On the Solvability of Boundary Value Problems for Systems of Generalized Ordinary Differential Equations and for Impulsive Equations

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Let $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$; $a_{mik} : [-a, a] \to R$ be continuous at the points $-a$ and a, nondecreasing functions for $m \in \{1, 2\}$ and $i, k \in \{1, \dots, n\}$; $a_{ik}(t) \equiv a_{1ik}(t) - a_{2ik}(t)$, $A = (a_{ik})_{i,k=1}^{n}$, $A_m = (a_{mik})_{i,k=1}^{n}$ $(m = 1, 2)$; $f = (f_k)_{k=1}^{n} : [-a, a] \times R^n \to R^n$ be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A, and $\varphi_i : \; BV_s([-a, a], R^n) \to R$ $(i = 1, \dots, n)$ be continuous functionals, which are nonlinear in general.

For the system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot f(t, x(t)), \tag{1}$$

where $x = (x_i)_{i=1}^{n}$, consider the multipoint boundary value problem

$$x_i(-\sigma_i a) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n). \tag{2}$$

Generalized ordinary differential equations are introduced by J.Kurzweil (1956). Quite a few questions of the theory (both linear and nonlinear)have been studied sufficiently well by J.Kurzweil, T.H.Hildebrandt, J.Groh, St.Schvabik, M.Tvrdy, N.Kekelia and other.

In this report we give necessary and sufficient as well effective sufficient conditions for the existence of solutions of the boundary value problem (1), (2). Analogous results are established for the multipoint boundary value problems for systems of ordinary differential equations by I.Kiguradze.

The theory of generalized ordinary differential equations enables one to investigate ordinary differential, impulsive and difference equations from the commonly accepted standpoint.

Throughout the report, the following notation and definitions will be used.

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$; $[a, b]$ $(a, b \in R)$ is a closed segment.

$R^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$;

$$R_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} : \; x_{ij} \geq 0 \; (i = 1, \dots, n; \; j = 1, \dots, m)\}.$$

$R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $R_+^n = R_+^{n\times 1}$.

$diag(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$; δ_{ij} is Kronecker symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ $(i, j = 1, \dots, n)$.

$\underset{a}{\overset{b}{V}}(X)$ is the total variation of the matrix-function $X : [a, b] \to R^{n\times m}$, i.e., the sum of total variations of the latter's components.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a, b] \to R^{n\times m}$ at the point t.We shall assume $X(t) = X(a)$ for $t \le a$ and $X(t) = X(b)$ for $t \ge b$, if necessary;

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t);$$

$$\|X\|_s = \sup\{\|X(t)\| : \ t \in [a, b]\}$$

.

$BV([a, b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a, b] \to R^{n\times m}$ (i.e., such that $\underset{a}{\overset{b}{V}}(X) < +\infty$);

$BV_s([a, b], R^n)$ is the normed space ($BV([a, b], R^n), \|\cdot\|_s$);

$\widetilde{C}([a, b], D)$, where $D \subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a, b] \to D$.

$\widetilde{C}_{loc}([a, b] \setminus \{\tau_k\}_{k=1}^m, D)$ is the set of all matrix-functions $X : [a, b] \to D$ whose restrictions to an arbitrary closed interval $[c, d]$ from $[a, b] \setminus \{\tau_k\}_{k=1}^m$ belong to $\widetilde{C}([c, d], D)$.

If B_1 and B_2 are normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if

$$g(\lambda x) = \lambda g(x)$$

for every $\lambda \in R_+$ and $x \in B_1$.

An operator $\varphi : \ BV([a, b], R^n) \to R^n$ is called nondecreasing if for every $x, y \in \ BV([a, b], R^n)$ such that $x(t) \le y(t)$ for $t \in [a, b]$ the inequality $\varphi(x)(t) \le \varphi(y)(t)$ holds for $t \in [a, b]$.

