(n) < 0$$

and (8) holds.

With respect to the inequality (9) we have

$$
\begin{aligned}
&f(n, c(n), u_1, \ldots, u_k) - c(n+1) + c(n) \\
&\qquad = f(n, \nu(n), u_1, \ldots, u_k) - \nu(n+1) + \nu(n) \\
&\qquad = -p_0(n)\nu(n) - \sum_{i=1}^{k} p_i(n)u_i - \nu(n+1) + \nu(n).
\end{aligned}
$$

Since $u_i \in \omega^*(n-i)$, then $u_i < \nu(n-i)$, $i = 1, \dots, k$, and due to (12), (13)

$$
\begin{aligned}
f(n, c(n), &u_1, \dots, u_k) - c(n+1) + c(n) \\
&> -p_0(n)\nu(n) - \sum_{i=1}^{k} p_i(n)\nu(n-i) - \nu(n+1) + \nu(n) \\
&= -\sum_{i=0}^{k} p_i(n)\nu(n-i) - \Delta\nu(n) \geq 0.
\end{aligned}
$$

Inequality (9) is valid. We conclude that all the assumptions of Theorem 1 are valid. With respect to the equation (1) (we change u with x) it means that there exists an initial function $\varphi\colon \mathbb{Z}_{a-k}^{a} \to \mathbb{R}$, $\varphi(n) \in \omega^*(n)$, $n \in \mathbb{Z}_{a-k}^{a}$ such that $x = x(n, a, \varphi)$ satisfies the inequalities

$$0 \equiv b(n) < u(n; a, \varphi) < c(n) \equiv \nu(n) \tag{14}$$

for every $n \in \mathbb{Z}_a^{\infty}$. Inequality (14) coincides with the conclusion of Theorem 2. □

For the proof of the main result we need a comparison result for the equation (1) and an equation

$$\Delta w(n) = -\sum_{i=0}^{k} P_i(n) w(n-i) \tag{15}$$

where $P_0\colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $P_i\colon \mathbb{Z}_a^\infty \to \mathbb{R}_+$, $i = 1, \dots, k$, $k \geq 1$ under assumption $P_i(n) \leq p_i(n)$, $i = 1, \dots, k$, $n \in \mathbb{Z}_a^\infty$.

Theorem 3. *Let*

$$\sum_{i=1}^{k} P_i(n) > 0 \tag{16}$$

for any $n \in \mathbb{Z}_{a-k}^\infty$. *Assume that equation* (1) *admits a positive solution* $x = \mu(n)$ *on* $\mathbb{Z}_{a-k}^\infty$ *and*

$$P_i(n) \leq p_i(n), \tag{17}$$

$i = 1, \dots, k$, $n \in \mathbb{Z}_a^\infty$. *Then the equation* (15) *admits a positive solution* $w = w(n)$ *on* $\mathbb{Z}_{a-k}^\infty$ *and, moreover,* $w(n) < \mu(n)$.

PROOF. We will use Theorem 1 with

$$f(n, u(n), u(n-1), \dots, u(n-k)) := -\sum_{i=0}^{k} P_i(n) u(n-i),$$

$$b(n) := 0,\ c(n) := \mu(n).$$

Then $\omega^*(n) \equiv \{(t) \colon t \in \mathbb{R}, 0 < t < \mu(n)\}$. We verify inequalities (8), (9). With respect to (8) we have

$$f(n, b(n), u_1, \dots, u_k) - b(n+1) + b(n) = f(n, 0, u_1, \dots, u_k)$$
$$= -\sum_{i=1}^{k} P_i(n) u_i.$$

Since $u_i \in \omega^*(n-i)$ we have $u_i > 0$, $i = 1, \dots, k$ and (we use (16))

$$f(n, b(n), u_1, \dots, u_k) - b(n+1) + b(n) \le -\sum_{i=1}^{k} P_i(n) < 0.$$

Inequality (8) holds.

With respect to (9) we have

$$\begin{aligned} & f(n, c(n), u_1, \ldots, u_k) - c(n+1) + c(n) \\ & = f(n, \mu(n), u_1, \ldots, u_k) - \mu(n+1) + \mu(n) \\ & = -P_0(n)\mu(n) - \sum_{i=1}^{k} P_i(n) u_i - \mu(n+1) + \mu(n). \end{aligned}$$

Since $u_i \in \omega^*(n-i)$, then $u_i < \mu(n-i)$, $i = 1, \ldots, k$, and due to (16), (17)

$$
\begin{aligned}
f(n,&c(n), u_1, \ldots, u_k) - c(n+1) + c(n) \\
&> -P_0(n)\mu(n) - \sum_{i=1}^{k} P_i(n)\mu(n-i) - \mu(n+1) + \mu(n) \\
&\geq -\sum_{i=0}^{k} p_i(n)\mu(n-i) - \Delta\mu(n) = 0.
\end{aligned}
$$

Inequality (9) is valid and Theorem 1 holds. With respect to the equation (15) (we change u with w) it means that there exists an initial function $\varphi\colon \mathbb{Z}_{a-k}^{a} \to \mathbb{R}$, $\varphi(n) \in \omega^*(n)$, $n \in \mathbb{Z}_{a-k}^{a}$ such that $w = w(n, a, \varphi)$ satisfies the inequalities

$$
0 \equiv b(n) < w(n; a, \varphi) < c(n) \equiv \mu(n) \tag{18}
$$

for every $n \in \mathbb{Z}_a^{\infty}$. Inequality (18) coincides with the conclusion of Theorem 3. □

Before formulation of the main result we need auxiliary results on asymptotic decompositions.

Definition 1. *Let us define the expression* $\ln_q n$, $q \geq 1$ *by the formula* $\ln_q n = \ln(\ln_{q-1} n)$, $\ln_0 n \equiv n$ *where* $n > \exp_{q-2} 1$ *and* $\exp_s n = \exp\left(\exp_{s-1} n\right)$, $s \geq 1$, $\exp_0 n \equiv n$ *and* $\exp_{-1} n \equiv 0$ *(with* $\ln_0 n$, $\ln_1 n$ *abbreviated to* n, $\ln n$ *in the sequel).*

The following lemmas (necessary for the proof of the main result) can be proved in an elementary way by the method of induction. The symbol "o" means the Landau order symbol.

Lemma 1. *For fixed* $r, \sigma \in \mathbb{R} \setminus \{0\}$ *and for* $n \to \infty$ *the asymptotic representation*

$$(n-r)^\sigma = n^\sigma \left[1 - \frac{\sigma r}{n} + \frac{\sigma(\sigma-1)r^2}{2n^2} - \frac{\sigma(\sigma-1)(\sigma-2)r^3}{6n^3} + o\left(\frac{1}{n^3}\right)\right] \tag{19}$$

holds for $n \to \infty$.

Lemma 2. *For fixed* $r, \sigma \in \mathbb{R} \setminus \{0\}$, $q \geq 1$ *and for* $n \to \infty$ *the asymptotic representation*

$$\frac{\ln_q^\sigma (n-r)}{\ln_q^\sigma n} = 1 - \frac{r\sigma}{n \ln n \ldots \ln_q n} - \frac{r^2\sigma}{2n^2 \ln n \ldots \ln_q n}$$

$$- \frac{r^2\sigma}{2(n \ln n)^2 \ln_2 n \ldots \ln_q n} - \cdots - \frac{r^2\sigma}{2(n \ln n \ldots \ln_{q-1} n)^2 \ln_q n}$$

$$+ \frac{r^2\sigma(\sigma-1)}{2(n \ln n \ldots \ln_q n)^2} - \frac{r^3\sigma(1+o(1))}{3n^3 \ln n \ldots \ln_q n} \quad (20)$$

holds for $n \to \infty$.

Let $\ell \geq 0$ be a fixed integer. We define auxiliary functions

$$p_\ell(n) = \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k}{8(n \ln n \ldots \ln_\ell n)^2}\right] \quad (21)$$

and

$$\nu_\ell(n) = \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln n \ln_2 n \ldots \ln_\ell n} \quad (22)$$

which play an important role in the investigation of positive solutions of an equation

$$\Delta x(n) = -p(n)x(n-k) \quad (23)$$

being a particular case of (1) (with $p_0 \equiv p_1 \equiv \cdots \equiv p_{k-1} \equiv 0$ and $p_k \equiv p$). We assume that n in (21) and (22) is sufficiently large such that p_ℓ and ν_ℓ are well defined.