$S_j : \ BV([a, b], R) \to \ BV([a, b], R)$ $(j = 0, 1, 2)$ are the operators defined, respectively, by

$$S_1(x)(a) = S_2(x)(a) = 0,$$

$$S_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad , \quad S_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad for \quad a < t \le b,$$

and

$$S_0(x)(t) = x(t) - S_1(x)(t) - S_2(x)(t) \quad for \quad t \in [a,b].$$

If $g : [a,b] \to R$ is a nondecreasing function, $x : [a,b] \to R$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,dS_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,dS_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(S_0(g))$ corresponding to the function $S_0(g)$.

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \quad for \quad s \le t.$$

$L([a,b],R;g)$ is the set of all functions $x : [a,b] \to R$, measurable and integrable with respect to the measures $\mu(g_i)$ $(i = 1,2)$, i.e. such that

$$\int_a^b |x(t)|\,dg_i(t) < +\infty \quad (i = 1,2).$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset R^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)dg_{ik}(\tau)\Big)_{i,j=1}^{l,m} \quad for \quad a \le s \le t \le b,$$

$$S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 0,1,2).$$

If $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, then $K([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$: a) the function $f_{kj}(\cdot,x) : [a,b] \to D_2$ is $\mu(g_{ik})$-measurable for every $x \in D_1$; b) the function $f_{kj}(t,\cdot) : D_1 \to D_2$ is continuous for $\mu(g_{ik})$-almost every $t \in [a,b]$, and $\sup\{|f_{kj}(\cdot,x)| : x \in D_0\} \in L([a,b], R; g_{ik})$ for every compact $D_0 \subset D_1$.

If $G_j : [a,b] \to R^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a,b] \to R^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \quad for \quad s \le t,$$

$$S_k(G) = S_k(G_1) - S_k(G_2) \quad (k = 0,1,2),$$

$$L([a,b], D; G) = \bigcap_{j=1}^{2} L([a,b], D; G_j),$$

$$K([a,b]\times D_1, D_2; G) = \bigcap_{j=1}^{2} K([a,b]\times D_1, D_2; G_j).$$

If $G(t) \equiv \mathit{diag}(t,\dots,t)$, then we omit G in the notations containing G.

The inequalities between the vectors and between the matrices are understood componentwise. A vector-function $x \in BV([-a,a], R^n)$ is said to be a solution of the system (1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau, x(\tau)) \quad for \quad -a \le s \le t \le a.$$

By a solution of the system of generalized ordinary differential inequalities

$$dx(t) \le dA(t)\cdot f(t, x(t)) \ \ (\ge)$$

we mean a vector-function $x \in BV([-a,a], R^n)$ such that

$$x(t) \le x(s) + \int_s^t dA(\tau)\cdot f(\tau, x(\tau)) \ \ (\ge) \quad for \quad -a \le s \le t \le a.$$

If $s \in R$ and $\beta \in \ BV[a, b],)$ are such that

$$1 + (-1)^j d_j \beta(t) \neq 0 \ \ for \ \ \ (-1)^j (t - s) < 0 \quad (j = 1, 2),$$

then by $\gamma_\beta(\cdot, s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t)\, d\beta(t), \ \ \gamma(s) = 1.$$

It is known that

$$\gamma_\beta(t, s) = \exp(S_0(\beta)(t) - S_0(\beta)(s)) \prod_{s<\tau\leq t} (1 - d_1\beta(\tau))^{sgn(s-t)} \times$$

$$\prod_{s\leq\tau<t} (1 + d_2\beta(\tau))^{sgn(t-s)} \ \ for \ \ t, s \in [a, b]. \tag{3}$$