Lemma 3. *Let* $\ell \geq 0$ *be a fixed integer. Then the inequality*

$$\Delta\nu(n) \leq -p_\ell(n)\nu(n-k) \tag{24}$$

has a (positive) solution $\nu \equiv \nu_\ell$ *provided* n *is sufficiently large.*

PROOF. We consider the left hand side of (24) and asymptotically (for $n \longrightarrow \infty$) decompose $\Delta\nu_\ell(n)$.

$$\Delta\nu_\ell(n) = \nu_\ell(n+1) - \nu_\ell(n)$$

$$= \left(\frac{k}{k+1}\right)^{n+1} \cdot \sqrt{(n+1)\ln(n+1)\ln_2(n+1)\ldots\ln_\ell(n+1)}$$

$$- \left(\frac{k}{k+1}\right)^{n} \cdot \sqrt{n \ln n \ln_2 n \ldots \ln_\ell n}$$

$$= \left(\frac{k}{k+1}\right)^{n} \cdot \sqrt{n \ln n \ln_2 n \ldots \ln_\ell n} \times \left[\left(\frac{k}{k+1}\right) \cdot \mathcal{V}_1 - 1\right]$$

where

$$\mathcal{V}_1 = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{\sqrt{\ln(n+1)}}{\sqrt{\ln n}} \cdot \frac{\sqrt{\ln_2(n+1)}}{\sqrt{\ln_2 n}} \cdot \dots \cdot \frac{\sqrt{\ln_\ell(n+1)}}{\sqrt{\ln_\ell n}}$$

Since by formula (19) with $\sigma = 1/2$ and $r = -1$

$$\frac{\sqrt{n+1}}{\sqrt{n}} = 1 + \frac{1}{2n} - \frac{1}{8n^2} + \frac{1}{16n^3} + o\left(\frac{1}{n^3}\right),$$

by formula (20) with $\sigma = 1/2$, $q = 1$ and $r = -1$

$$\frac{\sqrt{\ln(n+1)}}{\sqrt{\ln n}} = 1 + \frac{1}{2n\ln n} - \frac{1}{4n^2\ln n} - \frac{1}{8(n\ln n)^2} + \frac{1+o(1)}{6n^3\ln n},$$

by formula (20) with $\sigma = 1/2$, $q = 2$ and $r = -1$

$$\frac{\sqrt{\ln_2(n+1)}}{\sqrt{\ln_2 n}} = 1 + \frac{1}{2n \ln n \ln_2 n} - \frac{1}{4n^2 \ln n \ln_2 n} - \frac{1}{4(n \ln n)^2 \ln_2 n} - \frac{1}{8(n \ln n \ln_2 n)^2} + \frac{1 + o(1)}{6n^3 \ln n \ln_2 n},$$

etc., and by formula (20) with $\sigma = 1/2$, $q = \ell$ and $r = -1$

$$\frac{\sqrt{\ln_\ell(n+1)}}{\sqrt{\ln_\ell n}} = 1 + \frac{1}{2n \ln n \ldots \ln_\ell n} - \frac{1}{4n^2 \ln n \ldots \ln_\ell n} - \ldots - \frac{1}{4(n \ln n \ldots \ln_{\ell-1} n)^2 \ln_\ell n} - \frac{1}{8(n \ln n \ldots \ln_\ell n)^2} + \frac{1 + o(1)}{6n^3 \ln n \ldots \ln_\ell n},$$

we have

$$\mathcal{V}_1 = 1 + \frac{1}{2n} + \frac{1}{2n\ln n} + \frac{1}{2n\ln n\ln_2 n} + \cdots + \frac{1}{2n\ln n\ldots\ln_\ell n} - \frac{1}{8n^2} - \cdots - \frac{1}{8(n\ln n\ldots\ln_\ell n)^2} + \frac{1}{16n^3} + o\left(\frac{1}{n^3}\right).$$