Definition 1 *Let* $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$. *We say the pair* $((c_{il})_{i,l=1}^n; (\varphi_{0i})_{i=1}^n)$, *consisting of a matrix-function* $(c_{il})_{i,l=1}^n \in \ BV([a, b], R^{n\times n})$ *and a positive homogeneous nondecreasing operator* $(\varphi_{0i})_{i=1}^n : \ \ BV_s([a, b], R_+^n) \rightarrow R_+^n$, *belongs to the set* $U^{\sigma_1,\dots,\sigma_n}$ *if the functions* c_{il} $(i \neq l;\ i, l = 1, \dots, n)$ *are nondecreasing on* $[a, b]$ *and continuous at the point* t_i,

$$d_j c_{ii}(t) \geq 0 \quad for \quad t \in [-a, a] \quad (j = 1, 2; \ \ i = 1, \dots, n)$$

and the problem

$$\sigma_i dx_i(t) \leq \sum_{l=1}^n x_l(t) dc_{il}(t) \ \ for \ \ t \in [-a, a] \backslash \{-\sigma_i a\} \ \ (i = 1, \dots, n),$$

$$(-1)^j d_j x_i(-\sigma_i a) \leq x_i(-\sigma_i a) d_j c_{ii}(-\sigma_i a) \quad (j = 1, 2; \ \ i = 1, \dots, n);$$

$$x_i(-\sigma_i a) \leq \varphi_{0i}(|x_1|, \dots, |x_n|) \quad (i = 1, \dots, n) \tag{4}$$

has no nontrivial non-negative solution.

The set $U^{\sigma_1,\dots,\sigma_n}$ has been introduced by I. Kiguradze for ordinary differential equations.

Theorem 1 *The problem* (1), (2) *is solvable if and only if there exist vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in BV([-a, a], R^n)$ $(m = 1, 2)$ *and matrix-functions* $(\beta_{mik})_{i,k=1}^n : [-a, a] \rightarrow R^{n\times n}$

$(m = 1,2)$ *such that* $\beta_{mik} \in L([-a,a],R;a_{jik})$ $(m,j = 1,2;$ $i,k = 1,\dots,n)$,

$$\alpha_{mi}(t) \equiv \alpha_{mi}(-\sigma_i a) + \sum_{l=1}^{n}\Big(\int_{-\sigma_i a}^{t} \beta_{mik}(\tau)\,da_{1ik}(\tau) -$$

$$\int_{-\sigma_i a}^{t} \beta_{3-mik}(\tau)\,da_{2ik}(\tau)\Big)(m = 1,2; i = 1,\dots,n);$$

$$\alpha_1(t) \le \alpha_2(t) \quad \textit{for} \;\; t \in [-a,a]; \tag{5}$$

$$(-1)^m\sigma_i(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n) - \beta_{mik}(t)) \le 0$$
$$\textit{for} \;\; \mu(a_{1+|m-j|ik}) - \textit{almost} \;\; \textit{all}\; t \in [-a,a],$$
$$\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \quad (m,j = 1,2; \;\; i,k = 1,\dots,n);$$
$$(-1)^m\Big(x_i - (-1)^j\sum_{k=1}^{n} f_k(t,x_1,\dots,x_n)d_j a_{ik}(t) - \alpha_{mi}(t) -$$
$$(-1)^j d_j\alpha_{mi}(t)\Big) \le 0 \; \textit{for}\; t \in [-a,a], \; \alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t),$$
$$(-1)^j\sigma_i > 0 \quad (m,j = 1,2; \;\; i = 1,\dots,n) \tag{6}$$

and the inequalities

$$\alpha_{1i}(-\sigma_i a) \le \varphi_i(x_l,\dots,x_n) \le \alpha_{2i}(-\sigma_i a) \;\; (i = 1,\dots,n) \tag{7}$$

are fulfilled on the set $\{(x_l)_{l=1}^n \in \; BV([a,b],R^n),\, \alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t)$ *for* $t \in [-a,a]\}$.