Now we (for $n \to \infty$) asymptotically decompose the right hand side of (24) with $\Delta\nu(n) \equiv \Delta\nu_\ell(n)$. We get

$$-p_\ell(n)\nu_\ell(n-k) = -\left(\frac{k}{k+1}\right)^n$$

$$\times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k}{8(n \ln n \ldots \ln_\ell n)^2}\right]$$

$$\times \left(\frac{k}{k+1}\right)^n \cdot \sqrt{(n-k)\ln(n-k)\ln_2(n-k)\ldots\ln_\ell(n-k)}$$

$$= -\left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln n \ln_2 n \ldots \ln_\ell n}$$

$$\times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k}{8(n \ln n \ldots \ln_\ell n)^2}\right] \times \mathcal{V}_2$$

where

$$\mathcal{V}_2 = \frac{\sqrt{n-k}}{\sqrt{n}} \cdot \frac{\sqrt{\ln(n-k)}}{\sqrt{\ln n}} \cdot \frac{\sqrt{\ln_2(n-k)}}{\sqrt{\ln_2 n}} \cdot \cdots \cdot \frac{\sqrt{\ln_\ell(n-k)}}{\sqrt{\ln_\ell n}}$$

Since by formula (19) with $\sigma = 1/2$ and $r = k$

$$\frac{\sqrt{n-k}}{\sqrt{n}} = 1 - \frac{k}{2n} - \frac{k^2}{8n^2} - \frac{k^3}{16n^3} + o\left(\frac{1}{n^3}\right),$$

by formula (20) with $\sigma = 1/2$, $q = 1$ and $r = k$

$$\frac{\sqrt{\ln(n-k)}}{\sqrt{\ln n}} = 1 - \frac{k}{2n\ln n} - \frac{k^2}{4n^2\ln n} - \frac{k^2}{8(n\ln n)^2} - \frac{k^3 + o(1)}{6n^3\ln n},$$

by formula (20) with $\sigma = 1/2$, $q = 2$ and $r = k$

$$\frac{\sqrt{\ln_2(n-k)}}{\sqrt{\ln_2 n}} = 1 - \frac{k}{2n \ln n \ln_2 n} - \frac{k^2}{4n^2 \ln n \ln_2 n} - \frac{k^2}{4(n \ln n)^2 \ln_2 n} - \frac{k^2}{8(n \ln n \ln_2 n)^2} - \frac{k^3 + o(1)}{6n^3 \ln n \ln_2 n},$$

etc., and by formula (20) with $\sigma = 1/2$, $q = \ell$ and $r = k$

$$\frac{\sqrt{\ln_\ell(n-k)}}{\sqrt{\ln_\ell n}} = 1 - \frac{k}{2n \ln n \ldots \ln_\ell n} - \frac{k^2}{4n^2 \ln n \ldots \ln_\ell n} - \ldots - \frac{k^2}{4(n \ln n \ldots \ln_{\ell-1} n)^2 \ln_\ell n} - \frac{k^2}{8(n \ln n \ldots \ln_\ell n)^2} - \frac{k^3 + o(1)}{6n^3 \ln n \ldots \ln_\ell n},$$

we have

$$\mathcal{V}_2 = 1 - \frac{k}{2n} - \frac{k}{2n \ln n} - \frac{k}{2n \ln n \ln_2 n} - \cdots - \frac{k}{2n \ln n \ldots \ln_\ell n}$$

$$- \frac{k^2}{8n^2} - \cdots - \frac{k^2}{8(n \ln n \ldots \ln_\ell n)^2} - \frac{k^3}{16n^3} + o\left(\frac{1}{n^3}\right).$$

Then

$$\left[\frac{1}{k+1}+\frac{k}{8n^2}+\frac{k}{8(n\ln n)^2}+\cdots+\frac{k}{8(n\ln n\ldots\ln_\ell n)^2}\right]\times \mathcal{V}_2$$

$$=\frac{1}{k+1}\left[1-\frac{k}{2n}-\cdots-\frac{k}{2n\ln n\ldots\ln_\ell n}\right]$$

$$+\frac{k}{k+1}\left[\frac{1}{8n^2}+\cdots+\frac{1}{8(n\ln n\ldots\ln_\ell n)^2}\right]$$

$$-\left(1+\frac{k}{k+1}\right)\frac{k^2}{16n^3}+o\left(\frac{1}{n^3}\right).$$

Now we see that for

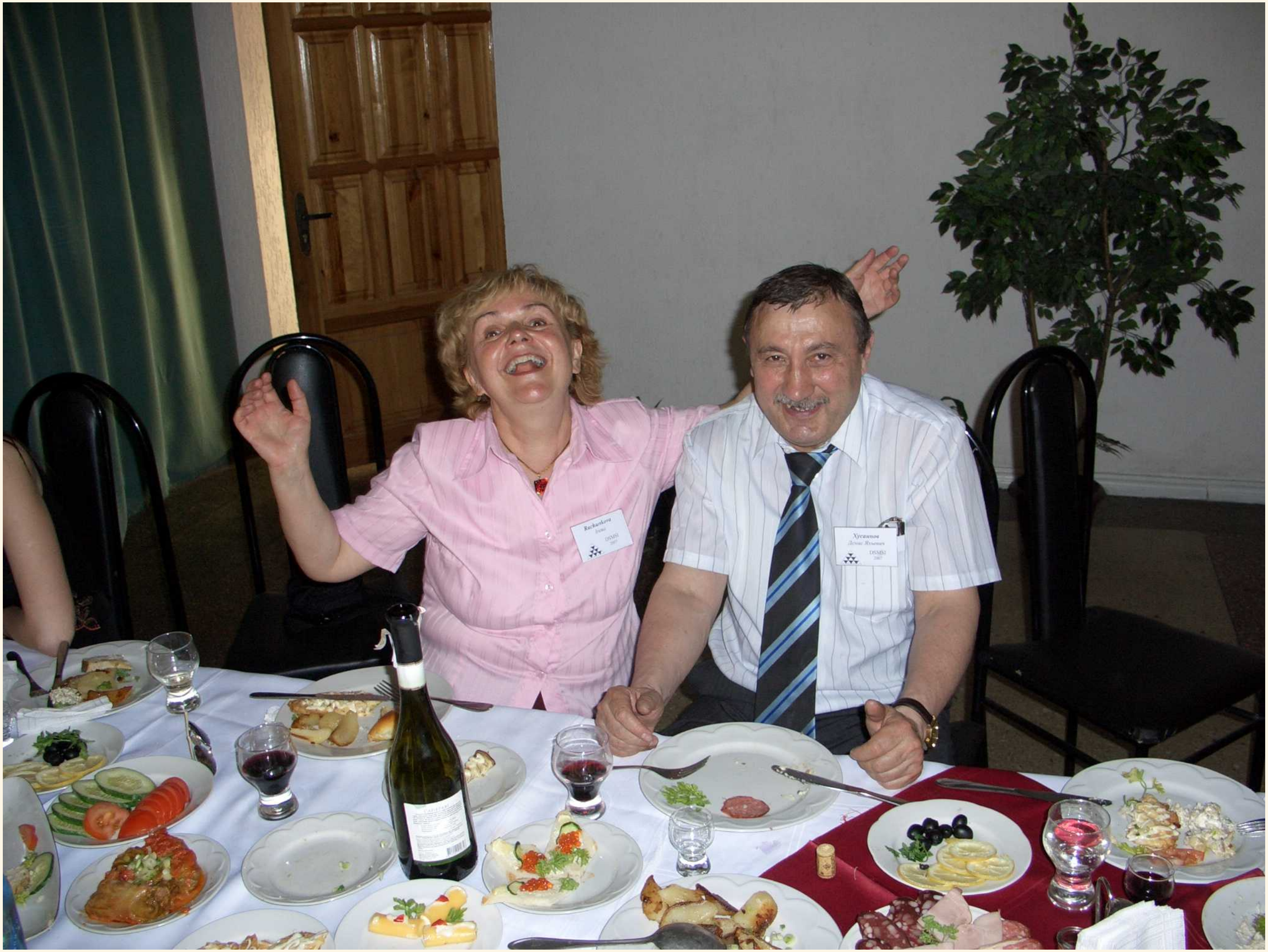
$$\Delta\nu_\ell(n) \le -p_\ell(n)\nu_\ell(n-k) \tag{25}$$