Corollary 1 *Let the matrix-function* $A(t) = (a_{ik})_{i,k=1}^n$ *be non-decreasing on* $[-a,a]$. *Then the problem* (1), (2) *is solvable if and only if there exist vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in BV([-a,a],R^n)$ $(m = 1,2)$ *and matrix-functions* $(\beta_{mik})_{i,k=1}^n :$ $[-a,a] \to R^{n\times n}$ $(m = 1,2)$ *such that* $\beta_{mik} \in L([-a,a],R;a_{jik})$ $(m,j = 1,2; i,k = 1,\dots,n)$,

$$\alpha_{mi}(t) \equiv \alpha_{mi}(-\sigma_i a) + \sum_{l=1}^{n}\Big(\int_{-\sigma_i a}^{t} \beta_{mik}(\tau)da_{1ik}(\tau)\Big)$$

$$(m = 1,2; \;\; i,k = 1,\dots,n);$$

the conditions (5)–(7) *hold, and the inequalities*

$$(-1)^m\sigma_i(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n)-\beta_{jik}(t))\le 0$$

are fulfilled for $\mu(a_{ik})$*-almost all* $t\in[-a,a]$ *and*
$\alpha_1(t)\le(x_l)_{l=1}^n\le\alpha_2(t)$.

Theorem 2 *Let the condition*

$$(-1)^{m+1}\sigma_i f_k(t,x_1,\dots,x_n)\ sign x_i\le\sum_{l=1}^n p_{mikl}(t)|x_l|+q_k(t)$$
$$for\ \ \mu(a_{mik})-almost\ \ all\ \ t\in[-a,a]$$
$$(m=1,2;\ \ i,k=1,\dots,n) \tag{8}$$

be fulfilled on R^n, *and let the inequalities*

$$|\varphi_i(x_1,\dots,x_n)|\le\varphi_{0i}(|x_1|,\dots,|x_n|)+\zeta_i\ \ (i=1,\dots,n) \tag{9}$$

be fulfilled on $BV([-a,a],R^n)$, *where* $(p_{mikl})_{k,l=1}^n\in L([-a,a],R^{n\times n};A_m)$ $(m=1,2;\ i=1,\dots,n)$, $q_k=(q_{ki})_{i=1}^n\in L([-a,a],R_+^n;A_m)$ $(m=1,2)$,$\zeta_i\in R_+$ $(i=1,\dots,n)$. *Let, moreover, there exists a matrix-function* $(c_{il})_{i,l=1}^n\in BV([-a,a],R^{n\times n})$ *such that*

$$((c_{il})_{i,l=1}^n;(\varphi_{0i})_{i=1}^n)\in U^{\sigma_1,\dots,\sigma_n} \tag{10}$$

and

$$\sum_{m=1}^2\sum_{k=1}^n\int_s^t p_{mikl}(\tau)da_{mik}(\tau)\le c_{il}(t)-c_{il}(s)$$
$$for\quad -a\le s<t\le a\quad (i,l=1,\dots,n). \tag{11}$$

Then the problem (1), (2) *is solvable.*

Corollary 2 *Let there exist* $m,m_1\in\{1,2\}$ *such that* $m+m_1=3$ *and the condition* (8) *and*

$$(-1)^{m_1+1}\sigma_i f_k(t,x_1,\dots,x_n)\ sgn x_i\le\sum_{l=1}^n\eta_{il}|x_l|+q_k(t)$$
$$for\ \ \mu_(a_{m_1ik})-almost\ all\ \ t\in[-a,a]\ \ (i,k=1,\dots,n)$$

be fulfilled on R^n, the inequalities

$$|\varphi_i(x_1,\dots,x_n)| \le \mu_i|x_i(s_i)| + \zeta_i \ \ (i=1,\dots,n) \tag{12}$$

be fulfilled on $BV_s([a,b],R^n)$,

$$0 \le d_j\alpha_i(t) < |\eta_{ii}|^{-1} \ for \ (-1)^j(t+\sigma_i a) > 0$$
$$(j=1,2; \ \ i=1,\dots,n) \tag{13}$$