(if $n \to \infty$) is sufficient

$$\left(\frac{k}{k+1}\right) \cdot \mathcal{V}_1 - 1$$
$$\le - \left[\frac{1}{k+1} + \frac{k}{8n^2} + \cdots + \frac{k}{8(n \ln n \ldots \ln_\ell n)^2}\right] \times \mathcal{V}_2$$

or

$$\left(\frac{k}{k+1}\right)\left[1+\frac{1}{2n}+\cdots+\frac{1}{2n\ln n\ldots\ln_\ell n}-\frac{1}{8n^2}\right.$$
$$\left.-\frac{1}{8(n\ln n)^2}-\cdots-\frac{1}{8(n\ln n\ldots\ln_\ell n)^2}+\frac{1}{16n^3}\right]-1+o\left(\frac{1}{n^3}\right)$$
$$\leq -\frac{1}{k+1}\left[1-\frac{k}{2n}-\cdots-\frac{k}{2n\ln n\ldots\ln_\ell n}\right]$$
$$-\frac{k}{k+1}\left[\frac{1}{8n^2}+\cdots+\frac{1}{8(n\ln n\ldots\ln_\ell n)^2}\right]$$
$$+\left(1+\frac{k}{k+1}\right)\frac{k^2}{16n^3}+o\left(\frac{1}{n^3}\right).$$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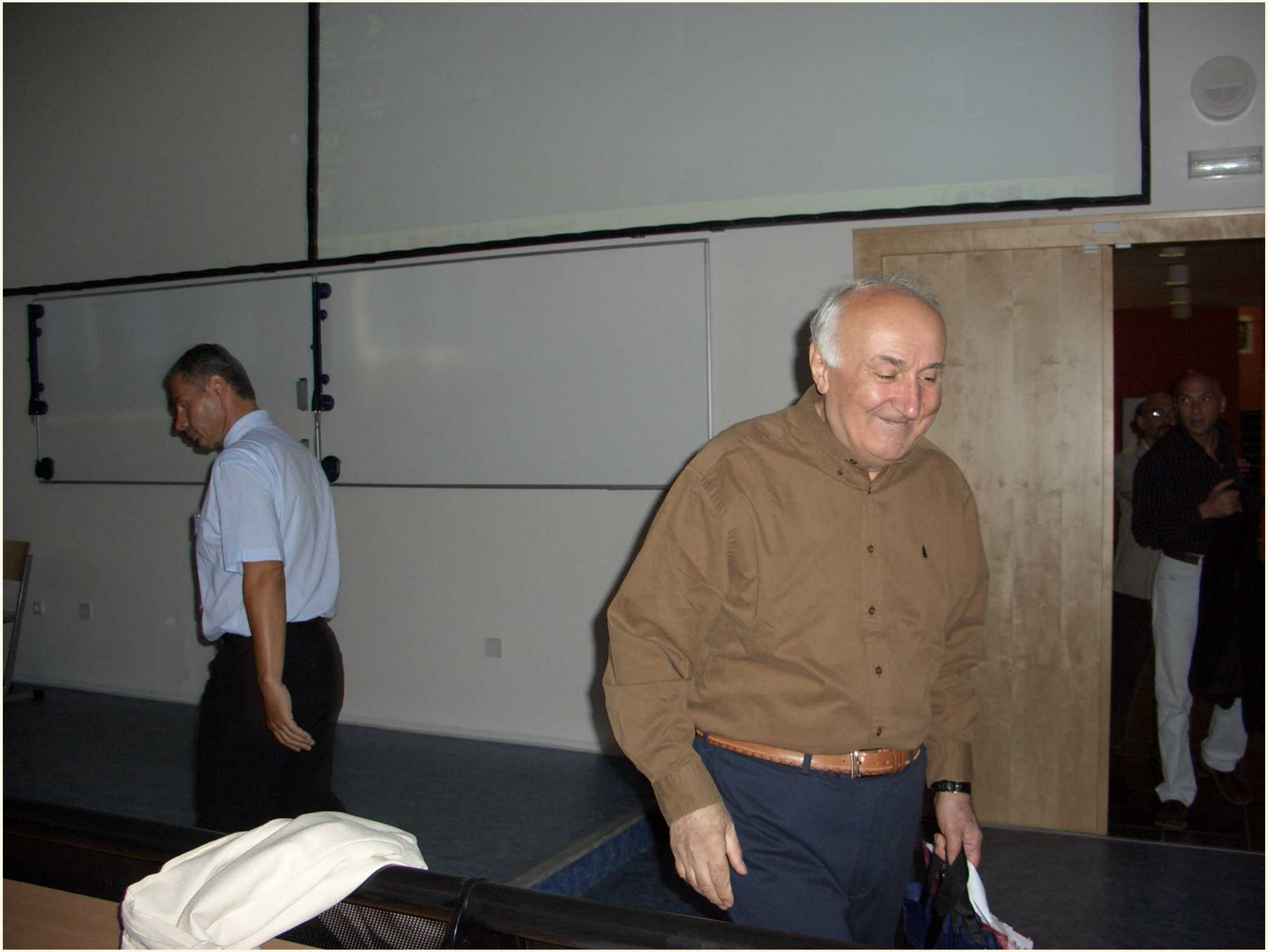