and let

$$\mu_i\gamma_i(s_i,-\sigma_i a) < 1 \ \ (i=1,\dots,n), \tag{14}$$

where $(p_{mikl})_{k,l=1}^n \in L([-a,a],R_+^{n\times n};A_m)$ $(i=1,\dots,n)$, $\eta_{il} \in R_+$ $(i \ne l;\ i,l=1,\dots,n)$, $\eta_{ii} < 0$ $(i=1,\dots,n)$, $q_k = (q_{ki})_{k=1}^n \in L([-a,a],R_+^n;A_m)$ $(m=1,2)$,$\zeta_i \in R_+$ $(i=1,\dots,n)$; $\mu_i \in R_+$ and $s_i \in [-a,a]$, $s_i \ne -\sigma_i a$ $(i=1,\dots,n)$,

$$\alpha_i(t) \equiv \sum_{k=1}^n a_{m_1ik}(t) \ \ (i=1,\dots,n), \gamma_i(t,s) \equiv \gamma_{a_i}(t,s) \ \ (i=1,\dots,n),$$

$$a_i(t) \equiv \eta_{ii}\sigma_i(\alpha_(t) - \alpha_i(-\sigma_i a)) \ \ (i=1,\dots,n),$$

and the functions γ_{a_i} $(i=1,\dots,n)$ are defined according to (3)*. Let, moreover,*

$$g_{ii} < 1 \ \ (i=1,\dots,n) \tag{15}$$

and the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^n$ be negative, where

$$\xi_{il} = \eta_{il}[\delta_{il} + (1-\delta_{il})h_i] - \eta_{ii}g_{il} \quad (i,l=1,\dots,n),$$

$$g_{il} = \mu_i(1-\mu_i\gamma_i(s_i,-\sigma_i a))^{-1}\gamma_{il}(s_i) + \max\{\gamma_{il}(-a),\gamma_{il}(a)\}$$
$$(i,l=1,\dots,n),$$

$$\gamma_{il}(-\sigma_i a) = 0, \quad \gamma_{il}(t) = |\beta_{il}(t) - \beta_{il}(-\sigma_i a)| - (1-\delta_{il})d_j\beta_{il}(-\sigma_i a)$$
$$for \ \ (-1)^j(t+\sigma_i a) > 0 \ \ (j=1,2; \ \ i=1,l\dots,n),$$

$$\beta_{il}(t) \equiv \sum_{k=1}^n \int_{-a}^t p_{mikl}(\tau)da_{mik}(\tau) \ \ (i=1,\dots,n),$$

$$h_i = 1 \ \ for \ \ \mu_i \le 1 \ \ and$$

$$h_i = 1 + (\mu_i - 1)(1-\mu_i\gamma_i(s_i,-\sigma_i a))^{-1} \ \ for \ \ \mu_i > 1 \ \ (i=1,\dots,n).$$

Then the problem (1), (2) *is solvable.*

Remark 1 . *In the Corollary 2 we take as matrix-function* $C = (c_{il})_{i,l=1}^n$*:*

$$c_{il}(-\sigma_i a) = 0 \quad (i, l = 1, \dots, n),$$

$$c_{il}(t) = \eta_{il}(a_i(t) - a_i(-\sigma_i a) - (-1)^j d_j a_i(-\sigma_i a)) +$$

$$\beta_{il}(t) - -\beta_{il}(-\sigma_i a) - (-1)^j d_j \beta_{il}(-\sigma_i a) \quad textfor \quad (-1)^j (t + \sigma_i a) > 0$$

$$(j = 1, 2; \ \ i, l = 1, \dots, n).$$

If the matrix-function $A = (a_{ik})_{i,k=1}^n : [-a, a] \to R^{n \times n}$ is nondecreasing then the Corollary has the following form.

Corollary 3 *Let the matrix-function* $A = (a_{ik})_{i,k=1}^n : [-a, a] \to R^{n \times n}$ *be nondecreasing, the conditions* (12)–(14) *hold, the condition*

$$\sigma_i f_k(t, x_1, \dots, x_n) \ sgn x_i \le \sum_{l=1}^n \eta_{il} |x_l| + q_k(t)$$

$$for \ \ \mu_(a_{ik}) - almost \ all \ \ t \in [-a, a] \ \ (i, k = 1, \dots, n) \tag{16}$$

be fulfilled on R^n *and let the real part of every characteristic value of the matrix* $(\eta_{il}(\delta_{il} + (1 - \delta_{il}) h_i))_{i,l=1}^n$ *be negative, where*

$$\alpha_i(t) \equiv \sum_{k=1}^n a_{ik}(t) \ \ (i = 1, \dots, n),$$

and the functions $\gamma_i(t, s)$ $(i = 1, \dots, n)$ *and* $a_i(t)$ $(i = 1, \dots, n)$ *and the numbers* h_i $(i = 1, \dots, n)$ *are defined as in the Corollary* 2*. Then the problem* (1), (2) *is solvable.*

Corollary 4 *Let the matrix-function* $A = (a_{ik})_{i,k=1}^n : [-a, a] \to R^{n \times n}$ *be nondecreasing and continues from the left, the conditions* (12),(14),(16) *and*

$$0 \le d_2 \alpha_i(t) < |\eta_{ii}|^{-1} \ \ for \ \ t \in]-a, a[\ \ (i = 1, \dots, n)$$

hold and let the real part of every characteristic value of the matrix $(\eta_{il}(\delta_{il} + (1 - \delta_{il}) h_i))_{i,l=1}^n$ *be negative, where the functions* $\alpha_i(t)$ $(i = 1, \dots, n)$, $\gamma_i(t, s)$ $(i = 1, \dots, n)$ *and* $a_i(t)$ $(i = 1, \dots, n)$ *and the numbers* h_i $(i = 1, \dots, n)$ *are defined as in the Corollary 3. Then the problem* (1), (2) *is solvable.*

Boundary value problems for impulsive systems.

In this section we realize the results given above for the impulsive system

$$\frac{dx}{dt} = f(t, x(t)) \;\; for \;\; a.\,e. \;\; t \in [a,b] \setminus \{\tau_k\}_{k=1}^{m_0}, \tag{17}$$

$$x(\tau_k+) - x(\tau_k-) = I_k(x(\tau_k-)) \;\; (k = 1, \dots, m_0), \tag{18}$$

where $f = (f_k)_{k=1}^n \in K([-a,a] \times R^n, R^n)$, $I_k = (I_{ki})_{i=1}^n : R^n \to R^n$ $(k = 1, \dots, n)$ are arbitrary operators, $-a < \tau_1 < \dots < \tau_{m_0} \le a$ (we will assume $\tau_0 = -a$ and $\tau_{m_0+1} = a$, if necessary). By a solution of the impulsive system (17), (18) we understand a continuous from the left vector-function $x \in \widetilde{C}_{loc}([-a,a] \setminus \{\tau_k\}_{k=1}^{m_0}, R^n) \cap BV_s([-a,a], R^n)$ satisfying both the system (17) for a.a. $t \in [-a,a] \setminus \{\tau_k\}_{k=1}^{m_0}$ and the relation (18) for every $k \in \{1, \dots, m_0\}$.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well. The results analogous to those are obtained by I.Kiguradze for ordinary differential equations. But such results are unknown for impulsive equations, as far as we know. Using the theory of generalized ordinary differential equations, we extend these results to the systems of impulsive equations.

By $\nu(t)$ $(-a < t \le a)$ we denote the number of the points τ_k $(k = 1, \dots, m_0)$ belonging to $[-a, t[$.

To establish the results dealing with the boundary value problems for the impulsive system (17), (18) we use the following concept.

It is easy to show that the vector-function x is a solution of the impulsive system (17), (18) if and only if it is a solution of the system (1), where

$$A(t) \equiv \; diag(a_{11}(t), \dots, a_{nn}(t)),$$

$$a_{ii}(t) = t \quad for \quad -a \le t \le \tau_1,$$

$$a_{ii}(t) = t + k \quad for \;\; \tau_k < t \le \tau_{k+1}$$

$$(k = 1, \dots, m_0; \;\; i = 1, \dots, n),$$

$$f(\tau_k, x) \equiv I_k(x) \quad (k = 1, \dots, m_0).$$

It is evident that the matrix-function A is continuous from the left, $d_2A(t) = 0$ if $t \in [-a, a]\setminus\{\tau_k\}_{k=1}^{m_0}$ and $d_2A(\tau_k) = 1$ ($k = 1, \dots, m_0$). Therefore the system (17), (18) is a particular case of the system (1).