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We see that the last inequality turns into

$$0 \leq \frac{k(k^2+k-1)}{16n^3(k+1)} + o\left(\frac{1}{n^3}\right)$$

and holds if $n \to \infty$. □

The following result is an interesting consequence of Lemma 3 and Theorem 2.

Lemma 4. *Let $\ell \geq 0$ be a fixed integer. Then the equation*

$$\Delta x(n) = -p_\ell(n)x(n-k) \tag{26}$$

has a positive solution $x = x(n) < \nu_\ell(n)$ provided n is sufficiently large.

PROOF. Since the inequality (24) has a (positive) solution $\nu \equiv \nu_\ell$ provided n is sufficiently large then the proof is a straightforward consequence of Theorem 2 (we assume a is sufficiently large) with

$$p_0 \equiv p_1 \equiv \cdots \equiv p_{k-1} \equiv 0$$

$$p_k \equiv \nu_\ell.$$

□

Theorem 4 (Main result). *Let $\ell \geq 0$ be a fixed integer and*

$$0 < p(n) \leq p_\ell(n) \tag{27}$$

($p_\ell(n)$ is defined by (21)) for $n \to +\infty$. Then the equation (23) has a positive solution

$$x = x(n) < \nu_\ell(n)$$

provided n is sufficiently large.

PROOF. It is a direct consequence of Theorem 3 (we assume a is sufficiently large) with

$$P_0 \equiv P_1 \equiv \cdots \equiv P_{k-1} \equiv 0$$
$$P_k \equiv p(n)$$

and Lemma 4 if we put

$$p_0 \equiv p_1 \equiv \cdots \equiv p_{k-1} \equiv 0$$
$$p_k \equiv p_\ell(n)$$

in (1). □

Comparisons and Concluding Remarks

We formulate the following known result

Györi, I., Ladas, G.: *Oscillation Theory of Delay Differential Equations,* Clarendon Press, 1991.

Theorem 5. *Assume* $k \in \mathbb{N} \setminus \{0\}$, $p(n) > 0$ *for* $n \geq 0$, *and*

$$p(n) \leq \frac{k^k}{(k+1)^{k+1}}. \tag{28}$$

Then the difference equation

$$\Delta u(n) = -p(n)u(n-k) \tag{29}$$

where $n = 0, 1, 2, \ldots$ *has a positive solution.*

Comparing this result with the result given by Theorem 4 we conclude that the inequality (27) where

$$p_\ell(n) = \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n \ldots \ln_\ell n)^2}\right]$$

is a substantial improvement over (28). Moreover, the inequality (27) unlike the inequality (28) involves the variable n on the right hand side. As noted in the cited book, for $p(n) \equiv p =$ const, the inequality (28) is sharp in a sense since in this case the necessary and sufficient condition for the oscillation of all solutions of (29) is the inequality

$$p > \frac{k^k}{(k+1)^{k+1}}.$$

In

Diblík J.: Positive and oscillating solutions of differential equations with delay in the critical case. *J. Comput. Appl. Math.*, **88** (1998), 185–202.

is similar problem discussed for

$$\dot{x}(t) = -a(t)x(t-\tau) \tag{30}$$

where $a \in C(I, \mathbb{R}^+)$, $\tau > 0$.

Conjecture 1. *Let* $\ell \geq 0$ *be a fixed integer,* $\theta > 1$ *and*

$$p(n) \geq p_{\ell,\theta}(n) \tag{31}$$

for $n \to +\infty$ *with*

$$p_{\ell,\theta}(n) := \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k}{8(n \ln n \ldots \ln_{\ell-1} n)^2} + \frac{k\theta}{8(n \ln n \ldots \ln_\ell n)^2}\right].$$

Then all solutions of (23) *are (for* $n \to \infty$*) oscillating.*