The definition of the set $U(t_1, \dots, t_n)$ has the following form in this case.

Definition 2 *Let* $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$ *and* $-a < \tau_1 < \dots \tau_{m_0} \le a$. *We say that the triple* $(Q, \{H_k\}_{k=1}^{m_0}, \varphi_0)$ *consisting of a matrix-function* $Q = (q_{il})_{i,l=1}^n \in L([-a, a], R^{n\times n})$, *a finite sequence of constant matrices* $H_k = (h_{kil})_{i,l=1}^n \in R^{n\times n}$ $(k = 1, \dots, m_0)$ *and a positive homogeneous nondecreasing continuous operator* $\varphi_0 = (\varphi_{0i})_{i=1}^n : BV([-a, a], R_+^n) \to R_+^n$ *belongs to the set* $U^{\sigma_1,\dots,\sigma_n}(\tau_1, \dots, \tau_{m_0})$ *if* $q_{il}(t) \ge 0$ $(i \ne l;\ i, l = 1, \dots, n)$ *for a.e.* $t \in [-a, a]$, $h_{kil} \ge 0$ $(i \ne l;\ i, l = 1, \dots, n;\ k = 1, \dots, k_{m_0})$, *and the system*

$$\sigma_i x_i'(t) \le \sum_{l=1}^n q_{il}(t)x_l(t) \ \ for \ \ a.a. \ \ t \in [a, b]\setminus\{\tau_k\}_{k=1}^{m_0} \ \ (i = 1, \dots, n),$$

$$x_i(\tau_k+) - x_i(\tau_k-) \le \sum_{l=1}^n h_{kil}x_l(\tau_k) \ \ (i = 1, \dots, n;\ \ k = 1, \dots, m_0)$$

has no nontrivial nonnegative solution satisfying the condition (4).

Theorem 3 *The impulsive problem* (17), (18); (2) *is solvable if and only if there exist continuous from the left vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in \widetilde{C}_{loc}([-a, a] \setminus \{\tau_k\}_{k=1}^{m_0}, R^n) \cap BV_s([-a, a], R^n)$ $(m = 1, 2)$ *such that the condition* (5) *holds,*

$$(-1)^j\sigma_i(f_i(t, x_1, \dots, x_{i-1}, \alpha_{ji}(t), x_{i+1}, \dots, x_n) - \alpha_{ji}'(t)) \le 0$$

$$for \ almost \ all \ \ t \in [-a, a] \setminus \{\tau_k\}_{k=1}^{m_0},$$

$$\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \ \ (j = 1, 2;\ \ i = 1, \dots, n);$$

$$(-1)^m(x_i - I_{ki}(x_1, \dots, x_n) - \alpha_{mi}(\tau_k+)) \le 0$$

$$for \ \ \alpha_1(\tau_k) \le (x_l)_{l=1}^n \le \alpha_2(\tau_k)$$

$$(m = 1, 2; \ \ i = 1, \dots, n; \ \ k = 1, \dots, m_0)$$

and the inequalities (7) *are fulfilled on the set*

$$\{(x_l)_{l=1}^n \in \widetilde{C}_{loc}([-a, a] \setminus \{\tau_k\}_{k=1}^{m_0}, R^n) \cap \ BV_s([-a, a], R^n),$$

$$\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \quad textfor \ \ t \in [-a, a]\}.$$

Theorem 4 *Let the conditions*

$$\sigma_i f_i(t, x_1, \dots, x_n) \ signx_i \le \sum_{l=1}^n p_{il}(t)|x_l| + q_i(t)$$

$$for \ almost \ all \ \ t \in [-a, a] \setminus \{\tau_k\}_{k=1}^{m_0} \ \ (i = 1, \dots, n)$$

and

$$\sigma_i I_{ki}(x_1, \dots, x_n) \ signx_i \le \sum_{l=1}^n h_{kil}(t)|x_l| + q_i(\tau_k)$$

$$(k=1, \dots, m_0; \ \ i=1, \dots, n) \tag{19}$$

be fulfilled on R^n, *and let the inequalities* (9) *be fulfilled on* $\widetilde{C}_{loc}([-a, a] \setminus \{\tau_k\}_{k=1}^{m_0}, R^n) \cap \ BV_s([-a, a], R^n)$ *and*

$$((p_{il})_{i,l=1}^n, \{(h_{kil})_{i,l=1}^n\}_{k=1}^{m_0}; (\varphi_{0i})_{i=1}^n) \in U^{\sigma_1, \dots, \sigma_n}(\tau_1, \dots, \tau_{m_0}),$$

where $q_i \in L([-a, a], R_+)$ $(i = 1, \dots, n)$, $\zeta_i \in R_+$ $(i = 1, \dots, n)$. *Then the impulsive problem* (17), (18); (2) *is solvable.*

Corollary 5 *Let the conditions* (19) *and*

$$\sigma_i f_i(t, x_1, \dots, x_n) \ signx_i \le \sum_{l=1}^n \eta_{il}(t)|x_l| + q_i(t)$$

$$for \ almost \ all \ \ t \in [-a, a] \setminus \{\tau_k\}_{k=1}^{m_0} \ \ (i = 1, \dots, n)$$

be fulfilled on R^n *and let*

$$-1 < \eta_{ii} < 0 \quad (i = 1, \dots, n),$$

$$\mu_i \exp(\eta_{ii}(s_i + a)) \cdot (1 + \eta_{ii})^{\nu(s_i)} < 1 \ \ (i = 1, \dots, n),$$

where η_{il} *and* $h_{ikl} \in R_+$ $(i \ne l;\ i, l = 1, \dots, n)$, μ_i *and* $\zeta_i \in R_+$ $(i = 1, \dots, n)$, $s_i \in [-a, a]$ *and* $s_i \ne -\sigma_i a$ $(i = 1, \dots, n)$.

Let, moreover, the condition (15) *hold and the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^{n}$ *be negative, where*

$$\xi_{il} = \eta_{il}[\delta_{il} + (1 - \delta_{il})h_{ij}] - \eta_{ii}g_{il} \quad (i, l = 1, \dots, n),$$

$$g_{il} = \mu_i(1 - \mu_i\gamma_i)^{-1}\gamma_{il}(s_i) + \gamma_{il}(a) \quad (i, l = 1, \dots, n),$$

$$\gamma_i = \exp(\eta_{ii}(s_i + a))(1 + \eta_{ii})^{\nu(s_i)} \quad (i = 1, \dots, n),$$

$$\gamma_{il}(-a) = 0, \;\; \gamma_{il}(t) = |\sum_{-a<\tau_k<t} h_{kil}| \;\; for \;\;]-a, a[\;\; (i, l = 1, \dots, n),$$

$$h_i = 1 \;\; for \;\; \mu_i \leq 1 \;\; and$$

$$h_i = 1 + (\mu_i - 1)(1 - \mu_i\gamma_i)^{-1} \;\; for \;\; \mu_i > 1 \;\; (i = 1, \dots, n).$$

Then the problem (1), (2) *is solvable.*

Remark 2 *In the Corollary 5 we take as matrix-function* $C = (c_{il})_{i,l=1}^{n}$*:*

$$c_{il}(t) \equiv \eta_{il}t + \beta_{il}(t) \quad (i, l = 1, \dots, n),$$

where

$$\beta_{il}(t) \equiv \eta_{il}t + \sum_{a<\tau_k<t} h_{kil} \quad (i, l = 1, \dots, n)$$

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